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by

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INCOMPRESSIBLE FLOWS BASED UPON STABILIZED METHODS

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SUMMARY

We give a brief overview of our efforts in developing stabilized finite element methods for incompressible flows. In particular, we review some suggestions for stability parameters and comment once more on the curious relationships between some stable Galerkin and stabilized finite element methods.

INTRODUCTION

Stabilized finite element methods are formed by adding perturbation terms to the standard Galerkin method, that are functions of the Euler-Lagrange equations evaluated elementwise, so that consistency is preserved. The perturbation terms are constructed to enhance stability of the original Galerkin formulation, allowing convergence of a variety of simple finite element interpolations in various applications.

For advective dominated equations it is well accepted to 'modify' the Galerkin method by adding 'artificial diffusion', whereas for elliptic problems with constraints, such as in the Stokes problem, approximation within the Galerkin method can be done employing the mixed method theory [2,4,6]. However, these two behaviors are present within the Navier-Stokes equations (i.e., locally advective dominated flows far from solid boundaries and locally diffusive dominated flows in stagnation regions) and the aforementioned numerical approaches seem conflicting one to the other: one admits addition of a perturbation term, the other suggests pursuing the Galerkin method framework.

Vis-à-vis these convictions, it could be considered a *sacrilège* to add a perturbation term to the Galerkin method to approximate the Stokes problem, as proposed in [19,20]. The rather intriguing result is that simple equal-low-order interpolations, as well as a wide range of combinations of simple piecewise polynomials for velocity and pressure are rendered convergent [14,16], an impossibility for the Galerkin method according to the mixed method theory. Nevertheless, this nonstandard methodology is actually related to the Galerkin method with a velocity space enriched by bubble functions [1], provided bubbles are eliminated at the element level and computations are performed in the reduced space without the bubbles [24]. In other words, static condensation of the MINI-element recovers the stabilized method [20] employing equal-order linear elements, a result that can be generalized for high order interpolations

[3]. Thus either the Galerkin method with bubbles and static condensation, or more directly stabilized methods can be used to deal with the Stokes problem.

Equally, it could be considered a *sacrilège* to employ the Galerkin method to approximate advective dominated equations [7,25]. However, this methodology falls within the artificial diffusion framework when we approximate the Galerkin method with the advected variable enriched by bubble functions and then eliminate the bubbles to compute in the reduced space [5]. Again, a stabilized method surfaces after condensation of the bubbles. Therefore either approach can be used to deal with advective dominated equations.

Therefore, our apparent dilemma to approximate the Navier-Stokes equations can be resolved by either employing a Galerkin method enriched with bubble functions (and using static condensation), or by using stabilized finite element methods. Both methods, separately, will deal with the two major numerical intricacies in these equations, namely advective-dominated flows and careful choices of velocity and pressure spaces. Both approaches are related, and therefore either could be used to solve our problem of interest: the Navier-Stokes equations.

Yet, neither one of these methods addresses the following question, phrased differently for the bubble and the artificial diffusion point-of-views, respectively: a) what are the shape of the bubbles?; or, b) how much perturbation should we add?

We know that there are 'virtual bubbles' [3] that can reproduce the asymptotic orders of the perturbation terms in the cases of locally advective dominated flows ($O(h)$) and locally diffusive dominated flows ($O(h^2)$) so that good accuracy properties are preserved. Still, this does not answer our question above.

Herein we review a few recent suggestions for the parameters of stabilized methods (the task seems considerably more complex for the corresponding appropriate bubble shape functions). We give designs including high order interpolations, that are amenable for p-adaptivity strategies. We review our suggestions of stability parameters for the model problems: Stokes equations and advective-diffusive scalar model, and combine them for the Navier-Stokes equations.

MODEL PROBLEMS

THE STOKES PROBLEM

Let us first consider the Stokes problem defined on an open bounded domain $\Omega \subset I\!R^N, N = 2, 3$ with a polygonal or polyhedral boundary Γ and given by

$$-2\mu\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{u}) + \nabla p = \mathbf{f} \qquad \text{in } \Omega, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{2}$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma \tag{3}$$

where $\mathbf{u}$ is the velocity, p is the pressure, μ is the viscosity, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetric part of the velocity gradient and $\mathbf{f}$ is the body force. The homogeneous boundary condition (3) suffices for our discussion, and can be simply generalized for more complex situations such as non-homogenous Dirichlet combined with Neumann boundary conditions.

We approximate velocity and pressure by the standard finite element spaces given by

$$\mathbf{V}_h = \{\mathbf{v} \in H_0^1(\Omega)^N \,|\, \mathbf{v}_{|K} \in R_k(K)^N,\ K \in \mathcal{C}_h\}\,, \tag{4}$$

$$P_h = \{p \in \mathcal{C}^0(\Omega) \cap L_0^2(\Omega) \,|\, p_{|K} \in R_l(K),\ K \in \mathcal{C}_h\}\,, \tag{5}$$

where $H_0^1(\Omega)$ is the Sobolev space of functions with square-integrable value and derivatives in Ω with zero value on the boundary Γ, the H^1-norm will be denoted by $\|\cdot\|_1$, $L_0^2(\Omega)$ is the space of functions with zero average and square-integrable values in Ω, the L^2-norm will be denoted by $\|\cdot\|_0$, $\mathcal{C}^0(\Omega)$ is the set of continuous functions in Ω, $\mathcal{C}_h$ is a partition of $\overline{\Omega}$ into elements K consisting of triangles (tetrahedrons in $\mathbb{R}^3$) or convex quadrilaterals (hexahedrons) performed in the usual way (i.e., no overlapping is allowed between any two elements of the partition; the union of all domains K reproduces $\overline{\Omega}$, etc.). Quasiuniformity is *not* assumed. We also employ the following notation:

$$R_m(K) = \begin{cases} P_m(K) & \text{if } K \text{ is a triangle or tetrahedron,} \\ Q_m(K) & \text{if } K \text{ is a quadrilateral or hexahedron.} \end{cases}$$

where for each integer $m \geq 0$, P_m and Q_m have the usual meaning.

A stabilized finite element method for this model can be written as: Find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in P_h$ such that

$$B(\mathbf{u}_h, p_h; \mathbf{v}, q) = F(\mathbf{v}, q) \qquad (\mathbf{v}, q) \in \mathbf{V}_h \times P_h \tag{6}$$

with

$$\begin{aligned} B(\mathbf{u}, p; \mathbf{v}, q) &= (2\mu\, \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v})) - (\nabla\cdot\mathbf{v}, p) + (\nabla\cdot\mathbf{u}, q) \\ &+ \sum_{K\in\mathcal{C}_h} \left(\nabla p - 2\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}), \tau(\nabla q - 2\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\right)_K \end{aligned} \tag{7}$$

and

$$F(\mathbf{v}, q) = (\mathbf{f}, \mathbf{v}) + \sum_{K\in\mathcal{C}_h} \left(\mathbf{f}, \tau(\nabla q - 2\mu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\right)_K \tag{8}$$

where $(\cdot,\cdot)$ denotes the $L^2(\Omega)$-inner product (between scalar, vector or second order tensor valued functions - context should be clear which is being used), and $(\cdot,\cdot)_K$ denotes the $L^2(K)$-inner product.

Following [20], the stability parameter can be written as

$$\tau = \frac{\alpha h_K^2}{2\mu} \tag{9}$$

where h_K is the element diameter and α is a non-dimensional stability constant recommended by numerical experience to be $O(1)$ for linear velocities and equal to $C_k/2$

for high order interpolations. C_k, in turn, is the inverse estimate constant defined to be the largest constant such that

$$C_k \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2 , \quad \forall \mathbf{v} \in \mathbf{V}_h \tag{10}$$

Such estimate holds for finite element polynomials defined on shape regular elements [9] (with no restrictions on size of elements) and computations of such constants can be found in [18].

Alternatively, following [15] one could define τ free from element diameters h_K and inverse estimate constants C_k by

$$\tau = \frac{1}{4\lambda_K \mu} \tag{11}$$

where

$$\lambda_K = \max_{0 \neq \mathbf{v} \in (R_k(K)/\mathbb{R})^N} \frac{\|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2}{\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2}, \qquad K \in \mathcal{C}_h \tag{12}$$

Remarks

1. The stabilized method (6)-(8) was proposed in [10] as a modification of the method introduced in [13,19,20]. From the stability analysis of this method any positive parameter τ renders this method stable [10,16]. However, from numerical experience [12], we know that better accuracy is obtained for a choice near the design given in (11)-(12).
2. The design given by (11)-(12) excludes linear velocity interpolations, in which case one should employ (9) with $\alpha = O(1)$. However, (11)-(12) can be employed for pressures that are linearly interpolated or higher, i.e., $l \geq 1$, provided that velocities are interpolated with piecewise polynomials of degree $k \geq 2$.
3. The parameter λ_K is computed as the largest eigenvalue of the following generalized eigenvalue problem defined for each K: Find $\mathbf{w}_h \in (R_k(K)/\mathbb{R})^N$ and λ_K such that

$$(\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{w}_h), \nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v}))_K - \lambda_K (\nabla \mathbf{w}_h, \nabla \mathbf{v})_K = 0 \qquad \forall\, \mathbf{v} \in (R_k(K)/\mathbb{R})^N \tag{13}$$

In practice to simulate the quotient space $(R_k(K)/\mathbb{R})^N$, we fix the degrees of freedom of a node to zero and solve for the remaining ones. The largest eigenvalue is computed using the power method. Once we have the computed value from the first element, this is used as the guess for the computation on the next and so on, until the whole mesh is covered. In the case of a non-linear problem, such as the Navier-Stokes equations, this computation is done once and for all, after the coordinates and type of interpolations are input, before entering any non-linear algorithm (Newton-like) to solve the nonlinear set of algebraic equations. Determining the largest eigenvalue on each element is *not* a costly procedure, as one may think at first. The generalized eigenvalue problem is to be solved only for the largest eigenvalue, and each problem is formulated at the element level, therefore involving relatively small matrices.

4. For this model the relevant inverse estimate given by (10) provides us with the link to the definition of λ_K, (12), through (13), i.e.:

$$\lambda_K^{-1} = C_k h_K^2 \qquad K \in \mathcal{C}_h \tag{14}$$

Therefore by using λ_K, we automatically carry the inverse estimate constant and the element diameter to our design of the stability parameter, and the design (11)-(12) is free from having to arbitrate parameters.

THE ADVECTIVE-DIFFUSIVE PROBLEM

Let us now consider the advective-diffusive problem of finding u such that

$$\mathbf{a} \cdot \nabla u - \kappa \Delta u = f \qquad \text{in } \Omega, \tag{15}$$

$$u = 0 \qquad \text{on } \Gamma, \tag{16}$$

where $\mathbf{a}(\mathbf{x})$ is the given velocity field with $\nabla \cdot \mathbf{a} = 0$, κ is the diffusivity and $f(\mathbf{x})$ is a source function, and all the notations and assumptions from the previous section for $\Omega, \Gamma, \mathcal{C}_h$, etc., remain in force throughout.

The scalar field u is approximated in the following standard finite element space:

$$V_h = \left\{ v \in H_0^1(\Omega) \,|\, v_{|K} \in R_k(K),\ K \in \mathcal{C}_h \right\}, \tag{17}$$

The stabilized finite element method we wish to consider is the Galerkin-least-squares method studied for this model in [21]: find $u_h \in V_h$ such that

$$B(u_h, v) = F(v) \qquad v \in V_h \tag{18}$$

with

$$B(u,v) = (\mathbf{a} \cdot \nabla u, v) + (\kappa \nabla u, \nabla v) + \sum_{K \in \mathcal{C}_h} (\mathbf{a} \cdot \nabla u - \kappa \Delta u, \tau\,(\mathbf{a} \cdot \nabla v - \kappa \Delta v))_K \tag{19}$$

and

$$F(v) = (f, v) + \sum_{K \in \mathcal{C}_h} (f, \tau\,(\mathbf{a} \cdot \nabla v - \kappa \Delta v))_K \tag{20}$$

Similarly to the last section we will consider two alternatives for the design of τ. The first proposed in [12] is given by:

$$\tau(\mathbf{x}, \mathrm{Pe}_K(\mathbf{x})) = \frac{h_K}{2|\mathbf{a}(\mathbf{x})|_p} \xi(\mathrm{Pe}_K(\mathbf{x})) \tag{21}$$

$$\mathrm{Pe}_K(\mathbf{x}) = \frac{m_k |\mathbf{a}(\mathbf{x})|_p h_K}{2\kappa(\mathbf{x})} \tag{22}$$

$$\xi(\mathrm{Pe}_K(\mathbf{x})) = \begin{cases} \mathrm{Pe}_K(\mathbf{x}) & , 0 \leq \mathrm{Pe}_K(\mathbf{x}) < 1 \\ 1 & , \mathrm{Pe}_K(\mathbf{x}) \geq 1 \end{cases} \tag{23}$$

$$|\mathbf{a}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^{N} |a_i(\mathbf{x})|^p\right)^{1/p} & , \quad 1 \le p < \infty \\ \max\limits_{i=1,N} |a_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{24}$$

$$m_k = \min\left\{\frac{1}{3},\ 2\widetilde{C}_k\right\} \tag{25}$$

$$\widetilde{C}_k \sum_{K \in \mathcal{C}_h} h_K^2 \|\Delta v\|_{0,K}^2 \le \|\nabla v\|_0^2 \qquad v \in V_h \tag{26}$$

The alternative design for $k \ge 2$ proposed in [15] is given by:

$$\tau = \frac{2}{\sqrt{\lambda_K}|\mathbf{a}(\mathbf{x})|_p} \xi(\mathrm{Pe}_K(\mathbf{x})) \tag{27}$$

$$\mathrm{Pe}_K(\mathbf{x}) = \frac{|\mathbf{a}(\mathbf{x})|_p}{4\sqrt{\lambda_K}\ \kappa(\mathbf{x})} \tag{28}$$

$$\xi(\mathrm{Pe}_K(\mathbf{x})) = \begin{cases} \mathrm{Pe}_K(\mathbf{x}) & , 0 \le \mathrm{Pe}_K(\mathbf{x}) < 1 \\ 1 & , \mathrm{Pe}_K(\mathbf{x}) \ge 1 \end{cases} \tag{29}$$

$$\lambda_K = \max_{0 \ne v \in R_k(K)/\mathbb{R}} \frac{\|\Delta v\|_{0,K}^2}{\|\nabla v\|_{0,K}^2} \qquad , K \in \mathcal{C}_h \tag{30}$$

$$|\mathbf{a}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^{N} |a_i(\mathbf{x})|^p\right)^{1/p} & , \quad 1 \le p < \infty \\ \max\limits_{i=1,N} |a_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{31}$$

Remarks

1. The design (27)-(31) excludes the linear interpolation case, $k = 1$. It is only valid for high order interpolations. In the case of linear interpolations, we recommend (21)-(25) with $m_1 = 1/3$.
2. The parameter λ_K is calculated by computing the largest eigenvalue of the following generalized eigenvalue problem defined for each K: Find $w_h \in R_k(K)/\mathbb{R}$ and λ such that

$$(\Delta w_h, \Delta v) - \lambda(\nabla w_h, \nabla v) = 0 \qquad \forall\, v \in R_k(K)/\mathbb{R} \tag{32}$$

   As before, the largest eigenvalue is computed using the power method.
3. By definition of λ_K in eq.(30), it follows from (26) and (32) that

$$\lambda_K^{-1} = \widetilde{C}_k h_K^2 \tag{33}$$

   This is the link between the λ_K parameter with the inverse estimate constant $\widetilde{C}_k$ and the mesh parameter h_K, similar to what we had for the Stokes problem.

The convergence analysis of GLS with the design of τ in (27)-(31) can be done similarly as in [12]. The main difference here is that inverse estimates are not needed. Instead, by definition of λ_K from eq.(30) we have

$$\lambda_K \|\nabla v\|_{0,K}^2 \geq \|\Delta v\|_{0,K}^2, \qquad K \in \mathcal{C}_h \qquad \forall\, v \in V_h \tag{34}$$

and this is used to establish stability as follows. First note that from (27)-(29), for $\mathrm{Pe}_K \geq 1$:

$$\tau = \frac{2}{\sqrt{\lambda_K}|\mathbf{a}(\mathbf{x})|_p} \frac{|\mathbf{a}(\mathbf{x})|_p}{4\sqrt{\lambda_K}\,\kappa} \frac{1}{\mathrm{Pe}_K} \leq \frac{1}{2\lambda_K \kappa(\mathbf{x})} \tag{35}$$

Since by definition the equality sign of (35) holds for $\mathrm{Pe}_K < 1$ therefore it follows that the bound (35) is valid for all $\mathrm{Pe}_K \geq 0$.

Now, by definition (eq.(19)), and using that $\nabla \cdot \mathbf{a} = 0$ and that κ is constant:

$$\begin{aligned} B(v,v) =& (\mathbf{a}\cdot\nabla v, v) + \kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - 2\sum_{K\in\mathcal{C}_h} (\mathbf{a}\cdot\nabla v, \tau\,\kappa\Delta v)_K \\ &+ \sum_{K\in\mathcal{C}_h} \|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 \\ \geq& \kappa\|\nabla v\|_0^2 + \frac{1}{2}\|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2 - \sum_{K\in\mathcal{C}_h} \|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 \end{aligned} \tag{36}$$

Note that from (34)-(35) the last term in (36) can be estimated as follows

$$\begin{aligned} \sum_{K\in\mathcal{C}_h} \|\tau^{1/2}\,\kappa\Delta v\|_{0,K}^2 =& \sum_{K\in\mathcal{C}_h} \|(\tau\,\kappa)^{1/2}\kappa^{1/2}\Delta v\|_{0,K}^2 \\ \leq& \sum_{K\in\mathcal{C}_h} \frac{\kappa}{2\lambda_K}\|\Delta v\|_{0,K}^2 \qquad \text{(by (35))} \\ \leq& \frac{\kappa}{2}\|\nabla v\|_0^2 \qquad \text{(by (34))} \end{aligned}$$

Therefore combining this estimate with (36) implies

$$B(v,v) \geq \frac{1}{2}(\kappa\|\nabla v\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla v\|_0^2) \tag{37}$$

which is the (numerical) stability result for this method. Combining with the intrinsic consistency of stabilized methods, then the following convergence of u_h solution of (18)-(20) to u solution of (15)-(16) follows in the the same norm of the stability result above (37):

$$\begin{aligned} &\kappa\|\nabla(u_h - u)\|_0^2 + \|\tau^{1/2}\,\mathbf{a}\cdot\nabla(u_h - u)\|_0^2 \\ &\leq C\sum_{K\in\mathcal{C}_h} h_K^{2k}|u|_{k+1,K}^2\Big(\mathrm{H}(\mathrm{Pe}_K - 1)h_K \sup_{\mathbf{x}\in K}|\mathbf{a}|_p + \mathrm{H}(1-\mathrm{Pe}_K)\kappa\Big) \end{aligned} \tag{38}$$

where H(·) is the Heaviside function given by

$$\mathrm{H}(x-y)=\begin{cases}0, & x<y;\\ 1, & x>y.\end{cases} \tag{39}$$

To establish (38), besides (37) we need an interpolation estimate for this particular design of τ that can be obtained as Lemma 3.2 of [12] using the relation between λ_K with C_k and h_K given by equation (33). This is the only instance that we need an inverse estimate in the analysis: to obtain the rates of convergence. Otherwise, the entire analysis goes through without ever needing (26) or (33).

THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

Let us consider the steady state incompressible Navier-Stokes given by:

$$(\nabla\mathbf{u})\mathbf{u}-2\nu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u})+\nabla p=\mathbf{f} \quad \text{in } \Omega, \tag{40}$$

$$\nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega, \tag{41}$$

$$\mathbf{u}=\mathbf{0} \quad \text{on } \varGamma, \tag{42}$$

where $\mathbf{u}$ is the velocity, p is the pressure, ν is the viscosity, $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetric part of the velocity gradient and $\mathbf{f}$ is the body force.

Let us consider a partition $\mathcal{C}_h$ of the domain Ω and the standard finite element spaces for velocity and pressure (4)-(5) as dicussed in the Stokes problem.

The stabilized finite element method as suggested in [11] can be written as: Find $\mathbf{u}_h\in\mathbf{V}_h$ and $p_h\in P_h$ such that

$$B(\mathbf{u}_h,p_h;\mathbf{v},q)=F(\mathbf{v},q), \qquad \forall(\mathbf{v},q)\in\mathbf{V}_h\times P_h, \tag{43}$$

with

$$\begin{aligned}B(\mathbf{u},p;\mathbf{v},q)&=((\nabla\mathbf{u})\mathbf{u},\mathbf{v})+(2\nu\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))-(\nabla\cdot\mathbf{v},p)+(\nabla\cdot\mathbf{u},q)\\&+(\nabla\cdot\mathbf{u},\delta\nabla\cdot\mathbf{v})\\&+\sum_{K\in\mathcal{C}_h}\big((\nabla\mathbf{u})\mathbf{u}+\nabla p-2\nu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{u}),\tau((\nabla\mathbf{v})\mathbf{u}+\nabla q-2\nu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\big)_K\end{aligned} \tag{44}$$

and

$$F(\mathbf{v},q)=(\mathbf{f},\mathbf{v})+\sum_{K\in\mathcal{C}_h}\big(\mathbf{f},\tau((\nabla\mathbf{v})\mathbf{u}+\nabla q-2\nu\nabla\cdot\boldsymbol{\varepsilon}(\mathbf{v}))\big)_K \tag{45}$$

Similarly to the previous sections, we consider two alternative designs of the stability parameters τ and δ. The first proposed in [11] is given by:

$$\delta=\lambda|\mathbf{u}(\mathbf{x})|_p h_K\xi(\mathrm{Re}_K(\mathbf{x})) \tag{46}$$

$$\tau(\mathbf{x},\mathrm{Re}_K(\mathbf{x}))=\frac{h_K}{2|\mathbf{u}(\mathbf{x})|_p}\xi(\mathrm{Re}_K(\mathbf{x})) \tag{47}$$

$$\mathrm{Re}_K(\mathbf{x}) = \frac{m_k |\mathbf{u}(\mathbf{x})|_p h_K}{4\nu(\mathbf{x})} \tag{48}$$

$$\xi(\mathrm{Re}_K(\mathbf{x})) = \begin{cases} \mathrm{Re}_K(\mathbf{x}) & , 0 \leq \mathrm{Re}_K(\mathbf{x}) < 1 \\ 1 & , \mathrm{Re}_K(\mathbf{x}) \geq 1 \end{cases} \tag{49}$$

$$|\mathbf{u}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^N |u_i(\mathbf{x})|^p\right)^{1/p} & , \quad 1 \leq p < \infty \\ \max\limits_{i=1,N} |u_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{50}$$

$$m_k = \min\left\{\frac{1}{3},\ 2C_k\right\} \tag{51}$$

$$C_k \sum_{K \in \mathcal{C}_h} h_K^2 \|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2 \leq \|\boldsymbol{\varepsilon}(\mathbf{v})\|_0^2 \qquad \mathbf{v} \in \mathbf{V}_h \tag{52}$$

The alternative design for $k \geq 2$ proposed in [15] is given by:

$$\delta = \frac{|\mathbf{u}(\mathbf{x})|_p}{\sqrt{\lambda_K}} \xi(\mathrm{Re}_K(\mathbf{x})) \tag{53}$$

$$\tau = \frac{\xi(\mathrm{Re}_K(\mathbf{x}))}{\sqrt{\lambda_K} |\mathbf{u}(\mathbf{x})|_p} \tag{54}$$

$$\mathrm{Re}_K(\mathbf{x}) = \frac{|\mathbf{u}(\mathbf{x})|_p}{4\sqrt{\lambda_K}\nu(\mathbf{x})} \tag{55}$$

$$\xi(\mathrm{Re}_K(\mathbf{x})) = \begin{cases} \mathrm{Re}_K(\mathbf{x}) & , 0 \leq \mathrm{Re}_K(\mathbf{x}) < 1 \\ 1 & , \mathrm{Re}_K(\mathbf{x}) \geq 1 \end{cases} \tag{56}$$

$$\lambda_K = \max_{0 \neq \mathbf{v} \in (R_k(K)/\boldsymbol{R})^N} \frac{\|\nabla \cdot \boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2}{\|\boldsymbol{\varepsilon}(\mathbf{v})\|_{0,K}^2} \qquad , K \in \mathcal{C}_h \tag{57}$$

$$|\mathbf{u}(\mathbf{x})|_p = \begin{cases} \left(\sum_{i=1}^N |u_i(\mathbf{x})|^p\right)^{1/p} & , \quad 1 \leq p < \infty \\ \max\limits_{i=1,N} |u_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{58}$$

Remarks

1. The stabilized formulation given was introduced in [11], where a convergence analysis is given for a linearized form of the incompressible Navier-Stokes equations and a few numerical simulations are presented. A related form is also proposed in [22]. For low order interpolations, see [8,17,26]. Herein we wish to emphasize the definitions of the stability parameters τ and δ. In particular, using larger values of τ than the ones presented above, may produce spurious oscillations when we employ high order interpolations (see [11] for a detailed discussion).
2. The design given by (53)-(58) excludes linear velocity interpolations. However, pressures may be linearly interpolated or higher, i.e., $l \geq 1$.

3. Similarly to the previous sections, the parameter λ_K is computed as the largest eigenvalue of the generalized eigenvalue problem given in equation (13). Again, by definition of λ_K in (57) combined with (13) and (52) yields

$$\lambda_K^{-1} = C_k h_K^2 \tag{59}$$

The same link of λ_K with C_k and h_K holds as in the Stokes model.

4. Convergence analysis taking into account the design in (53)-(58) of the stability parameters can be performed for a linearized model similarly as in [11]. As pointed out before, inverse estimates are no longer needed to establish stability and carry out the entire analysis, up to the point where interpolation estimates results are needed to characterize the rates of convergence. The analysis considerations for this case are similar to what is described in the advective-diffusive model section, which combined with the analysis presented in [11] yields a similar convergence result.
5. The design of the stability parameter τ for the Stokes flow given by (11)-(12) can be obtained by taking the limit as $\mathrm{Re}_K \to 0$ in (53)-(58). It can be viewed as the diffusive limit of the general situation of advective-diffusive incompressible flows governed by the Navier-Stokes equations.
6. Discretization of (43) is carried out by expanding the trial functions $\mathbf{u}_h, p_h$ and the test functions in terms of their finite element basis or shape functions. This leads to a set of nonlinear algebraic equations, parametrized, in particular, by the Reynolds number. To solve this set of equations for a fixed Reynolds number, we employ the standard Newton-Raphson method, combined with a Quasi-Newton strategy. The Quasi-Newton part consists of "freezing" the matrices if the norm of the residuals of the algebraic equations are monotonically decreasing. Otherwise, the matrices are updated with the latest increment, and the strategy is restarted. On the top of this algorithm we also make a continuation on the Reynolds number, which roughly consists in: solve the problem for a low Reynolds number, and use this solution as the initial guess for a larger Reynolds number problem, until the Reynolds number we wish to calculate is reached.
7. The matrix system for each Newton iteration is solved by a direct method (Gaussian elimination) for small two-dimension benchmark flows, and with GMRES for large two-dimensional and three-dimensional flows. We refer to [11,23] for numerical experiments employing this methodology.

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