Conditional Chromatic Numbers with Forbidden Cycles

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Revised December, 1993

Abstract

The *conditional chromatic number* $\chi(G, P)$ of a graph G with respect to a graphical property P is the minimum number of colors needed to color the nodes of G such that each color class induces a subgraph of G with property P. The conditional chromatic number with respect to various properties has been studied by numerous authors. We investigate $\chi(G, P)$ where P is the property of having no cycle of length j for fixed $j \geq 3$. We find the conditional chromatic number with forbidden cycles for graphs of large size and for all graphs with order at most six.

1 Introduction

A *coloring* of a graph is an assignment of colors to its nodes so that no two adjacent nodes have the same color. The *chromatic number* $\chi(G)$ is the minimum k for which G has a coloring with k colors. We consider the following generalization of the chromatic number: A P-*coloring* of a graph is an assignment of colors to its nodes so that the subgraph induced by each color class satisfies the property P. The P-*conditional chromatic number*, or briefly P-*chromatic number*, $\chi(G, P)$ of a graph G is the minimum k for which G has a P-coloring with k colors. When P is the property that a graph consists entirely of isolated nodes, the P-chromatic number coincides with the usual chromatic number. This terminology was introduced by F. Harary [12] in 1985. However, the P-chromatic number has been studied under various guises. Other terms referring to the conditional chromatic number of a graph include the generalized chromatic number [5,6], the subchromatic number [1] and the P conditional partition number [15].

When P is the property that a graph contains no subgraph (not necessarily induced) isomorphic to a graph F, we write $\chi(G, \neg F)$ for the P-chromatic number and refer to a P-coloring as a $\neg F$-coloring and the P-chromatic number as the $\neg F$-chromatic number. A variation of the $\neg F$-chromatic number has been studied in [17] and [4]. Broere and Mynhardt study the general property that a graph contains no *induced* subgraph isomorphic to a given graph F. Various authors have studied the $\neg F$-chromatic number for special types of graphs F. For example, when F is a path of length k, the $\neg F$-chromatic number was studied in [7]; when F is $K_{1,m}$ (or equivalently the graph contains no node of degree $\geq m$), in [13], [14], [2], and [3]; and when F is the complete graph of order k, in [19]. See [1] for a survey of these and other results related to conditional chromatic numbers.

Here we investigate $\chi(G, \neg F)$ where F is the cycle (or j-cycle) C_j on $j \geq 3$ nodes. A closely related conditional chromatic number is $\chi(G, P_a)$ where P_a is the property that a graph be acyclic. This conditional chromatic number was studied in [15], [9], and [8]. The P_a-chromatic number is also called the *point arboricity* of the graph.

In the remainder of the paper a graph $G = (V, E)$ will be simple without loops and we will generally utilize the graphical definitions and notation of [11]. However, we write $|V| = n = n(G)$, $|E| = e = e(G)$. In addition we let mG denote m disjoint copies of the graph G; and $G + H$, the disjoint union of the graphs G and H. For $X \subseteq V$, $N(X) = \{u | uv \in E \text{ for some } v \in X\}$, $N[X] = N(X) \cup \{X\}$, and $H(X) = V - N[X]$. When X is a single node x, the preceding definitions correspond respectively to the usual definitions of the open neighborhood of x, closed neighborhood of x, and to the non-neighbors of x. If S is a subset of either E or V, then $\langle S \rangle$ is the subgraph of G induced by S. The minimum degree and the maximum degree of any node are $\delta(G)$ and $\Delta(G)$ respectively.

2 Preliminary Results

We present some elementary results about the $\neg C_j$-chromatic numbers and some implications for these numbers that follow from the research of other authors. In the following theorem, we summarize some results about $\chi(G, \neg C_j)$ whose proofs are straightforward. Several of these results follow from observations of previous authors.

Theorem 2.1 *(a)* $\chi(K_n, \neg C_j) = \lceil n/(j-1) \rceil$.

(b) If H is a subgraph of the graph G, then $\chi(H, \neg C_j) \leq \chi(G, \neg C_j)$.

(c) For every graph G, $\chi(G, \neg C_j) \leq \lceil n/(j-1) \rceil$.

(d) If $G = G_1 + G_2 + \cdots + G_r$, then $\chi(G, \neg C_j) = \max_i\{\chi(G_i, \neg C_j)\}$.

(e) $\chi(G, \neg C_j) = 1$ if and only if G contains no C_j. Consequently, for $n < j$, $\chi(G, \neg C_j) = 1$.

(f) $\chi(G, \neg C_j) \leq \chi(G, P_a) \leq \chi(G)$.

(g) If $p \leq q$, then $\chi(K_{p,q}, \neg C_j) = \begin{cases} 1 & \textit{if } j \textit{ is odd or } j > 2p, \textit{ and} \\ 2 & \textit{if } j \textit{ is even and } 4 \leq j \leq 2p. \end{cases}$

For each $j \geq 3$ and $k \geq 1$, can we find graphs with $\neg C_j$-chromatic number k? Clearly Theorem 2.1(a) implies that the complete graphs with $k(j-1)$ nodes are a family of such graphs. For the usual chromatic number the existence of k-chromatic graphs with arbitrarily high girth is well known. (See for example [10].) In [5] Brown and Corneil prove the corresponding result for P-chromatic numbers when P is a nontrivial hereditary property. Since the property of containing no j-cycle is hereditary, it follows that for all $k \geq 1$ and $j \geq 3$ there are graphs of arbitrarily high girth and $\neg C_j$-chromatic number k.

Hedetniemi [15], Chartrand and Kronk [8], and Chartrand, Kronk and Wall [9] independently studied the conditional chromatic number with respect to the property P_a. ¿From Theorem 2.1(f) it follows that an upper bound on the point arboricity for a family of graphs is also an upper bound for the $\neg C_j$-chromatic numbers for the given family and each value of $j \geq 3$.

In [9] Chartrand, Kronk and Wall showed that $\chi(G, P_a) \leq \lceil (1 + \Delta(G))/2 \rceil$. Later in [8], Chartrand and Kronk showed that the upper bound can be improved to $\lfloor (1 + \Delta(G))/2 \rfloor$ for connected non-regular graphs. This immediately gives the following upper bounds for the $\neg C_j$-chromatic numbers.

Theorem 2.2 *For all $j \geq 3$ and all graphs G, $\chi(G, \neg C_j) \leq \lceil (1 + \Delta(G))/2 \rceil$. If in addition G is connected and non-regular, then $\chi(G, \neg C_j) \leq \lfloor (1 + \Delta(G))/2 \rfloor$.*

For j-cycles, Theorem 2.2 is tight. However, for planar graphs with large maximum degree, this bound is not very strong. For example, the graph $K_{1,m}$ contains no C_j and hence has $\neg C_j$-chromatic number 1, but the bound from the theorem is $\lceil (m+1)/2 \rceil$. In fact, for planar graphs the bound can be greatly improved.

In [15] and [9] the authors proved independently that if G is a planar graph, $\chi(G, P_a) \leq 3$. In [15] it was shown that if in addition G is an outerplanar graph, $\chi(G, P_a) \leq 2$. From these results, we immediately have the following:

Theorem 2.3 *(a) If G is a planar graph, then $\chi(G, \neg C_j) \leq 3$ for all $j \geq 3$.*
(b) If G is an outerplanar graph, then $\chi(G, \neg C_j) \leq 2$ for all $j \geq 3$.

For each $j \geq 3$, the j-cycle is an outerplanar graph with $\neg C_j$-chromatic number 2. In [15] it was shown that the dual of the Tutte graph has P_a-chromatic number 3. However, the $\neg C_3$-chromatic number of this graph is easily shown to be 2. This leads us to ask: For each $j \geq 3$, does there exist a planar graph whose $\neg C_j$-chromatic number is 3?

3 The Value of $\chi(\mathbf{G}, \neg \mathbf{C_j})$ when $|\mathbf{E(G)}|$ is Large

We investigate the value of $\chi(G, \neg C_j)$ when the number of edges in G is large. We will often make use of two standard results about Hamiltonian graphs, which we will now state for the reader's convenience. In the following for conciseness we will always assume that n denotes the number of nodes of a graph G.

Theorem 3.1 Ore [18] *In a graph G with $n \geq 3$, if for every pair u and v of nonadjacent vertices $\deg u + \deg v \geq n$, then G is Hamiltonian.*

Theorem 3.2 Ore [18] *If G is a graph with $n \geq 3$ and $|E(G)| \geq (n^2 - 3n + 6)/2$, then G is Hamiltonian.*

Now we present a series of theorems in which the value of $\chi(G, \neg C_j)$ is given when G is missing a small number of edges. First we find the value of the conditional chromatic number when G is missing $j - 3$ or fewer edges.

Theorem 3.3 *If $e(G) \geq \binom{n}{2} - (j-3)$, then $\chi(G, \neg C_j) = \lceil n/(j-1) \rceil$ for all $j \geq 3$.*

Proof. From Theorem 2.1(c), it follows that $\chi(G, \neg C_j) \leq \lceil n/(j-1) \rceil$. Suppose by way of contradiction that G is $\neg C_j$-colored with fewer than $\lceil n/(j-1) \rceil$ colors. Then some color class A contains at least j nodes. Let $B \subseteq A$ with $|B| = j$. Then $e(B) \geq j(j-1)/2 - (j-3) = (j^2 - 3j + 6)/2$. It follows from Theorem 3.2 that B is Hamiltonian. Therefore A contains a j-cycle, a contradiction. □

Next we want to find the values of $\chi(G, \neg C_j)$ when G is missing $j-2$ edges. On the way to this result, we show that when the complement of G contains edges that induce particular types of subgraphs, the values are readily found.

Theorem 3.4 *If G is a graph for which $\langle E(\bar{G}) \rangle = K_{1,m}$ for $j - 2 \leq m \leq n - 1$, then $\chi(G, \neg C_j) = \lceil (n-1)/(j-1) \rceil$ for all $j \geq 3$.*

Proof. Let v be the node of degree m in the subgraph of $\langle E(\bar{G}) \rangle$ induced by the $K_{1,m}$. Let $k = \chi(G, \neg C_j)$ and let A be a color class in a $\neg C_j$-coloring of G with k colors. If $v \notin A$, then A induces a complete subgraph of G and hence contains an $|A|$-cycle. Thus, every color class not containing v has at most $j-1$ elements. If $v \in A$ and $|A| \geq j+1$, then $A - \{v\}$ induces a complete graph on j nodes and hence contains a j-cycle. Thus the color class containing v contains at most j elements. Since there are $k-1$ color classes with at most $j-1$ elements and one color class with at most j elements, $n \le (k-1)(j-1)+j = k(j-1)+1$. Thus $k \ge \lceil (n-1)/(j-1) \rceil$.

On the other hand, we can color V as follows: Color v and $j-2$ of its non-neighbors plus one additional node with color 1. Since the degree of v in this color class is at most 1 and the class contains j nodes, clearly this class has no j-cycle. Color the remaining $n-j$ nodes so that no color class contains more than $j-1$ nodes. This gives a $\neg C_j$-coloring of V with $1 + \lceil (n-j)/(j-1) \rceil = \lceil (n-1)/(j-1) \rceil$ colors. □

Theorem 3.5 *If* $\langle E(\bar{G}) \rangle = mK_2$ *where* $1 \le m \le \lfloor n/2 \rfloor$*, then* $\chi(G, \neg C_3) = \lceil (n-m)/2 \rceil$.

Proof. Let $M = E(\bar{G})$ and let A be a color class in a minimum $\neg C_3$-coloring $\mathcal{C}$ of G. Since any subgraph induced by five nodes is missing at most two edges of M and therefore must contain a triangle, $|A| \le 4$. If $|A| = 4$, then to be C_3-free, A must be missing exactly two edges of M. Also if $|A| = 3$, then A must be missing exactly one edge of M. Let a be the number of color classes in $\mathcal{C}$ with four nodes; b, the number with three nodes; and c, the number with no more than two nodes. Then $b = m - 2a - r$ for some nonnegative integer r and since $\mathcal{C}$ is a minimum coloring, $c = \lceil (n-4a-3b)/2 \rceil$. The number of colors in $\mathcal{C}$ is $a+b+c$, so that $\chi(G, \neg C_3) = a + b + \lceil (n-4a-3b)/2 \rceil = \lceil (n-2a-b)/2 \rceil = \lceil (n-2a-(m-2a-r))/2 \rceil = \lceil (n-m+r)/2 \rceil \ge \lceil (n-m)/2 \rceil$.

Now we produce a $\neg C_3$-coloring for G with $\lceil (n-m)/2 \rceil$ colors. Let $n = 2m+s$ for $s \ge 0$, and suppose first that m is even. Form $m/2$ classes of four nodes, each class consisting of the endpoints of two edges of M. ¿From the remaining nodes, form $\lceil s/2 \rceil$ classes with at most two nodes. This gives a coloring with $m/2 + \lceil s/2 \rceil = \lceil (s+m)/2 \rceil = \lceil (n-m)/2 \rceil$ colors.

Now suppose m is odd. Form $(m-1)/2$ classes of four nodes from the endpoints of edges of M, form $\lceil (s-1)/2 \rceil$ classes of two nodes from the nodes of $\overline{\langle M \rangle}$, and form one additional class consisting of the endpoints of the remaining edge in M and the remaining node of $\overline{\langle M \rangle}$, if there is one. Then we have used $(m-1)/2 + 1 + \lceil (s-1)/2 \rceil = \lceil (m+s)/2 \rceil = \lceil (n-m)/2 \rceil$ colors. □

The next theorem confirms that missing m independent edges from G does not reduce the value of $\chi(G, \neg C_j)$ when $j \ge 4$.

Theorem 3.6 *If* $\langle E(\bar{G}) \rangle = mK_2$ *where* $1 \le m \le \lfloor n/2 \rfloor$*, then* $\chi(G, \neg C_j) = \lceil n/(j-1) \rceil$ *for all* $j \ge 4$.

Proof. Let A be a set of j nodes and let $v \in A$. Since the missing edges are independent, at most one of them is incident to v. Hence $\deg_{\langle A \rangle} v \ge j-2 \ge j/2$ for $j \ge 4$. It follows from Theorem 3.1 that A induces a Hamiltonian subgraph, i.e., a subgraph with a j-cycle, and hence that no color class contains more than $j-1$ nodes. Thus $\chi(G, \neg C_j) \ge \lceil n/(j-1) \rceil$ and equality follows from Theorem 2.1(c). □

However, when G is missing disjoint copies of K_3, the values of the $\neg C_j$-chromatic numbers may be smaller for $j \ge 4$.

Theorem 3.7 *If $\langle E(\bar{G})\rangle = mK_3$, where $1 \le m \le \lfloor n/3 \rfloor$ and $n = 3m + s$, then*

$$\chi(G, \neg C_4) = \begin{cases} \lceil n/4 \rceil & \text{if } 0 \le s \le m, \text{ and} \\ \lceil (n-m)/3 \rceil & \text{if } s > m. \end{cases}$$

Proof. Let A be a color class in a minimum $\neg C_4$-coloring $\mathcal{C}$ of G. Suppose $|A| \ge 5$. Then in the subgraph B induced by any five nodes of A, the maximum number of missing edges is 4 and they must occur in $K_3 + K_2$. But even in this case B contains a 4-cycle. Thus $|A| \le 4$. If $|A| = 4$, then to be C_4-free, A must be missing three edges in a single copy of K_3. Clearly, if $|A| \le 3$, A contains no 4-cycle.

Let a be the number of classes with four nodes and b the number with three or fewer. Since each color class with four nodes must be missing the edges of a K_3, $a \le m$. In addition, since $\mathcal{C}$ is minimum, $b = \lceil (n-4a)/3 \rceil$ so that $\chi(G, \neg C_4) = a + b = a + \lceil (n-4a)/3 \rceil = \lceil (n-a)/3 \rceil \ge \lceil (n-m)/3 \rceil$.

Now we produce a minimum $\neg C_4$-coloring. Let $n = 3m + s$ for some nonnegative number s and let S be the set of nodes that are not an endpoints of an edge in a K_3. If $0 \le s \le m$, then each of the nodes in S can be added to the three nodes in a K_3 to form a color class of order 4. Form $\lceil (n-4s)/4 \rceil$ color classes of order at most 4 from the remaining $n - 4s$ nodes by including all three nodes of one K_3 in each class and dividing the remaining nodes among these classes. This yields a $\neg C_4$-coloring with $\lceil n/4 \rceil$ colors.

Now suppose $m < s$. Then we use each of the m copies of K_3 together with a node in S to form m color classes of order four. We partition the remaining $s - m$ nodes into as many 3-node classes as possible. Thus we have $m + \lceil (s-m)/3 \rceil = \lceil (s+2m)/3 \rceil = \lceil (n-m)/3 \rceil$ color classes. □

A theorem corresponding to Theorem 3.7 when $j = 5$ appears to be quite complicated. For the proof of our main result in this section, we only need to know the value of the $\neg C_5$-chromatic number when G is missing a single copy of K_3.

Theorem 3.8 *If $\langle E(\bar{G})\rangle = K_3$, then $\chi(G, \neg C_5) = \lceil (n-1)/4 \rceil$.*

Proof. Let the nodes of the K_3 in $\langle E(\bar{G})\rangle$ be u, v and w. Let $k = \chi(G, \neg C_5)$. Suppose A is a color class in a $\neg C_5$-coloring with k colors and $B \subseteq A$ with $|B| = 5$. If B contains no two nodes of $\{u, v, w\}$, then B induces a complete graph and hence contains a 5-cycle. Suppose $u, v \in B$ and $w \notin B$. Then $B = K_5 - e$ and clearly B contains a 5-cycle. If $u, v, w \in B$, then it is easy to see that B is not Hamiltonian. Thus every color class must contain at most four nodes with the exception of one class, which contains five nodes only if it contains all three of the nodes u, v, and w. Thus $n \le 4(k-1) + 5$ from which it follows that $k \ge \lceil (n-1)/4 \rceil$. Clearly V can be colored by assigning one color to u, v, w and two other nodes and assigning colors to the remaining $n - 5$ nodes so that no color class contains more than four nodes. This gives a $\neg C_5$-coloring with $1 + \lceil (n-5)/4 \rceil = \lceil (n-1)/4 \rceil$ colors. □

For the proof of Theorem 3.10, we also need to find the value of the conditional chromatic number when G is missing a copy of P_4, the path on four nodes, and when G is missing $K_{1,2} + K_2$.

Theorem 3.9 *If* $\langle E(\bar{G})\rangle = P_4$ *or* $\langle E(\bar{G})\rangle = K_{1,2} + K_2$, *then* $\chi(G, \neg C_5) = \lceil n/4 \rceil$.

Proof. Suppose that V is $\neg C_5$-colored and let B be a subset of order 5 contained in a color class. Even if all four nodes of the missing P_4 or if all five nodes of the missing $K_{1,2} + K_2$ are contained in B, still B will be Hamiltonian. Thus each color class can contain at most four nodes, and it follows from Theorem 2.1(c) that $\chi(G, \neg C_5) = \lceil n/4 \rceil$. □

Theorem 3.10 *If* $e(G) = \binom{n}{2} - (j-2)$, *then* $\chi(G, \neg C_j) = \lceil n/(j-1) \rceil$ *for all* $j \geq 3$, *except when* $\langle E(\bar{G})\rangle = K_{1,j-2}$ *or when* $j = 5$ *and* $\langle E(\bar{G})\rangle = K_3$. *In the exceptional cases,* $\chi(G, \neg C_j) = \lceil (n-1)/(j-1) \rceil$.

Proof. If $j = 3$, then G is missing one edge, $\langle E(\bar{G})\rangle = K_{1,1}$, and the result follows from Theorem 3.5. If $j = 4$, then G is missing two edges; so $\langle E(\bar{G})\rangle$ is either $K_{1,2}$ or $2K_2$, and the result follows from Theorems 3.4 and 3.5. If $j = 5$, the graph is missing three edges and $\langle E(\bar{G})\rangle \in \{K_3, K_{1,3}, 3K_2, P_4, K_{1,2} + K_2\}$. The result follows from Theorems 3.8, 3.4, 3.6, and 3.9.

So we may assume $j \geq 6$. If $\langle E(\bar{G})\rangle = K_{1,j-2}$, the result follows from Theorem 3.4. Thus we assume that $\langle E(\bar{G})\rangle \neq K_{1,j-2}$. Let V be $\neg C_j$-colored and let A be a color class. Let $B \subseteq A$ with $|B| = j$. We will show that for every pair of nonadjacent nodes u and v in B, either u and v belong to a Hamiltonian cycle of B or that $\deg_B u + \deg_B v \geq j$. It will then follow from Theorem 3.1, that B is Hamiltonian and hence that A contains a j-cycle, a contradiction. From this it follows that no color class may contain more than $j-1$ nodes and hence $\chi(G, \neg C_j) \geq \lceil n/(j-1) \rceil$ with equality following from Theorem 2.1(c).

First we suppose that at most $j-3$ of the missing edges are incident to at least one of u and v. Assume $\deg_B u = j - 1 - s$ for some $s > 0$, that is, $|H(u) \cap B| = s$. There are at most $j-3$ edges missing at u and v, of which s, counting the edge uv, are missing at u. So there are at most $j - 3 - (s-1)$ edges missing at v. Thus $\deg_B v \geq j - 1 - (j - s - 2) = s + 1$. Now $\deg_B u + \deg_B v \geq j$.

This leaves the case where all $j-2$ missing edges are incident to at least one of u and v. Let $H = H(u,v) \cap B$. Since uv is a missing edge, $|H| \leq j-3$. Suppose $|H| = j-3 \geq 3$. Then there is exactly one node x in $B - H - \{u,v\}$, and each node of H is non-adjacent to exactly one of u or v and adjacent to the other. Moreover, since we assumed $\langle E(\bar{G})\rangle \neq K_{1,j-2}$, there is at least one node $y \in H$ incident to u and at least one $z \in H$ incident to v. Then the path z, v, x, u, y together with a Hamiltonian (y,z)-path in the complete graph $B - \{u,v,x\}$ is a Hamiltonian cycle in B. Suppose $|H| = j - 4 \geq 2$. Then there are two nodes x and y in $B - H - \{u,v\}$, and since $|H(u) \cap H(v)| = 1$, there is a $z \in H$ adjacent to one of u or v, say u. Now the path z, u, x, v, y together with a Hamiltonian (y,z)-path in the complete graph $B - \{u,v,x\}$ forms a Hamiltonian cycle in B. Finally, if $|H| \leq j-5$, then there are at least three nodes x, y, z in $B - H - \{u,v\}$. Then x, u, y, v, z together with a (z,x)-path in the complete graph $B - \{u,v,y\}$ is a Hamiltonian cycle in B. □

Combining Theorems 3.3 and 3.10, we get our main result.

Theorem 3.11 *If* $e(G) \geq \binom{n}{2} - (j-2)$*, then* $\chi(G, \neg C_j) = \lceil n/(j-1) \rceil$ *for all* $j \geq 3$*, except when* $\langle E(\bar{G}) \rangle = K_{1,j-2}$ *or when* $j = 5$ *and* $\langle E(\bar{G}) \rangle = K_3$*. In the exceptional cases,* $\chi(G, \neg C_j) = \lceil (n-1)/(j-1) \rceil$.

4 The value of $\chi(\mathbf{G}, \neg \mathbf{C_j})$ for graphs of small order

Now we use the results in the previous section to calculate the conditional chromatic numbers for graphs of order at most 6.

¿From Theorem 2.1(e) and our assumption that $j \geq 3$, it follows that $\chi(G, \neg C_j) = 1$ for all graphs G of order 1 or 2 and for all $j \geq 3$.

Clearly, a graph G of order 3 contains a cycle if and only if $G = K_3$. Thus if G is a 3-node graph, $\chi(G, \neg C_j) = 1$ for all $j \geq 3$ with the single exception that $\chi(K_3, \neg C_3) = 2$.

Now we consider the conditional chromatic numbers for graphs of order 4. It follows from Theorem 2.1(e) that for every 4-node graph G, $\chi(G, \neg C_j) = 1$ for all $j \geq 5$. We give the values when $j = 3$ and $j = 4$ in the next two theorems.

Theorem 4.1 *If G is a graph of order 4, then*

$$\chi(G, \neg C_3) = \begin{cases} 2 & \text{if } G = K_4 \text{ or } \langle E(\bar{G}) \rangle \in \{K_2, K_{1,2}, K_{1,3}\}, \text{ and} \\ 1 & \text{otherwise,} \end{cases}$$

Proof. From Theorem 2.1(a) $\chi(K_4, \neg C_3) = 2$ and it follows from Theorem 2.1(c) that all values are at most 2. The only other 4-node graphs that contain C_3 are those whose complements are $K_{1,1}, K_{1,2}$ and $K_{1,3}$. These graphs have conditional chromatic number 2 and the rest have value 1. □

Theorem 4.2 *If G is a graph of order 4, then*

$$\chi(G, \neg C_4) = \begin{cases} 2 & \text{if } G = K_4 \text{ or } \langle E(\bar{G}) \rangle \in \{K_2, 2K_2\}, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

The proof is similar to that of Theorem 4.1.

Now we consider graphs with order five. From Theorem 2.1(e) it follows that $\chi(G, \neg C_j) = 1$ for all $j \geq 6$. Thus we only need to find the values of the $\neg C_j$-chromatic numbers for $j \in \{3, 4, 5\}$. These are given in the next three theorems.

Theorem 4.3 *If G is a graph of order 5, then*

$$\chi(G, \neg C_3) = \begin{cases} 3 & \text{if } e(G) = 10, \text{ i.e. } G = K_5, \\ 2 & \text{if } 7 \leq e(G) \leq 9, \\ 2 & \text{if } 3 \leq e(G) \leq 6 \text{ and } C_3 \subseteq G, \\ 1 & \text{if } 3 \leq e(G) \leq 6 \text{ and } C_3 \not\subseteq G, \text{ and} \\ 1 & \text{if } e(G) \leq 2. \end{cases}$$

Proof. Theorem 2.1(a) implies $\chi(K_5, \neg C_3) = 3$ and Theorem 3.11 implies $\chi(K_5 - e, \neg C_3) = 2$. Thus from Theorem 2.1(b) it follows that the rest of the values are 2 or 1 depending on whether the graph does or does not contain C_3. It follows from the classic theorem of Turán [20] that for any graph G on n nodes, if $e(G) > n^2/4$, then G contains K_3. Thus if G has five nodes and $7 \leq e(G) \leq 9$, then $\chi(G, \neg C_3) = 2$. For each value k, $3 \leq k \leq 6$, there is a 5-node graph of size k that contains C_3 and one that does not. □

Theorem 4.4 *If G is a graph on 5 nodes, then*

$$\chi(G, \neg C_4) = \begin{cases} 2 & \text{if } 7 \leq e(G) \leq 10, \\ 2 & \text{if } 4 \leq e(G) \leq 6 \text{ and } C_4 \subseteq G, \\ 1 & \text{if } 4 \leq e(G) \leq 6 \text{ and } C_4 \not\subseteq G, \text{ and} \\ 1 & \text{if } e(G) \leq 3. \end{cases}$$

Proof. Since $\chi(K_5, \neg C_4) = 2$, the value for every 5-node graph G is 2 or 1 depending on whether G does or does not contain C_4. Theorem 3.11 implies that if $e(G) \geq 8$, then $\chi(G, \neg C_4) = 2$. However, it is not hard to verify that every 5-node graph G with seven edges contains a 4-cycle. (See for example the list on page 217 of [11].) For every value k, $4 \leq k \leq 6$, there is a graph on five nodes with k edges that has value 2 and another that has value 1. □

Theorem 4.5 *If G is a graph of order 5, then*

$$\chi(G, \neg C_5) = \begin{cases} 2 & \text{if } 7 \leq e(G) \leq 10 \text{ and } \langle E(\bar{G}) \rangle \neq K_3, \\ 1 & \text{if } \langle E(\bar{G}) \rangle = K_3, \\ 2 & \text{if } G \in \{C_5, C_5 + e\}, \text{ and} \\ 1 & \text{if } e(G) \leq 6 \text{ and } G \notin \{C_5, C_5 + e\}. \end{cases}$$

The proof is straightforward, using Theorem 3.11 to get the exact lower bound on size for graphs with value 2.

Now we calculate the values when $n = 6$. Of course, if $j \geq 7$, then $\chi(G, \neg C_j) = 1$ for all graphs G on 6 nodes. So we only need calculate the values for $3 \leq j \leq 6$. We give these values in the following four theorems.

Theorem 4.6 *If G is a graph of order 6, then*

$$\chi(G, \neg C_3) = \begin{cases} 3 & \text{if } G = K_6, \text{ or } \langle E(\bar{G}) \rangle = K_{1,m}\ 2 \leq m \leq 5, \\ 2 & \text{if } 10 \leq e(G) \leq 14 \text{ and } G \neq K_{1,m}\ 2 \leq m \leq 5, \\ 2 & \text{if } 3 \leq e(G) \leq 9 \text{ and } G \supseteq C_3, \\ 1 & \text{if } 3 \leq e(G) \leq 9 \text{ and } G \not\supseteq C_3, \text{ and} \\ 1 & \text{if } e(G) \leq 2. \end{cases}$$

Proof. Since $\chi(K_6, \neg C_3) = 3$, we know that $\chi(G, \neg C_3) \leq 3$ for all 6-node graphs G. From Theorem 3.11, it follows that $\chi(K_6 - e, \neg C_3) = 3$. When $e(G) \leq 13$, either $\langle E(\bar{G}) \rangle = K_{1,m}$ for some $m, 2 \leq m \leq 5$, or G is a subset of the graph whose complement is $2K_2$. From Theorem 3.4, the graphs of the first type have $\chi(G, \neg C_3) = 3$; from Theorems 3.5 and 2.1(b), those of the second type have $\chi(G, \neg C_3) \leq 2$. Turán's Theorem implies that every graph on six nodes with size 10 contains a triangle. Thus those graphs of the second type with at least ten edges have value exactly 2. The 6-node graph whose complement is $2K_3$ has size 9 and no K_3. □

Theorem 4.7 *If G is a graph of order 6, then*

$$\chi(G, \neg C_4) = \begin{cases} 2 & \text{if } 8 \leq e(G) \leq 15, \\ 2 & \text{if } 4 \leq e(G) \leq 7 \text{ and } G \supseteq C_4, \\ 1 & \text{if } 4 \leq e(G) \leq 7 \text{ and } G \not\supseteq C_4, \text{ and} \\ 1 & \text{if } e(G) \leq 3. \end{cases}$$

Proof. We have $\chi(K_6, \neg C_4) = 2$, so that for every 6-node graph G, $\chi(G, \neg C_4) \leq 2$. Neither Turán's Theorem nor Theorem 3.11 gives the exact lower bound on size for graphs with value 2. However, examination of the graphs of order 6 and size 8 in [11] reveals that each of them has a 4-cycle. The graph constructed by identifying one node in each of two triangles and an edge is a graph with seven edges and no 4-cycle. □

Theorem 4.8 *If G is a graph of order 6, then*

$$\chi(G, \neg C_5) = \begin{cases} 2 & \text{if } 10 \leq e(G) \leq 15, \\ 2 & \text{if } 5 \leq e(G) \leq 9 \text{ and } G \supseteq C_5, \\ 1 & \text{if } 5 \leq e(G) \leq 9 \text{ and } G \not\supseteq C_5, \text{ and} \\ 1 & \text{if } e(G) \leq 4. \end{cases}$$

Proof. We calculate $\chi(K_6, \neg C_5) = 2$, so that the values are 2 or 1 depending on whether G does or does not contain C_5. From Theorem 3.11, we can only conclude that if G has 12 edges, then $\chi(G, C_5) = 2$. However, examining the graphs on six nodes with ten edges, we find that each contains C_5. The graph on six nodes whose complement is $2K_3$ has nine edges and no 5-cycle. □

Theorem 4.9 *If G is a graph of order 6, then*

$$\chi(G, \neg C_6) = \begin{cases} 2 & \text{if } 11 \leq e(G) \leq 15 \text{ and } \langle E(\bar{G}) \rangle \neq K_{1,4}, \\ 1 & \text{if } \langle E(\bar{G}) \rangle = K_{1,4}, \\ 2 & \text{if } 6 \leq e(G) \leq 10 \text{ and } G \supseteq C_6, \\ 1 & \text{if } 6 \leq e(G) \leq 10 \text{ and } G \not\supseteq C_6, \text{ and} \\ 1 & \text{if } e(G) \leq 5. \end{cases}$$

The proof is straightforward, using Theorem 3.11 to find the lower bound on size of graphs with value 2.

Calculation of the conditional chromatic numbers for graphs of order 7 should be straightforward. However, one expects that this calculation will rapidly become more difficult as the order of the graph grows.

There are many open questions related to these numbers. In calculating the $\neg C_j$-chromatic numbers for graphs of small order, we notice that Theorem 3.11 is tight for some j but not for others. Can we characterize those j for which the bound is tight? Along these lines, can Theorem 3.11 be improved for certain values of j and n?

Clearly, the $\neg C_{2k+1}$-chromatic number is 1 for all bipartite graphs and the $\neg C_{2k}$-chromatic number is either 2 or 1 depending on whether the graph does or does not contain C_{2k}. Are there other interesting classes of graphs for which the conditional chromatic numbers are easy to find?

References

1 M. Albertson, R. Jamison, S. Hedetniemi, and S. Locke, The subchromatic number of a graph, *Discrete Math.* 74:33–49 (1989).

2 J. Andrews and M. Jacobson, On a generalization of chromatic number. *Congr. Numer.* 47:33–48 (1985).

3 J. Andrews and M. Jacobson, On a generalization of chromatic number and two kinds of Ramsey numbers, *Ars Combin.* 23:97–102 (1987).

4 I. Broere and C. Mynhardt, Generalized colorings of outerplanar and planar graphs, in *Graph Theory with Applications to Algorithms and Computer Science*, Wiley, 1985.

5 J. Brown and D. Corneil, On generalized graph colorings, *J. Graph Theory* 11:87–89 (1987).

6 J. Brown and D. Corneil, On uniquely $-G$ k-colorable graphs, *Quaestiones Mathematicae* 15:477–487 (1992).

7 G. Chartrand, D. Geller and S. Hedetniemi, A generalization of the chromatic number, *Proc. Cambridge Philos. Soc.* 64:265–271 (1968).

8 G. Chartrand and H. Kronk, The point-arboricity of planar graphs, *J. London Math. Soc.* 44:612–616 (1969).

9 G. Chartrand, H. Kronk and C. Wall, The point-arboricity of a graph, *Israel J. Math.* 6:169–175 (1968).

10 P. Erdös, Graph theory and probability, *Canad. J. Math.* 11:34–38 (1959).

11 F. Harary, *Graph Theory*, Addison-Wesley, 1969.

12 F. Harary, Conditional colorability in graphs, in *Graphs and Applications*, Proc. First Colo. Symp. Graph Theory (F. Harary, J. Maybee eds.) Wiley-intersci. Publ., N.Y., 1985.

13 F. Harary and K. Fraughnaugh (Jones), Conditional colorability II: bipartite variations, *Congr. Numer.* 50:205–218 (1985).

14 F. Harary and K. Fraughnaugh (Jones), Degree conditional bipartition numbers in graphs, *Congr. Numer.* 55:39–50 (1986).

15 S. Hedetniemi, On partitioning planar graphs, *Canad. Math. Bull.* 11:203-211 (1968).

16 L. Lovász, On chromatic number of finite set systems, *Acta Math. Acad. Sci. Hungar.* 19:59–67 (1968).

17 C. Mynhardt and I. Broere, Generalized colorings of graphs, in *Graph Theory with Applications to Algorithms and Computer Science*, Wiley Intersci. Publ.,New York, 1985.

18 O.Ore, Arc coverings of graphs, *Ann. Mat. Pura Appl.* 55:315–322 (1961).

19 H. Sachs and M. Schäuble, Konstruction von Graphen mit gewissen Färbungseigenschafte, in *Beiträge zur Graphentheorie*, (Kolloquium, Maneback, 1967) Teubner, Leipzig, 1968.

20 P. Turán, An extremal problem in graph theory (Hungarian) *Mat. Fiz. Lapok* 48:436–452 (1941).