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The Use of the Optimal Partition in a Linear Programming Solution for Postoptimal Analysis

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Abstract

Over the years we have learned to use an optimal basic solution to perform sensitivity analysis. Recently, the importance of an optimal partition, induced by a strictly complementary solution, has surfaced in connection with the interior point method. This paper gives examples where the partition is what is needed or desired to perform the analysis.

Keywords: linear programming, complementary slackness, linear systems, sensitivity analysis, computational economics

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Until recently, it has been generally believed that it is necessary to obtain an optimal basic solution in order to perform postoptimal sensitivity analysis correctly. Adler and Monteiro [1992] correct this belief and remind us of an old theorem in linear programming that has not appeared in papers and textbooks for decades because of this belief (also see [Güler et al, 1992; Jansen et al, 1992]).

First, the issue is put into a mathematical framework, and the basic theory is described. Second, some applications are given that demonstrate that an optimum obtained from an interior point method, which is generally not basic owing to alternative optima, is what we need to perform the analysis in each case.

Mathematical Foundations

Let the primal and dual linear programs be given by;

<u>Primal</u>	<u>Dual</u>
Min cx: $x, s \geq 0, Ax - s = b$	Max πb: $\pi, d \geq 0, \pi A + d = c$

For feasible solutions, the *duality gap* $\equiv cx - \pi b = xd + s\pi \geq 0$. This is zero if, and only if, (x, s) is optimal in the primal and (π, d) is optimal in the dual. Equivalently, (x, s, π, d) satisfies *complementary slackness*: $x_j d_j = 0$ for all j, and $s_i \pi_i = 0$ for all i. The solution is *strictly complementary* if one member of each complementary pair is positive: $x + d > 0$ and $s + \pi > 0$. Define the *support* (σ) of a non-negative vector as the set of indexes for which the coordinate is positive: $\sigma(x) = \{j: x_j > 0\}$. Then, in general, a solution satisfies $|\sigma(x)| + |\sigma(d)| \leq n$ and $|\sigma(s)| + |\sigma(\pi)| \leq m$. A solution is strictly complementary if, and only if, these hold with equality – that is, where the supports yield partitions.

It is easy to prove that every strictly complementary solution yields a partition since

$$\sigma(x) = \{j: x_j > 0\} = \{j: d_j = 0\} = N \backslash \sigma(d) \Longrightarrow \sigma(x) \cup \sigma(d) = N$$

and

$$\sigma(s) = \{i: s_i > 0\} = \{i: \pi_i = 0\} = M \backslash \sigma(\pi) \Longrightarrow \sigma(s) \cup \sigma(\pi) = M,$$

where $N \equiv \{1, ..., n\}$ and $M \equiv \{1, ..., m\}$.

Strict Complementarity Theorem. If the primal has an optimal solution, there exist optimal primal and dual solutions that are strictly complementary. Further, the support partition is the same for all strictly complementary primal-dual solutions.

This theorem says first that there is at least one strictly complementary solution, and second that there is only one partition induced by their supports. To prove the uniqueness of the support partition, let (x, s, π, d) and (x', s', π', d') be two strictly complementary solutions. Suppose $j \in \sigma(x)$ and $j \notin \sigma(x')$ – that is, $x_j > 0$, $d_j = 0$, $x'_j = 0$, and $d'_j > 0$. Since the set of optimal solutions is convex, $\frac{1}{2}(x, s, \pi, d) + \frac{1}{2}(x', s', \pi', d')$ must be optimal; however, $\frac{1}{2}(x_j+x'_j) > 0$ and $\frac{1}{2}(d_j+d'_j) > 0$, which contradicts complementarity. Thus, $\sigma(x)$ induces a unique partition of *N*. The uniqueness of the partition of *M*, $\sigma(s) \cup \sigma(\pi)$, follows by similar argument.

The existence part of the theorem was originally proven by Goldman and Tucker [1956], using the self-dual theorem by Tucker [1956]. (See also [Good, 1959].) Another proof, based on the theory of the interior point method, is given by Adler and Monteiro [1992] and Güler et al [1992]. In a related paper, Jansen et al [1992] prove that the break points in the optimal objective value (as a function of b and c) occur where the optimal partition changes. The intervals between these break points are the unions of the ranges obtained from individual optimal bases from a simplex method (this was previously proven by Adler and Monteiro [1992]). In connection with this, they demonstrate that the true range of a right-hand side (b_i) or objective coefficient (c_j) is obtained from partitions, not from limits imposed by a basic solution, which is what simplex-based optimizers give (an excellent reference is [Gal, 1979]; more recently Ward and Wendell [1990] presented a comprehensive analysis of methods to obtain correct ranges). In the next section, we present additional examples where the optimal partition is what is needed to perform certain types of analysis of a solution.

Clearly, there is no issue when the primal and dual optima are unique, which is generally not the case in practice. Let (x, s) be a basic feasible solution in the primal with associated complementary dual solution (π, d) (defined in the usual way). The basis is an *equilibrium solution* if (π, d) is feasible in the dual. Postoptimal analysis uses only equilibrium solutions, which are obtained by the simplex method. Then, another characterization of uniqueness is given by the following (proven in [Greenberg, 1986] in the context of a *theory of compatable bases*, designed to obtain correct ranges and directional derivatives).

Uniqueness Theorem. A primal-dual solution is uniquely optimal in their respective linear programs if, and only if, it is a strictly complementary basic (equilibrium) solution.

Thus, strict complementarity of a basic optimal solution is equivalent to uniqueness. This means that the simplex method terminates without a strictly complementary solution unless it is the only optimal (primal and dual) solution. Non-uniqueness is the typical result in practice, especially for the dual, owing to primal degeneracy, and that is the situation we consider in the next section.

Needs for the Optimal Partition

The need for the (unique) optimal partition arises when we need to know whether a variable is positive in *some* optimal solution. This could be obtained by traversing the alternative optimal basic solutions, but the interior point method gives this partition since its solution is strictly complementary, thus avoiding the computationally expensive traversal of basic optima. (The traversal can be done systematically, using the theory of compatable basis [Greenberg, 1986] and Gal's [1986] TNP rule to avoid revisiting an optimal basis.)

First, consider identifying the set of critical jobs in a project scheduling problem. The problem is to select start times, $\{s_i\}$ for $i=1, ..., n \equiv$ number of jobs, that satisfy precedence constraints, $P = \{<i, j>\}$, which means job i must be finished before job j can begin. Following Ahuja et al [1989], introduce dummy jobs, 0 and n+1, and enter $<0, j>$, $<j, n+1> \in P$ for $j = 1, ..., n$. Then, letting t_i denote the time it takes to perform job i with $t_0 \equiv t_{n+1} \equiv 0$, the linear program is:

$$\text{Min } T = s_{n+1} - s_0 \text{ subject to: } s_j \geq s_i + t_i \text{ for } <i, j> \in P.$$

Let $T^* \equiv$ minimum completion time. A job is critical if its completion time directly affects the project's minimum completion time. This means T^* changes with a small change in t_i.

The dual is a longest path problem, so a job is critical if it is in some longest path. The dual flow variable, x_{ij}, is associated with the precedence constraint, $s_j \geq s_i + t_i$ for $<i, j> \in P$, so what is needed is the optimal partition to identify $\{i: \Sigma_j x_{ij} > 0$ in some optimal dual solution$\}$. (In the dual, $x_{ij} = 0$ or 1 in each solution, which denotes whether arc $<i, j>$ is in the critical path; in a basic solution, $\Sigma_j x_{ij} = 0$ or 1.)

A reason for identifying critical jobs is to allocate resources that can reduce at least one of their completion times in order to reduce the minimum total completion time. If we obtain just one critical path when there are many, we might concentrate putting resources into reducing the processing time of a job only to discover that the minimum completion time is unchanged. The optimal partition gives us the set of jobs, called the *critical partition*, whose completion time reductions ensures a reduction in the total completion time.

A basic optimal solution gives us a necessary condition for completion time reduction: some job in the critical path must have its time reduced in order to decrease T^*. This applies to the critical partition too. Unlike a basic solution, a critical partition also gives us a sufficient condition for completion time reduction: T^* is reduced if the times of all jobs in the critical partition are reduced. Neither solution gives us everything we want, which is a minimal set of jobs whose time reductions are necessary and sufficient for reducing the total completion time, T*. But, choosing a job from the critical partition is no less effective than choosing a job from the critical path in a basic solution. In both cases, choosing one such job is necessary and need not be sufficient to reduce the total completion time.

With the critical partition, obtained from an interior point solution, the method of Jansen et al can be applied to perform the requisite sensitivity analysis. This generally avoids enumerating the critical paths, which is what is needed if only basic solutions are obtained by the optimizer. The real point, however, is that a basic solution gives only a necessary condition for time reduction, whereas an interior point solution gives not only the same kind of necessary condition, but also it gives a sufficient condition. There is thus an advantage with no offsetting disadvantage to obtain the optimal partition with an interior point solution, rather than a basic optimum.

A second example is the peer group problem described by Morey et al [1992]. A linear program is defined to find a convex combination of factors that measure a hospital's performance. There are 19 factors, such as: number of beds, number of cases, number of ER visits, dollars expended for medical education, and salaries. These factors for 300 hospitals comprise the data, and each hospital is selected for evaluation as follows.

First, solve a linear program that minimizes a hospital's total cost subject to constraints of the form:

$$\Sigma_{j \neq k} A_{ij} x_j \geq A_{ik} \text{ for } i \in G, \quad \Sigma_{j \neq k} A_{ij} x_j \leq A_{ik} \text{ for } i \in L, \quad \Sigma_{j \neq k} x_j = 1, \quad x \geq 0,$$

where A_{ij} = value of factor i for hospital j; G, L = partition of factors for which the constraint ensures a peer relation. Factors in G are performance measures, such as the number of ER visits, and factors in L are resources, such as the number of beds. Then, let $z = \Sigma_{j \neq k} A_j x_j$ be the peer equivalent to be compared with the performance of the k-th hospital. Constraints in G ensure that the peer performance is at least as great as that of the k-th hospital, and constraints in L ensure that the peer resources are no greater. For example, the peer equivalent cannot have more beds and must have at least as many ER visits.

The approach solves one linear program for each hospital. When the k-th hospital is one of the extremes, such as the only one with the greatest number of beds, the above LP is infeasible (since its factor vector does not lie in the convex hull of the other factor vectors). There are, however, modifications, such as letting x_k appear in the summation and seeking its minimum as part of the objective. What remains true is that the notion of peer equivalent is what is used to evaluate a hospital.

The *peer group* for hospital k is defined as the set of hospitals for which $x_j > 0$ $(j \neq k)$ – that is, $\sigma(x)$ for x optimal in the LP for the k-th hospital. This is a critical step, as the peer group determines whether the j-th hospital's performance is above or below what it should be, and the peer group is cited for the k-th hospital to review its operations. In the study, solutions were obtained from a simplex method, and there were typically alternative optima. The peer group, therefore, was determined somewhat arbitrarily, as another peer group would be obtained from an alternative basic optimum. The optimal partition, on the other hand, defines $\sigma(x)$ as the unique peer group composed of hospitals that critically relate to the k-th hospital. Because of the sensitivity of the qualitative conclusions (how well the hospital is doing) to the choice of peer group, this partition better fits its meaning.

The exact form of the total cost is not relevant to this point: there is less arbitrariness to defining a peer group from the unique optimal partition than to a somewhat arbitrary basic (optimal) solution.

Our next example is forming a subsystem to reduce the effort in finding implied equalities. Let $S \equiv \{Ax \geq b\}$, and let $X(S) = \{x: x \text{ satisfies } S\}$. The constraint $A_{i.}x \geq b_i$ is an implied equality if $x \in X(S)$ implies $A_{i.}x = b_i$ (equivalently, if its surplus variable is not *viable*[Chinneck, 1992]; see, also, [Greenberg, 1994] for connections with redundancy and more general forcing substructures).

Implied equalities could be found by systematically traversing the basic feasible solutions [Gal, 1979; Karwan et al, 1983]. One can initialize the constraints of interest with a label, 'Unknown', and proceed to maximize some surplus variable, called a *target variable*. Not only will the target variable's status be resolved at optimality (or sooner, if each basis is examined), but also some others might be determined by the tableau values. At a general iteration, each constraint of interest is labelled as 'Unknown', 'Implied equality', or 'Not implied equality'. Then, one can choose any constraint labelled 'Unknown' as the target variable and continue until no label is 'Unknown'.

This can be an exhaustive search, which is generally impractical, as there are many thousands of constraints in a medium size linear program. Huynh et al [1992] point out that we can avoid unnecessary computation if we first solve

$$\text{LP: } \max\{\pi b: \pi A = 0, \pi \geq 0, e\pi = 1\},$$

where e denotes a vector of 1's.

If this has an optimal solution with $\pi b = 0$, then $X(S) \neq \phi$ and S contains implied equalities. Otherwise, either $X(S) = \phi$ (if $\pi b > 0$), or S has a solution with $Ax > b$ (if LP is infeasible), in which case there are no implied equalities.

To see this, note the dual of LP is:

$$\min\{x_0: Ax + ex_0 \geq b\}.$$

This is feasible since x_0 can be made arbitrarily large. If a solution exists with $x_0 = 0$, $X(S) \neq \phi$; and, if a solution exists with $x_0 < 0$, there are no implied equalities because then $Ax > b$. If an optimal solution has $x_0 = 0$, there exists π for which $\pi A = 0$, $\pi b = 0$, $\pi \geq 0$, and $\pi_i > 0$ for some i (since $e\pi = 1$). This proves $A_{i.}x \geq b_i$ is an implied equality since π is a solution to its alternative.

This formulation was presented earlier by Stuckey [1991], who suggested an iterative algorithm to solve LP, removing $\sigma(\pi)$ each iteration. The next theorem [Freund et al, 1985], however, shows that only one LP needs to be solved if the optimal partition is obtained.

Implied Equality Partition Theorem. Suppose $X(S) \neq \phi$. Then, if S' is formed from an optimal partition of LP, S' contains all implied equalities.

Proof. If there are any implied equalities, LP has a feasible, and hence optimal, solution, say π, in which $\pi b = 0$. Since we assume $X(S) \neq \phi$, its dual has a solution with $x \in X(S)$; further, every $x \in X(S)$ is optimal since $x_0 = 0$ is feasible with $Ax \geq b$. Since LP and its dual have optimal solutions, there is a strictly complementary optimal solution, (x, π). If $\pi_i > 0$, we are done since $A_{i.}x \geq b_i$ is then contained in S'. If $\pi_i = 0$, we must have $A_{i.}x > b_i$ by strict complementarity, so $A_{i.}x \geq b_i$ is not an implied equality.

Algorithmically, it is prudent to solve LP to avoid solving the large number of linear programs when there is no implied equality. Otherwise, if the LP solution has zero objective value, $S' \equiv \{A_{i.}x \geq b_i : i \in \sigma(\pi)\}$ is a subset of S for which there are implied equalities. Since LP can have alternative optima, it is best to obtain the optimal partition from a strictly complementary solution in forming S'. The reason is that S' then contains all implied equalities, as shown with the above theorem.

In dealing with the more general system, $S = \{Ax \geq b \text{ and } Ex = f\}$, the idea is to shift implied equalities into the $Ex = f$ portion. The LP extends directly, with additional (sign unrestricted) variables associated with the equations. Without a strictly complementary solution, we have $\pi_i = 0$ and $A_{i.}x = b_i$ for some i. Then, $A_{i.}x \geq b_i$ is not included in S', but this could be an implied equality. This then requires multiple solutions of LPs to obtain all implied equalities, and each LP must contain the inequalities that had $\pi_i = 0$ and $A_{i.}x = b_i$ in the previous solution. By obtaining a strictly complementary solution, all implied equalities can be identified at once.

Our last example pertains to localizing the cause of inconsistency. Given S is inconsistent, an approach to localize the problem is to find an inconsistent subsystem with as few inequalities as possible. One approach, introduced by van Loon [1981], is that of an *irreducible infeasible subsystem* (IIS): an inconsistent subsystem is irreducible if every proper of subsystem of it is consistent. Various greedy algorithms are possible to obtain an IIS, and Chinneck [1991, 1992] has developed these for use with a large-scale linear programming system.

One of the approaches taken by Chinneck [1991] (also see [Chinneck and Dravnieks, 1991]), is to solve a succession of elastic programs, as follows. Let $S = \{Ax \geq b\}$, and let I index any subsystem of S that contains an IIS. We could let $I = M \equiv \{1, ..., m\}$, but we could also discard constraints for which the phase 1 solution has zero price (since the remaining system of inequalities still has a solution to the alternative), so in general $I \subseteq M$. Let $S(I) \equiv \{A_{i.}x \geq b_i : i \in I\}$, and let $S_i(I)$ denote $\{A_{k.}x \geq b_k : k \in I, k \neq i\}$.

A greedy algorithm to obtain an IIS in I is as follows. For each $i \in I$, solve the phase 1 LP for $S_i(I)$. If LP is feasible, put $A_{i.}x \geq b_i$ back into S; else, permanently discard the i-th inequality. After one pass through I, the surviving inequalities comprise an IIS.

Define the *elastic system*, $E(I) = \{Ax + v \geq b, v \geq 0 \text{ and } v_i = 0 \text{ for } i \in I\}$. $E(\phi)$ is feasible since we can choose $v = (b - Ax)^+$ for any x, and $E(M)$ is infeasible since the feasible region is X(S) when $v=0$. Starting with $I=\phi$ (or the complement of any inconsistent subsystem), we solve the elastic program,

$$EP(I): \text{ Min } ev : Ax + v \geq b, \; v \geq 0, \text{ and } v_i = 0 \text{ for } i \in I.$$

If EP(I) is feasible, it must have a positive minimum, and we augment $\sigma(v)$ to I to obtain $I' = I \cup \sigma(v)$. Then, we solve the next elastic program, where I has expanded by at least one index (since $|\sigma(v)| \geq 1$). Eventually, we reach an infeasible EP(I), at which point the set of constraints defined by I must contain an IIS.

The reason for solving the succession of elastic programs is to reduce the total number of (phase 1) linear programs solved. The greedy algorithm is applied to I, which requires $|I|$ linear programs. The terminal set from the elastic programming phase has $|I| < |M|$ (typically much less). The total number of linear programs solved, therefore, is $|I|$ + number of elastic programs solved.

Chinneck solves each elastic program by MINOS, which uses a simplex method, so the augmented constraints are those associated with the positive elastic variables in the particular basic solution found. The underlying theory, however, allows us to augment a constraint whose elastic variable is positive in any optimal solution of EP(I). We can therefore augment more constraints to I with a strictly complementary solution since $|\sigma(v)|$ is then the greatest number.

The reduction in the number of elastic programs can be significant. For example, suppose S has already been reduced (say by deleting constraints with a phase 1 zero price), so that it is irreducible. The optimal partition will have $\sigma(v) = M$, so only one elastic program is solved. On the other hand, each basic optimum can have $|\sigma(v)| = 1$, so that m elastic programs are solved before discovering that S is not reduced.

One such example is given by three inequalities and two variables: $S = \{x_1 - x_2 \geq 1, -x_1 - x_2 \geq -2, x_2 \geq 1\}$. These comprise an IIS, as any pair of constraints has a feasible solution. The elastic program (with $I = \phi$) has 3 basic optimal solutions:

$$x^1 = (1, 1), v^1 = (1, 0, 0);\; x^2 = (2, 1), v^2 = (0, 1, 0);\; x^3 = (1, 0), v^3 = (0, 0, 1).$$

Whichever basic solution we use, the set I now contains one index, and we solve a second elastic program with that elastic variable removed. The resulting basic solution is one of the other basic optima, so again only one index is added to I. In this example, EP(I) is feasible until $I = \{1, 2, 3\}$, so we obtain the original system after solving three elastic programs (which is the maximum possible). On the other hand, the optimal partition for $E(\phi)$ is $\sigma(v) = \{1, 2, 3\}$, so using it results in solving only one elastic program (the minimum possible).

More generally, the overall effort is not necessarily less because the terminal set, I, could have more inequalities with strictly complementary augmentations, but it does require fewer elastic programs (if alternative optima prevail). This is an example where the use of an interior point method might, or might not, result in less overall effort. Although examples, such as the one above, are easy to produce for which the total computational effort is less with the optimal partition, experiments are currently in progress to determine whether it is overall best to use an interior method, which yields the optimal partition, or than a simplex method, which yields a basic solution.

Conclusions

Although there are analysis questions for which it is best to have a basic optimal solution to an LP, there are other analysis questions for which a strictly complementary solution is needed. This occurs when we use the LP solution to form a partition of the variables, such as critical jobs or a peer group. The partition desired is the (unique) optimal partition, which is obtained by an interior point method because the computed optimum is strictly complementary. In some cases, the optimal partition reduces the computational effort, such as finding implied equalities. In other cases, it might reduce the computational effort, such as using elastic programming to find an infeasible subsystem, from which an IIS is found by a greedy algorithm.

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