BUBBLE FUNCTIONS PROMPT UNUSUAL STABILIZED FINITE ELEMENT METHODS

by

Leopoldo P. Franca
Department of Mathematics
University of Colorado at Denver
P.O.Box 173364, Campus Box 170
Denver, CO 80217-3364

Charbel Farhat
Department of Aerospace Engineering Sciences
and Center for Aerospace Structures
University of Colorado at Boulder
Boulder, CO 80309-0429

Abstract

A second order linear scalar differential equation including a zero-th order term is approximated using first the standard Galerkin method enriched with bubble functions. Static condensation of the bubbles suggest an unusual stabilized finite element method for which we establish a convergence study and obtain successful numerical simulations. The method is generalized to allow for a convection operator in the equation. This work may be employed as a starting point for simulation of nonlinear equations governing turbulence phenomena, flows with chemical reactions, and other important problems.

Submitted to:
Computer Methods in Applied Mechanics and Engineering

Preprint
June 1994

1. INTRODUCTION

We have pointed out in a recent communication [9] that for a certain model problem, bubble functions added to the usual finite element polynomials seem to subtract stability from the formulation. This finding contrasts our experience with other models such as the advective-diffusive equations and saddle-point problems [2,5,6,20,21], where bubble functions are key to obtaining stable formulations.

In this note we show that the apparent contradiction raised in [9] is in fact an inspiration to develop *unusual* stabilized methods. The subtraction prescribed by the static condensation of the bubbles indicates that stability can be achieved in a nonstandard manner by stabilized methods that we present herein.

In Section 2, we describe the model problem, the approximation of its solution with a Galerkin method using piecewise linear polynomials enriched with bubble functions, and the consequences of eliminating the bubbles. In the subsequent section we present a stabilized method and its error analysis. In Section 4 we add convection to the model equation and suggest a stabilized method to deal with all effects simultaneously. Finally, we report on some numerical simulations that confirm the arguments presented herein.

2. A MODEL PROBLEM AND BUBBLES' EFFECTS

We start with the model problem given by: find a scalar valued function $u(\mathbf{x})$ defined in $\Omega \subset I\!R^2$ such that

$$\sigma u - \kappa \Delta u = f \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma = \partial\Omega \tag{2}$$

where σ and κ are given positive constants and $f(\mathbf{x})$ is a given source function assumed to be square-integrable in Ω (i.e., $f \in L_2(\Omega)$).

The variational formulation corresponding to (1)-(2) is: find $u \in H_0^1(\Omega)$ such that

$$(\sigma u, v) + (\kappa \nabla u, \nabla v) = (f, v) \qquad v \in H_0^1(\Omega) \tag{3}$$

where $H_0^1(\Omega)$ is the Hilbert space of functions with square-integrable value and derivative in Ω satisfying (2), and we use the notation $(f, g) = \int_\Omega fg\, d\Omega$ or $(\mathbf{f}, \mathbf{g}) = \int_\Omega \mathbf{f} \cdot \mathbf{g}\, d\Omega$ depending on whether the underlying inner product is performed between scalar or vector fields, respectively.

The standard Galerkin method is formulated on a subspace $V_h \subset H_0^1(\Omega)$ employing the variational formulation (3): find $u_h \in V_h$ such that

$$(\sigma u_h, v) + (\kappa \nabla u_h, \nabla v) = (f, v) \quad v \in V_h \tag{4}$$

We consider the following subspace $V_h = V_h^b$

$$V_h^b = \{ v \in H_0^1(\Omega) \,|\, v_{|K} \in P_1(K) \oplus B(K),\ K \in \mathcal{C}_h \} , \tag{5}$$

where $\mathcal{C}_h$ is a partition of Ω into regularly shaped triangles, $P_1(K)$ denotes the space of linear functions defined on the triangle K and $B(K)$ denotes the space of bubble functions (e.g., spanned by a cubic function). The bubble basis function $\varphi \in B(K)$ satisfies

$$\begin{aligned} \varphi(\mathbf{x}) > 0 \qquad \forall \mathbf{x} \in K \\ \varphi(\mathbf{x}) = 0 \qquad \forall \mathbf{x} \in \partial K \end{aligned} \tag{6}$$

and $\varphi = 1$ at the baricenter of the triangle.

We now proceed to eliminate the bubbles from (4) using the static condensation procedure. First we obtain the "bubble equation" by selecting in (4) $v = \varphi(\mathbf{x})$ for $\mathbf{x} \in K$ and $v = 0$ elsewhere in Ω to obtain

$$(\sigma u_h, \varphi)_K + (\kappa \nabla u_h, \nabla \varphi)_K = (f, \varphi)_K \tag{7}$$

where the subscript K indicates integration over K.

Denoting by V_1 the space of piecewise linears, i.e.,

$$V_1 = \{v \in H_0^1(\Omega) \,|\, v_{|K} \in P_1(K), \;\; K \in \mathcal{C}_h\} \tag{8}$$

we may now decompose the unknown solution to (4) u_h into its linear part $u_1 \in V_1$ and its part spanned by the bubble, i.e.,

$$u_h = u_1 + \sum_{K \in \mathcal{C}_h} u_b^K \varphi \tag{9}$$

where u_b^K is the unknown bubble coefficient.

Substituting (9) into (7) we get

$$(\sigma u_1, \varphi)_K + u_b^K(\sigma\varphi, \varphi)_K + (\kappa\nabla u_1, \nabla\varphi)_K + u_b^K(\kappa\nabla\varphi, \nabla\varphi)_K = (f, \varphi)_K\,. \tag{10}$$

However, $\forall w_1 \in V_1$ we have by integration by parts

$$(\nabla w_1, \nabla\varphi)_K = -(\Delta w_1, \varphi)_K + (\nabla w_1 \cdot \mathbf{n}, \varphi)_{\partial K} = 0 \qquad w_1 \in V_1 \tag{11}$$

Therefore, using (11) for $w_1 = u_1$, we reduce (10) to

$$u_b^K[\sigma\|\varphi\|_{0,K}^2 + \kappa\|\nabla\varphi\|_{0,K}^2] = (f - \sigma u_1, \varphi)_K\,, \tag{12}$$

where we have used the notation $\|v\|_{0,K}^2 = \int_K v^2 \, d\,\Omega$. Solving (12) for the bubble coefficient in each element leads to

$$u_b^K = \frac{-1}{\sigma\|\varphi\|_{0,K}^2 + \kappa\|\nabla\varphi\|_{0,K}^2}(\sigma u_1 - f, \varphi)_K\,. \tag{13}$$

The procedure can be repeated to each element $K \in \mathcal{C}_h$ and equation (13) gives the value of each unknown coefficient u_b^K as a function of the chosen bubble basis function

φ, the data (σ, κ and f) and the unknown linear part u_1 of the solution u_h. The next step of static condensation consists in selecting in (4) $v = v_1 \in V_1$, and by using (9) and (11) we get:

$$(\sigma u_1, v_1) + \sum_{K \in \mathcal{C}_h} u_b^K (\sigma\varphi, v_1)_K + (\kappa \nabla u_1, \nabla v_1) = (f, v_1) \quad v_1 \in V_1 \tag{14}$$

Therefore the resulting variational equation (14) is equivalent to using the standard Galerkin method for piecewise linear functions (i.e., choosing $V_h = V_1$ in (4)) "plus" a term that in view of (13) can be written as

$$\sum_{K \in \mathcal{C}_h} u_b^K (\sigma\varphi, v_1)_K = - \sum_{K \in \mathcal{C}_h} \frac{1}{\sigma \|\varphi\|_{0,K}^2 + \kappa \|\nabla\varphi\|_{0,K}^2} (\sigma u_1 - f, \varphi)_K (\varphi, \sigma v_1)_K \,. \tag{15}$$

For the cubic bubble we have (see [20])

$$\int_K \varphi \, d\Omega = C_1 h_K^2, \qquad \varphi_{|K} \in B(K), \qquad K \in \mathcal{C}_h \tag{16}$$

$$\|\varphi\|_{0,K} = C_2 h_K^2, \qquad \varphi_{|K} \in B(K), \qquad K \in \mathcal{C}_h \tag{17}$$

$$\|\nabla\varphi\|_{0,K} = C_3, \qquad \varphi_{|K} \in B(K), \qquad K \in \mathcal{C}_h \tag{18}$$

where C_1,C_2 and C_3 are positive constants, independent of the element parameter h_K.

Using the results in [4] together with (16)-(18), (15) can be rewritten as

$$\sum_{K \in \mathcal{C}_h} u_b^K (\sigma\varphi, v_1)_K = - \sum_{K \in \mathcal{C}_h} \frac{C h_K^2}{\sigma h_K^2 + \kappa \bar{C}} (\sigma u_1 - f, \sigma v_1)_K \,. \tag{19}$$

where C and $\bar{C}$ are positive constants.

We now present a numerical simulation of the problem described in Figure 1, with $\sigma = 1$, $\kappa = 10^{-6}$, $f = 1$, the finite element space described by (5) and bubble functions of the cubic type. After static condensation the resulting elevation plot of the piecewise-linear function is presented in Figure 2. To contrast, we next take $V_h = V_1$ in (4), and plot the numerical results in Figure 3. We note that the addition of the bubbles attenuate the oscillations present using the standard Galerkin method with piecewise linears.

Remarks:

1. The variational formulation (14) governing u_1, the piecewise-linear part of the solution u_h to (4), indicates that a term should be subtracted from the standard Galerkin method. Furthermore, by substituting v_1 by u_1 we can immediately see that the term

$$\sum_{K \in \mathcal{C}_h} \frac{C h_K^2}{\sigma h_K^2 + \bar{C}\kappa} (\sigma u_1, \sigma u_1)_K$$

   is being subtracted from the Galerkin term $(\sigma u_1, u_1)$. In other words, the bubble elimination suggests that to control a large L_2-norm emanating from a large parameter σ, one should stabilize by *subtracting* a contribution of the same type. This suggestion is the basis for the introduction of the unusual stabilized method in the next section. A stability argument and numerical results presented in Section 3 confirm the suitability of such methods.

2. The Poisson equation is obtained by taking $\sigma = 0$. In this case the additional term (15) is zero, which translates into no advantage in enriching the subspace of piecewise linears with bubble functions. This result is discussed in detail in [10].

3. For the advective-difusive model, following a similar argument, it is shown in [6] that the additional term plays the role of upwinding in a consistent fashion, such as

the SUPG method proposed by Hughes and Brooks [7] (see also [11,12] for different designs of the stability parameters).

4. The relationship between the Galerkin method using bubble functions and stabilized finite element methods was first noticed in the context of the Stokes problem by Pierre [20] in relating the MINI-element for the Galerkin method [2] with the stabilized method proposed in [17]. This connection can be extended to other saddle-point problems, such as the Reissner-Mindlin bending plate model, as pointed out by Arnold [1] in relating the element proposed by Arnold and Falk [3] with the stabilized method of Franca and Stenberg [15].

5. If the reduced space of polynomials are high order ($k \geq 2$) then equation (11) is replaced by

$$(\nabla w_k, \nabla \varphi)_K = -(\Delta w_k, \varphi)_K \qquad w_k \in V_k \tag{20}$$

where

$$V_k = \{v \in H_0^1(\Omega) \,|\, v_{|K} \in P_k(K),\ K \in \mathcal{C}_h\} \tag{21}$$

and the bubble equation in this case gives

$$u_b^K = \frac{-1}{\sigma\|\varphi\|_{0,K}^2 + \kappa\|\nabla\varphi\|_{0,K}^2}(\sigma u_k - \kappa\Delta u_k - f, \varphi)_K\,. \tag{22}$$

In the second part of static condensation, i.e., selecting $v = v_k \in V_k$, instead of (14) we have now

$$\begin{aligned}(\sigma u_k, v_k) + \sum_{K\in\mathcal{C}_h} u_b^K(\varphi, \sigma v_k)_K + (\kappa\nabla u_k, \nabla v_k)& \\ - \sum_{K\in\mathcal{C}_h} u_b^K(\varphi, \kappa\Delta v_k)_K = (f, v_k) \quad v_k \in V_k&\end{aligned} \tag{23}$$

and in view of (22), the additional term is in this case

$$\sum_{K\in\mathcal{C}_h} u_b^K(\varphi, \sigma v_k - \kappa\Delta v_k)_K =$$

$$-\sum_{K\in\mathcal{C}_h}\frac{1}{\sigma\|\varphi\|_{0,K}^2+\kappa\|\nabla\varphi\|_{0,K}^2}(\sigma u_k-\kappa\Delta u_k-f,\varphi)_K(\varphi,\sigma v_k-\kappa\Delta v_k)_K \quad (24)$$

Using (16)-(18) and the theory in [4] we can rewrite (24) as

$$\sum_{K\in\mathcal{C}_h}u_b^K(\varphi,\sigma v_k-\kappa\Delta v_k)_K=-\sum_{K\in\mathcal{C}_h}\frac{Ch_K^2}{\sigma h_K^2+\kappa\bar{C}}(\sigma u_k-\kappa\Delta u_k-f,\sigma v_k-\kappa\Delta v_k)_K\,. \quad (25)$$

This is the starting point of the stabilized method presented next.

3. STABILIZED FINITE ELEMENT METHOD

The stabilized finite element we wish to consider for (1)-(2) can be written as: find $u_h\in V_1$ such that

$$B(u_h,v)=F(v)\qquad v\in V_1 \quad (26)$$

where V_1 is given in eq. (8) and

$$B(u,v)=(\sigma u,v)+(\kappa\nabla u,\nabla v)-\sum_{K\in\mathcal{C}_h}(\sigma u-\kappa\Delta u,\tau_K(\sigma v-\kappa\Delta v))_K \quad (27)$$

$$F(v)=(f,v)-\sum_{K\in\mathcal{C}_h}(f,\tau_K(\sigma v-\kappa\Delta v))_K \quad (28)$$

with the stability parameter τ_K given by

$$\tau_K=\frac{h_K^2}{\sigma h_K^2+\kappa} \quad (29)$$

Remarks:

1. The 'unusual' feature of this stabilized method is the subtraction of a term $\sum_{K\in\mathcal{C}_h}(\sigma u,\tau_K\sigma v)_K$ from $(\sigma u,v)$ of the Galerkin method. In previous works [11-19], the idea was to add a term to stabilize the discrete solution in the underlying

H_1-norm. In particular, for this model this can be achieved by adding the gradient least-squares of the governing Euler-Lagrange equation [8].

2. The method is considered for piecewise linears only, and therefore $\Delta v = 0 \quad v \in V_1$ and (27)-(28) can be simplified accordingly. We left them in the formulation to allow for high order interpolation generalizations and to consider an error analysis with consistency built in the method.

3. This method is a particular case of the general theory studied in [4]. Here we obtained an indication from the previous section on how to select τ_K and we now verify that improved stability and a global convergence analysis can be established for this method given by (26)-(29).

Let us now consider the following preliminary result to establish global convergence.

LEMMA 3.1 (Numerical Stability) :

$$B(v,v) = \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa} \|v\|_{0,K}^2 + \kappa \|\nabla v\|_0^2 \qquad v \in V_1$$

Proof: By definition (27)

$$\begin{aligned} B(v,v) &= \sigma\|v\|_0^2 + \kappa\|\nabla v\|_0^2 - \sum_{K \in \mathcal{C}_h} \tau_K \sigma^2 \|v\|_{0,K}^2 \\ &= \sum_{K \in \mathcal{C}_h} \Big(\sigma - \frac{\sigma^2 h_K^2}{\sigma h_K^2 + \kappa}\Big) \|v\|_{0,K}^2 + \kappa\|\nabla v\|_0^2 \\ &= \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa} \|v\|_{0,K}^2 + \kappa\|\nabla v\|_0^2 \qquad \blacksquare \end{aligned}$$

Before the next result, recall that by standard approximation theory, for a regular family of elements, there exists an interpolant $\widetilde{u}_{h|K} \in P_1(K)$ such that

$$\|\eta\|_{m,K} = \|u - \widetilde{u}_{h|K}\|_{m,K} \le C h_K^{2-m} |u|_{2,K} \qquad \forall u \in H^2(K)\,,\ 0 \le m \le 2 \tag{30}$$

We may now establish the following convergence result.

THEOREM 3.1. *Assume that the solution to (1)-(2) satisfies* $u \in H^2(\Omega) \cap H^1_0(\Omega)$. *Then the solution* u_h *of the method given by eqs. (26)-(29) converges to* u, *solution to eqs. (1)-(2) as follows:*

$$\|\nabla(u_h - u)\|_0^2 + \sum_{K \in \mathcal{C}_h} \frac{\sigma}{\sigma h_K^2 + \kappa} \|u_h - u\|_{0,K}^2 \le C h^2 |u|_2^2 \qquad \blacksquare$$

Proof: Let $e_h = u_h - \widetilde{u}_h$ and $e = e_h + \eta$. Then:

$$\begin{aligned}
\kappa\|\nabla e_h\|_0^2 &+ \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}\|e_h\|_{0,K}^2 \\
&\le B(e_h, e_h) \qquad \text{(by Lemma 3.1)} \\
&= B(e - \eta, e_h) \\
&= -B(\eta, e_h) \qquad \text{(consistency)} \\
&\le |B(\eta, e_h)| \\
&= |(\sigma\eta, e_h) + \kappa(\nabla\eta, \nabla e_h) - \sum_{K \in \mathcal{C}_h} (\sigma\eta - \kappa\Delta\eta, \tau_K(\sigma e_h - \kappa\Delta e_h))_K| \\
&= |\sum_{K \in \mathcal{C}_h} (\sigma - \frac{\sigma^2 h_K^2}{\sigma h_K^2 + \kappa})(\eta, e_h)_K + \kappa(\nabla\eta, \nabla e_h) + \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa h_K^2}{\sigma h_K^2 + \kappa}(\Delta\eta, e_h)_K| \\
&= |\sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}(\eta, e_h)_K + \kappa(\nabla\eta, \nabla e_h) + \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}(h_K^2\Delta\eta, e_h)_K| \\
&\le \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}\|\eta\|_{0,K}^2 + \frac{1}{4}\sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}\|e_h\|_{0,K}^2 + \frac{\kappa}{2}\|\nabla\eta\|_0^2 + \frac{\kappa}{2}\|\nabla e_h\|_0^2 \\
&+ \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa h_K^4}{\sigma h_K^2 + \kappa}\|\Delta\eta\|_{0,K}^2 + \frac{1}{4}\sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}\|e_h\|_{0,K}^2 \\
&= \sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}\left(\|\eta\|_{0,K}^2 + h_K^4\|\Delta\eta\|_{0,K}^2\right) + \frac{\kappa}{2}\|\nabla\eta\|_0^2 \\
&+ \frac{\kappa}{2}\|\nabla e_h\|_0^2 + \frac{1}{2}\sum_{K \in \mathcal{C}_h} \frac{\sigma\kappa}{\sigma h_K^2 + \kappa}\|e_h\|_{0,K}^2
\end{aligned}$$

Thus, the last two terms of the inequality above are half of the left-hand-side and after dividing by κ we are led to

$$\begin{aligned}&\frac{1}{2}\Bigg(\|\nabla e_h\|_0^2 + \sum_{K\in\mathcal{C}_h}\frac{\sigma}{\sigma h_K^2+\kappa}\|e_h\|_{0,K}^2\Bigg)\\ &\qquad\le \sum_{K\in\mathcal{C}_h}\frac{\sigma}{\sigma h_K^2+\kappa}\big(\|\eta\|_{0,K}^2 + h_K^4\|\Delta\eta\|_{0,K}^2\big)+\frac{1}{2}\|\nabla\eta\|_0^2\end{aligned} \tag{31}$$

And if we sum to both left and right hand sides of (31)

$$\frac{1}{2}\Bigg(\|\nabla \eta\|_0^2 + \sum_{K\in\mathcal{C}_h}\frac{\sigma}{\sigma h_K^2+\kappa}\|\eta\|_{0,K}^2\Bigg)$$

then we have by triangle inequality

$$\begin{aligned}&\frac{1}{4}\Bigg(\|\nabla e\|_0^2 + \sum_{K\in\mathcal{C}_h}\frac{\sigma}{\sigma h_K^2+\kappa}\|e\|_{0,K}^2\Bigg)\\ &\qquad\le \frac{3}{2}\sum_{K\in\mathcal{C}_h}\frac{\sigma}{\sigma h_K^2+\kappa}\big(\|\eta\|_{0,K}^2 + h_K^4\|\Delta\eta\|_{0,K}^2\big)+\|\nabla\eta\|_0^2\\ &\qquad\le \frac{3}{2}\sum_{K\in\mathcal{C}_h}\frac{\sigma h_K^2}{\sigma h_K^2+\kappa}\big(\frac{1}{h_K^2}\|\eta\|_{0,K}^2 + h_K^2\|\Delta\eta\|_{0,K}^2\big)+\|\nabla\eta\|_0^2\\ &\qquad\le \frac{3}{2}\sum_{K\in\mathcal{C}_h}\big(\frac{1}{h_K^2}\|\eta\|_{0,K}^2 + h_K^2\|\Delta\eta\|_{0,K}^2\big)+\|\nabla\eta\|_0^2\\ &\qquad\le Ch^2|u|_2^2\end{aligned}$$

The last line follows by (30). ■

Remark:

The above theorem establishes optimal order of convergence in the H^1-seminorm uniformly (i.e., independent on the values for σ and κ). This uniform convergence result is not attainable for the Galerkin method using piecewise linears. Furthermore for $\sigma h_K^2/\kappa >> 1$ optimal order is obtained in L^2 as well as in the H^1 norms, without the need of a duality argument.

We now illustrate the suitability of the method given by (26)-(29) using it to simulate the problem described in Figure 1 with $\sigma = 1$, $\kappa = 10^{-6}$ and $f = 1$. We present the result for the stabilized method using the finite element space spanned by piecewise linears in Figure 4.

4. CONVECTION INCLUDED

Let us now consider the model problem given by: find a scalar valued function $u(\mathbf{x})$ defined in $\Omega \subset I\!\!R^2$ such that

$$\sigma u + \mathbf{a} \cdot \nabla u - \kappa \Delta u = f \qquad \text{in } \Omega \tag{32}$$

$$u = 0 \qquad \text{on } \Gamma = \partial\Omega \tag{33}$$

where $\mathbf{a}$ is a given velocity field, σ and κ are given positive constants and $f(\mathbf{x})$ is a given source function.

If we repeat the static condensation procedure described in Section 2 for the Galerkin method corresponding to (32)-(33), then we are led to the following stabilized finite element method: find $u_h \in V_1$ such that

$$B(u_h, v) = F(v) \qquad v \in V_1 \tag{34}$$

where V_1 is given by (8) and

$$B(u,v) = (\sigma u, v) + (\mathbf{a}\cdot\nabla u, v) + (\kappa \nabla u, \nabla v) - \sum_{K \in \mathcal{C}_h} (\sigma u + \mathbf{a}\cdot\nabla u - \kappa\Delta u, \tau_K(\sigma v - \mathbf{a}\cdot\nabla v - \kappa \Delta v))_K \tag{35}$$

$$F(v) = (f, v) - \sum_{K \in \mathcal{C}_h} (f, \tau_K(\sigma v - \mathbf{a} \cdot \nabla v - \kappa \Delta v))_K \tag{36}$$

with the stability parameter τ_K given by

$$\tau_K = \frac{h_K}{2|\mathbf{a}|_p}\xi(\mathrm{Pe}_K) \tag{37}$$

$$\mathrm{Pe}_K = \frac{2|\mathbf{a}|_p h_K}{\sigma h_K^2 + \kappa} \tag{38}$$

$$\xi(\mathrm{Pe}_K) = \begin{cases} \mathrm{Pe}_K & , 0 \leq \mathrm{Pe}_K < 1 \\ 1 & , \mathrm{Pe}_K \geq 1 \end{cases} \tag{39}$$

$$|\mathbf{a}|_p = \begin{cases} \left(\sum_{i=1}^N |a_i(\mathbf{x})|^p\right)^{1/p} & , \quad 1 \leq p < \infty \\ \max\limits_{i=1,N} |a_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{40}$$

Remarks:

1. The case in which $\sigma = 0$, the method above reduces to the one introduced in [7].

2. The case in which $\mathbf{a} = \mathbf{0}$, the method above reduces to the method given in Section 3.

3. This method deals with the more general possibility of having the differential operators of order 2, 1 and zero present in the same equation. This is of interest in applying these methods to the transport equations of turbulent quantities (such as the ones found in the $k - \epsilon$ model), or to applications involving chemical reactions, etc.

4. In closing, we would like to reiterate that bubbles prompt unusual stabilized in that instead of adding a least-squares form of the Euler-Lagrange equations to the Galerkin method, we subtract a term of the type

$$\sum_{K \in \mathcal{C}_h} (Lu - f, L^* v)_K$$

where L is the differential operator associated with the scalar PDE and L^* its adjoint. See [4] for an abstract theory. It should be noted from this work that such unusual methods keep the desired additional stability characteristics of Galerkin-least-squares methods and do have a nontrivial counterpart within the framework of the Galerkin method using 'virtual' bubbles.

ACKNOWLEDGMENT

The authors acknowledge the support by the National Science Foundation under Grant ASC-9217394.

REFERENCES

[1] D.N. Arnold, "Innovative finite element methods for plates," Report No. AM 61, Department of Mathematics, PennState, 1990. To appear in Matemática Aplicada e Computacional.

[2] D.N. Arnold, F. Brezzi and M. Fortin, "A stable finite element for the Stokes equations," Calcolo 23/4 (1984) 337-344.

[3] D.N. Arnold and R.S. Falk, "A uniformly accurate finite element method for the Reissner-Mindlin plate," SIAM J. Numer. Anal. 26/6 (1989) 1276-1290.

[4] C. Baiocchi, F. Brezzi and L.P. Franca, "Virtual bubbles and the Galerkin-least-squares method", *Computer Methods in Applied Mechanics and Engineering*, Vol.**105** (1993) 125-141.

[5] R.E. Bank and B.D. Welfert, "A comparison between the mini-element and the Petrov-Galerkin formulations for the generalized Stokes problem," *Computer Methods in Applied Mechanics and Engineering*, Vol.**83** (1990) 61-68.

[6] F. Brezzi, M.O. Bristeau, L.P. Franca, M. Mallet and G. Roge, "A relationship between stabilized finite element methods and the Galerkin method with bubble functions," *Computer Methods in Applied Mechanics and Engineering*, Vol.**96** (1992) 117-129.

[7] A.N. Brooks and T.J.R. Hughes, "Streamline upwind/Petrov-Galerkin formulations for convective dominated flows with particular emphasis on the incompressible Navier-Stokes equations," *Computer Methods in Applied Mechanics and Engineering*, Vol.**32** (1982) 199-259.

[8] L.P. Franca and E.G.Dutra do Carmo, "The Galerkin gradient least-squares method," *Computer Methods in Applied Mechanics and Engineering*, Vol.**74** (1989) 41-54.

[9] L.P. Franca and C. Farhat, "Anti-stabilizing effects of bubble functions", *Proceedings of the Third World Congress on Computational Mechanics*, Extended Abstracts of Lectures, Chiba, Japan, August 1994, to appear.

[10] L.P. Franca and C. Farhat, "On the limitations of bubble functions", *Computer Methods in Applied Mechanics and Engineering*, to appear.

[11] L.P. Franca and S.L. Frey, "Stabilized finite element methods: II. The incompressible Navier-Stokes Equations", *Comput. Methods Appl. Mech. Engrg.*, Vol.**99** (1992) 209-233.

[12] L.P. Franca, S.L. Frey and T.J.R. Hughes, "Stabilized finite element methods: I. Application to the advective-diffusive model", *Comput. Methods Appl. Mech. Engrg.* Vol.**95** (1992) 253-276.

[13] L.P. Franca and T.J.R. Hughes, "Two classes of mixed finite element methods",

Comput. Methods Appl. Mech. Engrg. Vol.**69** (1988) 89-129.

[14] L.P. Franca, T.J.R. Hughes and R. Stenberg, "Stabilized finite element methods for the Stokes problem", pp. 87-107 in *Incompressible Computational Fluid Dynamics-Trends and Advances*, M.D. Gunzburger and R.A. Nicolaides eds., Cambridge University Press, 1993.

[15] L.P. Franca and R. Stenberg, "A modification of a low-order Reissner-Mindlin plate bending element", pp. 425-436 in *The Mathematics of Finite Elements and Applications VII*, (Whiteman, ed.), Academic Press (1991).

[16] T.J.R. Hughes and L.P. Franca, "A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces", *Comput. Methods Appl. Mech. Engrg.* Vol.**65** (1987) 85-96.

[17] T.J.R. Hughes, L.P. Franca and M. Balestra, "A new finite element formulation for computational fluid dynamics: V. Circumventing the Babuška-Brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations", *Comput. Methods Appl. Mech. Engrg.* Vol.**59** (1986) 85-99.

[18] T.J.R. Hughes, L.P. Franca and G.M. Hulbert, "A new finite element formulation for computational fluid dynamics: VIII. The Galerkin-least-squares method for advective-diffusive equations", *Comput. Methods Appl. Mech. Engrg.* Vol.**73** (1989) 173-189.

[19] C. Johnson, *Numerical solution of partial differential equations by the finite element method*, Cambridge University Press, Cambridge, 1987.

[20] R. Pierre, "Simple C^0 approximations for the computation of incompressible flows,"

Comput. Methods Appl. Mech. Engrg. Vol.**68** (1988) 205-227.

[21] R. Pierre, "Regularization procedures of mixed finite element approximations of the Stokes problem," *Num. Methods Partial Diff. Equations* Vol.**5** (1989) 241-258.

Figure 1. Boundary conditions for test problem with $f = 1$, $\sigma = 1$ and $\kappa = 10^{-6}$.

Figure 2. Elevation plot for the Galerkin method using piecewise linears plus bubbles (The plot shows the results after static condensation).

Figure 3. Elevation plot for the Galerkin method using piecewise linears.

Figure 4. Elevation plot for the stabilized finite element method using piecewise linears.