On the Limitations of Bubble Functions

by

Leopoldo P. Franca *
Department of Mathematics
University of Colorado at Denver
P.O.Box 173364, Campus Box 170
Denver, CO 80217-3364

Charbel Farhat
Department of Aerospace Engineering Sciences
and Center for Aerospace Structures
University of Colorado at Boulder
Boulder, CO 80309-0429

Abstract

We present two examples that demonstrate no advantage in enriching a finite element subspace with bubble functions.

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* Visiting Associate Professor from LNCC, Rua Lauro Müller 455, 22290 Rio de Janeiro, Brazil.

1. Quadratics or linears?

Let us consider the problem of finding the scalar valued function $u(x)$ defined on the unit interval and satisfying

$$-u_{,xx} = f \quad \text{on } (0,1) \tag{1}$$

$$u(0) = u(1) = 0 \tag{2}$$

Multiplying (1) by an arbitrary function $v \in H_0^1(\Omega)$ — where $H_0^1(\Omega)$ denotes the Hilbert space of functions satisfying (2) with square integrable value and derivative on the unit interval — and integrating on $(0,1)$ by parts, yields the variational formulation:

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v) \qquad v \in H_0^1(\Omega) \tag{3}$$

with

$$a(u,v) = (u_{,x}, v_{,x}) \tag{4}$$

$$f(v) = (f,v) \tag{5}$$

and $(\,\cdot\,,\,\cdot\,)$ denotes the integral on $(0,1)$.

The standard Galerkin method is obtained by computing with the same variational formulation (3) on a subspace of $H_0^1(\Omega)$ consisting of continuous functions that are piecewise polynomials on a partition of the unit interval. On each subinterval (or element) of the partition we define basis functions in a fixed reference coordinate ξ that varies on $(-1,1)$. For piecewise linears we have two basis functions on each element, that mapped to the ξ coordinate are given by:

$$\psi_1(\xi) = \frac{1}{2}(1-\xi) \tag{6}$$

$$\psi_2(\xi) = \frac{1}{2}(1+\xi) \tag{7}$$

Similarly, for quadratics the three basis functions employed are:

$$\varphi_1(\xi) = \frac{1}{2}\xi(\xi-1) \tag{8}$$

$$\varphi_2(\xi) = 1-\xi^2 \tag{9}$$

$$\varphi_3(\xi) = \frac{1}{2}\xi(\xi+1) \tag{10}$$

Before continuing note that φ_2 is zero on the boundaries of the element and positive on its interior. φ_2 is referred to as a *bubble function*. Note also that from (6)-(10) we can write:

$$\begin{aligned} \varphi_1(\xi) &= \psi_1(\xi) - \frac{\varphi_2(\xi)}{2} \\ \varphi_3(\xi) &= \psi_2(\xi) - \frac{\varphi_2(\xi)}{2} \end{aligned} \tag{11}$$

The stiffness matrix (or global matrix) of the problem can be obtained by assembling the contributions from each element. In the case of the linear element, this element level matrix is given by:

$$\mathbf{A}^l_K = \begin{bmatrix} a(\psi_1,\psi_1)_K & a(\psi_1,\psi_2)_K \\ symm. & a(\psi_2,\psi_2)_K \end{bmatrix} \tag{12}$$

where the subscript K on the bilinear form $a(\,\cdot\,,\,\cdot\,)$ given in (4) denotes the range of the integral of $a(\,\cdot\,,\,\cdot\,)$ is on the subinterval K instead of (0,1). The element load vector is given by

$$\mathbf{F}^l_K = \begin{bmatrix} f(\psi_1)_K \\ f(\psi_2)_K \end{bmatrix} \tag{13}$$

and the same convention applies for the subscript K.

Similarly for the quadratic approximation we have

$$\mathbf{A}_K^q = \begin{bmatrix} a(\varphi_1,\varphi_1)_K & a(\varphi_1,\varphi_2)_K & a(\varphi_1,\varphi_3)_K \\ & a(\varphi_2,\varphi_2)_K & a(\varphi_2,\varphi_3)_K \\ symm. & & a(\varphi_3,\varphi_3)_K \end{bmatrix} \tag{14}$$

and

$$\mathbf{F}_K^q = \begin{bmatrix} f(\varphi_1)_K \\ f(\varphi_2)_K \\ f(\varphi_3)_K \end{bmatrix} \tag{15}$$

The static condensation procedure consists in realizing that the unknown value corresponding to the bubble function φ_2 can be "eliminated" at the element level. More precisely, we can take $v = \varphi_2$ on K and zero elsewhere and write for each element K:

$$a(u_h,\varphi_2)_K = f(\varphi_2)_K \tag{16}$$

where u_h is the solution of the Galerkin method that can be written on an element K as

$$u_h|_K = \sum_{a=1}^{3} \varphi_a u_a \tag{17}$$

with $u_a, a = 1,2,3$, denoting the unknown values at the nodes corresponding to each shape (or basis) function defined on element K.

Substituting (17) into (16) yields

$$\sum_{a=1}^{3} a(\varphi_a,\varphi_2)_K\, u_a = f(\varphi_2)_K \tag{18}$$

or solving for u_2:

$$u_2 = \frac{1}{a(\varphi_2,\varphi_2)_K}(f(\varphi_2)_K - a(\varphi_2,\varphi_1)_K\, u_1 - a(\varphi_2,\varphi_3)_K\, u_3) \tag{19}$$

Eq. (19) gives an explicit formula for computing the unknown value associated with the mid-side node, and it clearly holds for any element K in our partition.

Let us now further simplify (19) using (11). Indeed,

$$a(\varphi_2,\varphi_1)_K = a(\varphi_2,\psi_1-\frac{\varphi_2}{2})_K = -\frac{1}{2}a(\varphi_2,\varphi_2)_K \tag{20}$$

since by integration by parts

$$\begin{aligned} a(\varphi_2,\psi_1)_K &= (\varphi_{2,x},\psi_{1,x})_K \\ &= -(\varphi_2,\psi_{1,xx})_K + (\varphi_2,\psi_{1,x})_{\partial K} \\ &= 0 \end{aligned} \tag{21}$$

Similarly,

$$a(\varphi_2,\varphi_3)_K = a(\varphi_2,\psi_2-\frac{\varphi_2}{2})_K = -\frac{1}{2}a(\varphi_2,\varphi_2)_K \tag{22}$$

Thus substituting (20) and (22) into (19) yields:

$$u_2 = \frac{1}{2}(u_1+u_3) + \frac{f(\varphi_2)_K}{a(\varphi_2,\varphi_2)_K} \tag{23}$$

In words, eq. (23) shows that if $f(\varphi_2)_K = 0$ then the bubble unknown value u_2 is the average of the vertex nodes. This occurs when there are point loads applied to vertex nodes, and in this particular circumstance there is no advantage in using quadratic approximation.

However even when $f(\varphi_2)_K \neq 0$ the vertex unknowns are the same as if we had computed with linears. Let us show this by further examining the element stiffness matrix and the element load vector in light of (11), (20)-(22), which reduces (14)-(15) to:

$$\mathbf{A}_K^q = \begin{bmatrix} a(\psi_1,\psi_1)_K + \frac{1}{4}a(\varphi_2,\varphi_2)_K & -\frac{1}{2}a(\varphi_2,\varphi_2)_K & a(\psi_1,\psi_2)_K + \frac{1}{4}a(\varphi_2,\varphi_2)_K \\ & a(\varphi_2,\varphi_2)_K & -\frac{1}{2}a(\varphi_2,\varphi_2)_K \\ symm. & & a(\psi_2,\psi_2)_K + \frac{1}{4}a(\varphi_2,\varphi_2)_K \end{bmatrix} \tag{24}$$

and

$$\mathbf{F}^q_K = \begin{bmatrix} f(\psi_1)_K - \frac{1}{2} f(\varphi_2)_K \\ f(\varphi_2)_K \\ f(\psi_2)_K - \frac{1}{2} f(\varphi_2)_K \end{bmatrix} \tag{25}$$

If we now take $\mathbf{A}^q_K \mathbf{u}$, where $\mathbf{u} = \{u_1 \; u_2 \; u_3\}^T$, then the second component is just the left-hand-side of (18), which we know by (18) equals the second component of $\mathbf{F}^q_K$. The first component of $\mathbf{A}^q_K \mathbf{u}$ is given by

$$(\mathbf{A}^q_K \mathbf{u})_1 = [a(\psi_1, \psi_1)_K + \frac{1}{4} a(\varphi_2, \varphi_2)_K] u_1 - \frac{1}{2} a(\varphi_2, \varphi_2)_K u_2 + [a(\psi_1, \psi_2)_K + \frac{1}{4} a(\varphi_2, \varphi_2)_K] u_3 \tag{26}$$

which using (23) reduces to

$$(\mathbf{A}^q_K \mathbf{u})_1 = a(\psi_1, \psi_1)_K u_1 + a(\psi_1, \psi_2)_K u_3 - \frac{1}{2} f(\varphi_2)_K \tag{27}$$

Similarly,

$$(\mathbf{A}^q_K \mathbf{u})_3 = a(\psi_2, \psi_1)_K u_1 + a(\psi_2, \psi_2)_K u_3 - \frac{1}{2} f(\varphi_2)_K \tag{28}$$

Therefore, neglecting the second row of $\mathbf{A}^q_K$ that has already been used to obtain (27)-(28), we can write

$$(\mathbf{A}^q_K \mathbf{u}) = \begin{bmatrix} a(\psi_1, \psi_1)_K & a(\psi_1, \psi_2)_K \\ symm. & a(\psi_2, \psi_2)_K \end{bmatrix} \begin{bmatrix} u_1 \\ u_3 \end{bmatrix} - \frac{1}{2} f(\varphi_2)_K \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{29}$$

which is equal to the linear element matrix, eq. (12), times the vertex unknowns minus the contribution to the right-hand-side that is also in (25).

Finally, assembling (29) and (25) to obtain the global equations, results in the *same* matrix and right-hand-side vector as if we had employed linear functions *ab initio*. Thus, the vertex unknowns on each element u_1 and u_3 will have the *same* value as if we had used linears, for any function $f(x)$.

Remark

An alternative explanation why there is no gain by using higher order interpolation for this particular model equation emanates from the well-known "superconvergence" result attributed to Douglas and Dupont that the finite element approximation nodally interpolates the exact solution at the vertex. Our argument is purely based on the observation that static condensation of the interior node of the quadratic polynomial yields the same matrix problem as if we had approximated directly with linears. Therefore the vertex node unknowns of the quadratic approximation give identical values as the node unknowns of the linear approximation, for any source function $f(x)$. A related result was briefly presented in [7].

2. The bubble function and the Poisson equation

Let us now consider the Poisson problem in two dimensions, i.e., we wish to find the scalar valued function $u(\mathbf{x})$ in $\Omega \subset I\!R^2$ such that

$$-\Delta u = f \qquad \text{in } \Omega \tag{30}$$

$$u = 0 \qquad \text{on } \Gamma = \partial\Omega \tag{31}$$

The variational formulation corresponding to (30)-(31) is:

Find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = f(v) \qquad v \in H_0^1(\Omega) \tag{32}$$

with

$$a(u,v) = (\nabla u, \nabla v) \tag{33}$$

$$f(v) = (f,v) \tag{34}$$

and $(\cdot\,,\cdot)$ denotes the integral on Ω.

Let us now consider a partition $\mathcal{C}_h$ of Ω into 'regular' triangles in the standard way (cf. Ciarlet [5]). We now consider the subspace of $H_0^1(\Omega)$ defined as

$$V_h = \{v \in H_0^1(\Omega) \,|\, v_{|K} \in P_1(K) \oplus B(K),\ K \in \mathcal{C}_h\}\,, \tag{35}$$

where $P_1(K)$ denotes the space of linear functions defined on the triangle K and $B(K)$ denotes the space of bubble functions spanned by, e.g., a cubic function. The bubble function $\varphi \in B(K)$ satisfies

$$\begin{aligned} \varphi(x) &> 0 \qquad \forall x \in K \\ \varphi(x) &= 0 \qquad \forall x \in \partial K \end{aligned} \tag{36}$$

and $\varphi = 1$ at the baricenter of the triangle.

Denoting by $\{\psi_i\}, i = 1,2,3$, the set of basis functions that span $P_1(K)$ and defined on the usual way (i.e., $\psi_i(\mathbf{a}_j) = \delta_{ij}, i,j = 1,2,3$, where $\mathbf{a}_j$ are the vertex coordinates of the triangle) then we can write $\forall v \in V_h$ in the triangle K:

$$v(x) = \sum_{i=1}^{3} v_i \psi_i(x) + v_0 \varphi(x) \tag{37}$$

where $v_i = v(\mathbf{a}_i)$.

Similarly to the 1-D case, we can eliminate the bubble unknown coefficient v_0 by taking $v = \varphi(x)$ in K and zero elsewhere in Ω, and write for each element K:

$$a(u_h, \varphi)_K = f(\varphi)_K \tag{38}$$

where u_h is the solution of the Galerkin method which by (37) can be written as

$$u_{h|K} = \sum_{i=1}^{3} u_i \psi_i + u_0 \varphi \tag{39}$$

Substituting into (38)

$$\sum_{i=1}^{3} a(\psi_i, \varphi)_K \, u_i + a(\varphi, \varphi)_K \, u_0 = f(\varphi)_K \tag{40}$$

Now, note that for $i = 1, 2$ or 3:

$$\begin{aligned} a(\psi_i, \varphi)_K &= (\nabla \psi_i, \nabla \varphi)_K \\ &= -(\Delta \psi_i, \varphi)_K + (\nabla \psi_i \cdot \mathbf{n}, \varphi)_{\partial K} \\ &= 0 \end{aligned} \tag{41}$$

Thus, the "bubble equation" (40) reduces to

$$a(\varphi, \varphi)_K \, u_0 = f(\varphi)_K \tag{42}$$

or

$$u_0 = \frac{f(\varphi)_K}{a(\varphi, \varphi)_K} \tag{43}$$

Note that in view of (41) the element stiffness matrix corresponding to this element can be written as:

$$\mathbf{A}_K = \begin{bmatrix} a(\psi_1, \psi_1)_K & a(\psi_1, \psi_2)_K & a(\psi_1, \psi_3)_K & 0 \\ & a(\psi_2, \psi_2)_K & a(\psi_2, \psi_3)_K & 0 \\ & & a(\psi_3, \psi_3)_K & 0 \\ symm. & & & a(\varphi, \varphi)_K \end{bmatrix} \tag{44}$$

When multiplying (44) by $\{u_1 \; u_2 \; u_3 \; u_0\}^T$ the last component gives the left-hand-side of (42), which equals to $f(\varphi)_K$. And the other first three components are the same as if we had used as a basis for V_h simply the set $\{\psi_i\}, i = 1, 2, 3,$. Therefore in assembling (44) the global matrix referring to the vertex unknowns is the *same* as if we had just used the linear support. Thus, the bubble function is unable to change the values of the unknowns in the vertex.

Remarks

1. The example presented in this section is closely related to the approximation using linear functions defined on subtriangles after a (not so careful) refinement (see Fig.1). The basis function associated with the baricenter node plays the role of a bubble function which satisfies (36) and is orthogonal to the linear functions associated to each vertex, as in (41). Using the relationship between the linear functions defined on subtriangles and their counterparts on the original triangle, one can show that the results with the original linear functions are the same as if there were no refinements as in Fig. 1. This observation was earlier noted in [6].

Fig. 1. A non productive mesh refinement

2. Static condensation of the bubble function in both sections gave us a "reduced" problem that had no distinction to the original one if approximated with linears

(a subspace of the the spaces with bubble functions). This will not be the case in general. If instead of the model equations treated herein, we consider for example the approximation of an advective-diffusive model by the Galerkin method using linears plus a bubble function, then static condensation of the bubble yields a stabilized finite element method of the SUPG type (see [2-4] and references therein). Similar result is obtained by eliminating the bubble from the MINI element [1] when approximating the Stokes problem (see [8-9]).

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