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FIRST-ORDER SYSTEM LEAST SQUARES FOR SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS: PART II

ZHIQIANG CAI* , THOMAS A. MANTEUFFEL† AND STEVEN F. MCCORMICK†

Abstract. This paper develops a least-squares functional that arises from recasting general second-order uniformly elliptic partial differential equations in $n = 2$ or 3 dimensions as a system of first-order equations. In part I [10] a similar functional was developed and shown to be elliptic in the $H(\mathrm{div}) \times H^1$ norm and to yield optimal convergence for finite element subspaces of $H(\mathrm{div}) \times H^1$. In this paper the functional is modified by adding a compatible constraint and imposing additional boundary conditions to the first-order system. The resulting functional is proved to be elliptic in the $(H^1)^{n+1}$ norm, and optimal convergence for finite element subspaces of $(H^1)^{n+1}$ is established. Further, multiplicative and additive multigrid algorithms for the numerical solution of the resulting discrete functional are derived and shown to be optimally convergent. Apparently, this is the first multigrid theory that applies directly to convection-diffusion-reaction and Helmholtz equations, avoiding the limitations of perturbation type results.

Key Words. least squares discretization, multigrid, second-order elliptic problems, iterative methods

AMS(MOS) subject classifications. 65F10, 65F30

1. Introduction.

The object of study of this paper, and its earlier companion [10], is the solution of elliptic equations (including convection-diffusion and Helmholtz equations) by way of a least-squares formulation for an equivalent first-order system. Such formulations have been considered by several researchers over the last few decades (see the historical discussion in [10]), motivated in part by the possibility of a well-posed variational principle for a general class of problems. In [10] a similar functional was developed and shown to be elliptic in the $H(\mathrm{div}) \times H^1$ norm and to yield optimal convergence for finite element subspaces of $H(\mathrm{div}) \times H^1$. In this paper the functional is modified by adding a compatible constraint and imposing additional boundary conditions to the first-order system. It is shown that the resulting functional is elliptic in the $(H^1)^{n+1}$ norm. Optimal convergence for finite element subspaces of $(H^1)^{n+1}$ is shown. Further, multiplicative and additive multigrid algorithms for the numerical solution of the resulting discrete functional are derived and shown to be optimally convergent. Apparently, this is the first multigrid theory that applies directly to convection-diffusion-reaction and Helmholtz equations, avoiding the limitations of perturbation type results.

The least-squares formulation considered in this paper, which follows Chang's work (see [12]) for Poisson's equation, differs from that of [10] (see also [20]) in that it incorporates a curl-free constraint and tangential boundary conditions on $\mathbf{u}$ (see

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uniformly symmetric positive definite and scaled appropriately: there exist positive constants

$$0 < \lambda \le 1 \le \Lambda \tag{2.2}$$

such that

$$\lambda \xi^T \xi \le \xi^T A \xi \le \Lambda \xi^T \xi \tag{2.3}$$

for all $\xi \in \Re^n$ and almost all $x \in \overline{\Omega}$.

Introducing the flux variable

$$\mathbf{u} = A\nabla p,$$

problem (2.1) may be rewritten as a first-order system of partial differential equations as follows:

$$\begin{cases} \mathbf{u} - A\nabla p &= \mathbf{0}, & \text{in} & \Omega, \\ \nabla^* \mathbf{u} + Xp &= f, & \text{in} & \Omega, \\ p &= 0, & \text{on} & \Gamma_D, \\ \mathbf{n}\cdot\mathbf{u} &= 0, & \text{on} & \Gamma_N, \end{cases} \tag{2.4}$$

where $\nabla^* : H^1(\Omega)^n \to L^2(\Omega)$ is the formal adjoint of $\nabla : H^1(\Omega) \to L^2(\Omega)^n$, that is, $\nabla^* \equiv -\nabla\cdot$. For any $f \in H^{-1}(\Omega)$, the associated weak form of the system (2.1) is uniquely solvable in $H^1(\Omega)$ if $\Gamma_D \ne \emptyset$. If $\Gamma_D = \emptyset$, then the associated weak form of (2.1) is uniquely solvable in $H^1(\Omega)/\Re$ (cf. [15]).

Let $\mathbf{curl} \equiv \nabla\times$ denote the curl operator. (Here and henceforth, we use notation for the case $n = 3$ and consider the special case $n = 2$ in the natural way by identifying $\Re^2$ with the (x_1, x_2)-plane in $\Re^3$. Thus, if $\mathbf{u}$ is two dimensional, then $\nabla\times\mathbf{u} = \mathbf{0}$ means $\partial_1 u_2 - \partial_2 u_1 = 0$, where u_1 and u_2 are the components of $\mathbf{u}$. In Section 2.1, we consider only the case $n = 2$, so there we will interpret $\nabla\times\mathbf{u}$ to mean $\partial_1 u_2 - \partial_2 u_1$.) Note that if $\mathbf{u}$ is sufficiently smooth, then the properly scaled solution, $A^{-1}\mathbf{u}$, of (2.4) is curl free, i.e., $\nabla\times(A^{-1}\mathbf{u}) = \mathbf{0}$, and that the homogeneous Dirichlet boundary condition on Γ_D implies the tangential flux or no-slip condition

$$\gamma_{\tau}(A^{-1}\mathbf{u}) = \mathbf{0}, \quad \text{on } \Gamma_D,$$

where $\gamma_{\tau}\mathbf{u} \equiv \mathbf{n}\times\mathbf{u}$. Here, τ represents the unit vector tangent to the boundary, $\partial\Omega$.

An equivalent extended system for (2.4) is

$$\begin{cases} \mathbf{u} - A\nabla p &= \mathbf{0}, & \text{in} & \Omega, \\ \nabla^* \mathbf{u} + Xp &= f, & \text{in} & \Omega, \\ \nabla\times A^{-1}\mathbf{u} &= \mathbf{0}, & \text{in} & \Omega, \\ p &= 0, & \text{on} & \Gamma_D, \\ \mathbf{n}\cdot\mathbf{u} &= 0, & \text{on} & \Gamma_N \\ \gamma_{\tau}(A^{-1}\mathbf{u}) &= \mathbf{0}, & \text{on} & \Gamma_D. \end{cases} \tag{2.5}$$

In [10] the following quadratic functional associated with system (2.4) was examined:

$$G_0(\mathbf{v}, q; f) = \|\mathbf{v} - A\nabla q\|_{0,\Omega}^2 + \|\nabla^*\mathbf{v} + Xq - f\|_{0,\Omega}^2, \tag{2.9}$$

for $(\mathbf{v}, q) \in \mathbf{W}_0(\text{div}; \Omega) \times V$. There it was shown that $G_0(\mathbf{v}, q; 0)$ is equivalent to the $H(\text{div}; \Omega) \times H^1(\Omega)$ norm on $\mathbf{W}_0(\text{div}; \Omega) \times V$ under the following *original* assumption:

Assumption A0: Either $\Gamma_D \neq \emptyset$ or an additional constraint is imposed on V, such as $\int_\Omega p\, dx = 0$, so that a Poincaré-Friedrichs inequality holds: there exists a constant $d > 0$ depending only on the domain Ω and the uniform bounds on A (see (2.3)) such that

$$\|p\|_{0,\Omega}^2 \leq d\|A^{1/2}\nabla p\|_{0,\Omega}^2 \tag{2.10}$$

for $p \in V$. Moreover, for any $f \in H^{-1}(\Omega)$ the associated weak form of (2.1) is invertible in $H^1(\Omega)$ and

$$\|Xp\|_{0,\Omega} \leq \eta\|A^{1/2}\nabla p\|_{0,\Omega} \tag{2.11}$$

for some $\eta > 0$ and every $p \in V$ for which $A\nabla p \in \mathbf{W}_0(\text{div}; \Omega)$.

The modified quadratic functional we study here is given by

$$G(\mathbf{v}, q; f) = \|\mathbf{v} - A\nabla q\|_{0,\Omega}^2 + \|\nabla^*\mathbf{v} + Xq - f\|_{0,\Omega}^2 + \|\nabla\times(A^{-1}\mathbf{v})\|_{0,\Omega}^2, \tag{2.12}$$

for $(\mathbf{v}, q) \in \mathbf{W} \times V$. Then the least-squares problem for (2.5) is to minimize this quadratic functional over $\mathbf{W} \times V$: find $(\mathbf{u}, p) \in \mathbf{W} \times V$ such that

$$G(\mathbf{u}, p; f) = \inf_{(\mathbf{v},q)\in\mathbf{W}\times V} G(\mathbf{v}, q; f). \tag{2.13}$$

It is easy to see that the variational form for (2.13) is to find $(\mathbf{u}, p) \in \mathbf{W} \times V$ such that

$$\mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) = f(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{W} \times V, \tag{2.14}$$

where the bilinear form $\mathcal{F}(\cdot\,;\cdot) : (\mathbf{W} \times V)^2 \longrightarrow \Re$ is defined by

$$\begin{aligned} \mathcal{F}(\mathbf{u}, p; \mathbf{v}, q) &= (\mathbf{u} - A\nabla p, \mathbf{v} - A\nabla q)_{0,\Omega} + (\nabla^*\mathbf{u} + Xp, \nabla^*\mathbf{v} + Xq)_{0,\Omega} \\ &+ (\nabla\times(A^{-1}\mathbf{u}), \nabla\times(A^{-1}\mathbf{v}))_{0,\Omega} \end{aligned} \tag{2.15}$$

and the linear functional $f(\cdot, \cdot) : \mathbf{W} \times V \longrightarrow \Re$ is defined by

$$f(\mathbf{v}, q) = (f, \nabla^*\mathbf{v} + Xq)_{0,\Omega}. \tag{2.16}$$

The first theorem establishes ellipticity and continuity of the bilinear form (2.15) with respect to the $(H(\text{div}; \Omega) \cap H(\text{curl } A; \Omega)) \times H^1(\Omega)$ norm under only assumption A0.

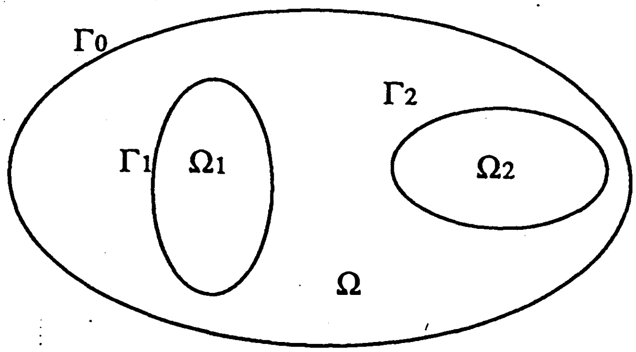

FIG. 1.

For $n = 2$, Γ_0 is divided into a finite number of connected pieces: $\Gamma_0 = \cup_{i=1,\dots,M}\Gamma_{0,i}$ such that $\Gamma_{0,i} \subseteq \Gamma_D$ for $i \in D_0$ and $\Gamma_{0,i} \subseteq \Gamma_N$ for $i \in N_0$; since Γ_0 is a simple closed curve, M is even; let D_0 be the odd indices and N_0 be the even indices. For $n = 3$, either $\Gamma_0 \subseteq \Gamma_D$ or $\Gamma_0 \subseteq \Gamma_N$.

Assumption A3: The matrix A is $C^{1,1}$. If $n = 2$ and $\mathbf{x} \in \Gamma_0$ is a point that separates Γ_D and Γ_N, then $\mathbf{x}$ must be a corner of Γ_0 and $\mathbf{n}_-^T A \mathbf{n}_+ \le 0$, where $\mathbf{n}_-$ and $\mathbf{n}_+$ are the outward unit normal vectors on the adjacent edges at $\underline{\mathbf{x}}$.

These additional assumptions, together with the original ones, are sufficient to guarantee that the boundary value problem

$$\left\{ \begin{array}{rcll} \nabla^*(A\nabla p) + Xp &=& f, & \text{in} \quad \Omega, \\ p &=& g, & \text{on} \quad \Gamma_D, \\ \mathbf{n}\cdot A\nabla p &=& h, & \text{on} \quad \Gamma_N, \end{array} \right. \tag{2.19}$$

is $H^2(\Omega)$ regular: there exists a constant C depending only on A, X, and Ω such that

$$\|p\|_{2,\Omega} \le C(\|f\|_{0,\Omega} + \|g\|_{3/2,\Gamma_D} + \|h\|_{1/2,\Gamma_N}). \tag{2.20}$$

For details, see [16].

Our main theorem establishes equivalence of the bilinear form (2.15) and the $H^1(\Omega)^{n+1}$ norm under the additional assumptions **A1 - A3**.

THEOREM 2.2. *Assume* **A0 - A3**. *Then there exist positive constants* α_2 *and* α_3 *such that*

$$\alpha_2 \left(\|\mathbf{v}\|_{1,\Omega}^2 + \|q\|_{1,\Omega}^2\right) \le \mathcal{F}(\mathbf{v},\, q;\, \mathbf{v},\, q) \tag{2.21}$$

for any $(\mathbf{v},\, q) \in \mathbf{W} \times V$ *and*

$$\mathcal{F}(\mathbf{u},\, p;\, \mathbf{v},\, q) \le \alpha_3 \left(\|\mathbf{u}\|_{1,\Omega}^2 + \|p\|_{1,\Omega}^2\right)^{\frac{1}{2}} \left(\|\mathbf{v}\|_{1,\Omega}^2 + \|q\|_{1,\Omega}^2\right)^{\frac{1}{2}} \tag{2.22}$$

LEMMA 2.2. *Let $\Omega \subset \Re^2$; then* $\mathbf{w} \in H(\mathbf{curl}\,;\Omega)$ *such that* $\nabla\times\mathbf{w} = 0$ *and* $\int_{\Gamma_i} \boldsymbol{\tau}\cdot\mathbf{w} = 0$ *for* $i = 1,\ldots,L$ *if and only if* $\mathbf{w} = \nabla q$ *with* $q \in H^1(\Omega)$.

Proof. The proof follows from Lemma 2.1 and (2.24). ∎

A result analogous to Green's formula also follows:

$$(\nabla\times\mathbf{z},\ \phi) = (\mathbf{z},\ \nabla^{\perp}\phi) - \int_{\Gamma}(\boldsymbol{\tau}\cdot\mathbf{z})\phi \tag{2.25}$$

for $\mathbf{z} \in H(\mathbf{curl}\,;\Omega)$ and $\phi \in H^1(\Omega)$.

The next lemma obtains sufficient conditions for a vector function in $\mathbf{W}$ to be zero.

LEMMA 2.3. *Let A be uniformly symmetric positive definite on Ω, which satisfies* **A1** *and* **A2**. *Let* $\mathbf{z} \in \mathbf{W}$ *satisfy*

$$\begin{array}{lrcll}
i) & \nabla^{*}\mathbf{z} & = & 0, & \textit{in } \Omega,\\
ii) & \nabla\times A^{-1}\mathbf{z} & = & 0, & \textit{in } \Omega,\\
iii) & \int_{\Gamma_i}\mathbf{n}\cdot\mathbf{z} & = & 0, & \textit{for } i\in D,\\
iv) & \int_{\Gamma_i}\boldsymbol{\tau}\cdot A^{-1}\mathbf{z} & = & 0, & \textit{for } i\in N,\\
\textit{and either} & & & & \\
v) & \int_{\Gamma_{0j}}\mathbf{n}\cdot\mathbf{z} & = & 0, & \textit{for } j\in D_0,\\
\textit{or} & & & & \\
vi) & \int_{\Gamma_{0j}}\boldsymbol{\tau}\cdot A^{-1}\mathbf{z} & = & 0, & \textit{for } j\in N_0.
\end{array} \tag{2.26}$$

Then $\mathbf{z} = \mathbf{0}$.

Proof. Assumptions (2.26) i), ii), iii), and iv) together with Lemmas 2.1 and 2.2 yield

$$\mathbf{z} = A\nabla p, \qquad \mathbf{z} = \nabla^{\perp}\phi,$$

with $p,\ \phi \in H^1(\Omega)$. Using Green's formula and assumption i), we have

$$\begin{aligned}
(\nabla^{*}\mathbf{z},\ p) &= (\mathbf{z},\ \nabla p) - \int_{\Gamma}(\mathbf{n}\cdot\mathbf{z})p\\
&= (A^{-1}\mathbf{z},\ \mathbf{z}) - \int_{\Gamma}(\mathbf{n}\cdot\mathbf{z})p = 0.
\end{aligned}$$

Thus,

$$(A^{-1}\mathbf{z},\ \mathbf{z}) = \int_{\Gamma_0}(\mathbf{n}\cdot\mathbf{z})p + \sum_{i\in D}\int_{\Gamma_i}(\mathbf{n}\cdot\mathbf{z})p + \sum_{i\in N}\int_{\Gamma_i}(\mathbf{n}\cdot\mathbf{z})p. \tag{2.27}$$

The last sum is zero because $\mathbf{n}\cdot\mathbf{z} = 0$ on Γ_N. The second sum is also zero because integration by parts on each of its terms yields

$$\int_{\Gamma_i}(\mathbf{n}\cdot\mathbf{z})p = -\int_{\Gamma_i}(\boldsymbol{\tau}\cdot\nabla\phi)p = \int_{\Gamma_i}(\boldsymbol{\tau}\cdot\nabla p)\phi = \int_{\Gamma_i}(\boldsymbol{\tau}\cdot A^{-1}\mathbf{z})\phi = 0,$$

and the functions p_{0j} for $j \in D_0$ that satisfy

$$(2.29) \qquad \begin{cases} \nabla^* A \nabla p_{0j} = 0, & \text{in } \Omega, \\ p_{0j} = 1, & \text{on } \Gamma_{0j}, \\ p_{0j} = 0, & \text{on } \Gamma_D \setminus \Gamma_{0j}, \\ \mathbf{n} \cdot A \nabla p_{0j} = 0, & \text{on } \Gamma_N. \end{cases}$$

Clearly, $A\nabla p_i$, $A\nabla p_{0j} \in \mathbf{W}$ and they satisfy (2.26) i) and ii). If **A1** – **A3** hold, then (2.28) and (2.29) are $H^2(\Omega)$ regular and

$$(2.30) \qquad \|p_i\|_{2,\Omega} \leq C_i, \quad \|p_{0j}\|_{2,\Omega} \leq C_{0j},$$

for $i \in D$ and $j \in D_0$, where C_i and C_{0j} depend only on Ω and A.

Next, note that

$$(2.31) \qquad \nabla \times A^{-1} \nabla^{\perp} = \nabla^* P^* A^{-1} P \nabla = \nabla^* B \nabla,$$

where

$$(2.32) \qquad B \equiv P^* A^{-1} P = \frac{1}{det(A)} A.$$

This relation is easily verified algebraically for any 2 by 2 symmetric matrix A. Now B is uniformly symmetric positive definite:

$$\frac{1}{\Lambda} \xi^T \xi \leq \xi^T B \xi \leq \frac{1}{\lambda} \xi^T \xi$$

for all $\xi \in R^n$ and $\mathbf{x} \in \bar{\Omega}$. Consider the functions ϕ_i for $i \in N$ that satisfy

$$(2.33) \qquad \begin{cases} \nabla^* B \nabla \phi_i = 0, & \text{in } \Omega, \\ \phi_i = 1, & \text{on } \Gamma_i, \\ \phi_i = 0, & \text{on } \Gamma_N \setminus \Gamma_i, \\ \mathbf{n} \cdot B \nabla \phi_i = 0, & \text{on } \Gamma_D, \end{cases}$$

and the functions ϕ_{0j} for $j \in N_0$ that satisfy

$$(2.34) \qquad \begin{cases} \nabla^* B \nabla \phi_{0j} = 0, & \text{in } \Omega, \\ \phi_{0j} = 1, & \text{on } \Gamma_{0j}, \\ \phi_{0j} = 0, & \text{on } \Gamma_N \setminus \Gamma_{0j}, \\ \mathbf{n} \cdot B \nabla \phi_{0j} = 0, & \text{on } \Gamma_D. \end{cases}$$

Since

$$\begin{aligned} \tau \cdot A^{-1} \nabla^{\perp} \phi_i &= \mathbf{n} \cdot B \nabla \phi_i = 0, && \text{on } \Gamma_D, \\ \mathbf{n} \cdot \nabla^{\perp} \phi_i &= \tau \cdot \nabla \phi_i = 0, && \text{on } \Gamma_N, \\ \tau \cdot A^{-1} \nabla^{\perp} \phi_{0j} &= \mathbf{n} \cdot B \nabla \phi_{0j} = 0, && \text{on } \Gamma_D, \\ \mathbf{n} \cdot \nabla^{\perp} \phi_{0j} &= \tau \cdot \nabla \phi_{0j} = 0, && \text{on } \Gamma_N, \end{aligned}$$

and.

$$\text{(2.42)} \qquad \text{either} \int_{\Gamma_{0j}} \mathbf{n} \cdot \mathbf{w} = 0, \ \ j \in D_0, \ \ \text{or} \ \int_{\Gamma_{0j}} \boldsymbol{\tau} \cdot A^{-1}\mathbf{w} = 0, \ \ j \in N_0.$$

A decomposition of the form (2.39) is accomplished by solving the linear system

$$\text{(2.43)} \quad \begin{array}{llll} \sum_{j \in D_0} (\int_{\Gamma_{0i}} \mathbf{n} \cdot A\nabla p_{0j})\alpha_{0j} & + \sum_{j \in D} (\int_{\Gamma_{0i}} \mathbf{n} \cdot A\nabla p_j)\alpha_j & + \sum_{j \in N} (\int_{\Gamma_{0i}} \mathbf{n} \cdot \nabla^{\perp}\phi_j)\beta_j & = \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{Z}, \quad \text{for } i \in D_0, \\ \sum_{j \in D_0} (\int_{\Gamma_i} \mathbf{n} \cdot A\nabla p_{0j})\alpha_{0j} & + \sum_{j \in D} (\int_{\Gamma_i} \mathbf{n} \cdot A\nabla p_j)\alpha_j & & = \int_{\Gamma_i} \mathbf{n} \cdot \mathbf{Z}, \quad \text{for } i \in D, \\ \sum_{j \in D_0} (\int_{\Gamma_i} \boldsymbol{\tau} \cdot \nabla p_{0j})\alpha_{0j} & & + \sum_{j \in N} (\int_{\Gamma_i} \mathbf{n} \cdot B\nabla\phi_j)\beta_j & = \int_{\Gamma_i} \boldsymbol{\tau} \cdot A^{-1}\mathbf{Z}, \quad \text{for } i \in N. \end{array}$$

Note that $\int_{\Gamma_i} \mathbf{n} \cdot \nabla^{\perp}\phi_j = \int_{\Gamma_i} \boldsymbol{\tau} \cdot \nabla\phi_j = 0$ for $i \in D$ and $\int_{\Gamma_i} \boldsymbol{\tau} \cdot \nabla p_j = 0$ for $i \in N$ because the integrations are carried out on a closed path.

To see that (2.43) has a solution, note first that it is a singular but consistent system of linear equations. Consider the 2×2 block in the upper left of the tableau. Since each $A\nabla p_i$ and $A\nabla p_{0j}$ is divergence free, then $\int_{\Gamma} \mathbf{n} \cdot A\nabla p_i = \int_{\Gamma} \mathbf{n} \cdot A\nabla p_{0i} = 0$. Thus, the sum of any column of this 2×2 block is zero. The sum of the first two blocks of the right-hand side is also zero by the same reasoning. The null space consists of setting $\alpha_i = \alpha$ for $i \in D$, $\alpha_{0i} = \alpha$ for $i \in D_0$, and $\beta_i = 0$ for $i \in N$. This corresponds to a constant function, which is in the null space of ∇. A reduced nonsingular system can be found by setting any α_i or α_{0i} to zero and deleting the corresponding row. To see that this reduced system is nonsingular, assume not; then, for some $\mathbf{z}$, there are two solutions whose α_i's differ by something other than a constant; their difference would yield a nonzero function of the form (2.37) that satisfies the hypotheses of Lemma 2.3, which is a contradiction.

With this choice for α_{0j}, α_i, and β_i, the function $\mathbf{w}$ satisfies the hypotheses of Lemma 2.3, which implies $\mathbf{w} = \mathbf{0}$. If the form (2.40) had been chosen, a similar argument would yield coefficients α_i, β_{0j}, and β_i with one β set to zero.

Now assume **A3**. Since the linear system represented by the left-hand side of (2.43) depends only upon Ω and A, then there exist constants $C_1 - C_4$ such that

$$\begin{aligned} \max_{j \in D \cup D_0} |\alpha_j| + \max_{j \in N \cup N_0} |\beta_j| &\leq C_1 \left(\max_{j \in D \cup D_0} \left| \int_{\Gamma_j} \mathbf{n} \cdot \mathbf{z} \right| + \max_{j \in N \cup N_0} \left| \int_{\Gamma_i} \boldsymbol{\tau} \cdot A^{-1}\mathbf{z} \right| \right) \\ &\leq C_2(\|\mathbf{n} \cdot \mathbf{z}\|_{-1/2,\Gamma} + \|\boldsymbol{\tau} \cdot A^{-1}\mathbf{z}\|_{-1/2,\Gamma}) \\ &\leq C_2(\|\mathbf{z}\|_{H(\mathrm{div};\Omega)} + \|A^{-1}\mathbf{z}\|_{H(\mathbf{curl};\Omega)}) \\ &\leq C_3\|\mathbf{z}\|_{0,\Omega}. \end{aligned}$$

Finally, (2.30), (2.35), and (2.37) yield

$$\|\mathbf{z}\|_{1,\Omega} \leq C_4(\max_{i \in D \cup D_0} |\alpha_i| + \max_{i \in N \cup N_0} |\beta_i|) \leq C\|\mathbf{z}\|_{0,\Omega},$$

and the lemma is proved. ∎

We remark that the decomposition of $\mathbf{z}$ is not unique. For example, any linear combination of (2.39) and (2.40) whose coefficients sum to one again yields zero.

2.2. Three Dimensions. Our additional assumptions for $n = 3$ restrict the boundary Γ_0 to be either Dirichlet or Neumann, that is, $\Gamma_0 \subseteq \Gamma_D$ or $\Gamma_0 \subseteq \Gamma_N$. Further, Γ_0 is now either $C^{1,1}$ or a convex polyhedron. The results in this section generalize Theorems 3.7, 3.8, and 3.9 in Chapter I of [15], where, in addition to the above restrictions on Γ_0, it is assumed that the entire boundary is either Dirichlet or Neumann and that $A = I$. Unlike the two-dimensional proof, we use the result in [15] in our three-dimensional proof, and thus make the same assumptions on Γ_0.

THEOREM 2.3. *Assume that $A = I$ and that either $\Gamma_N = \Gamma$ or $\Gamma_D = \Gamma$. Then* Theorem 2.2 *holds.*

Proof. The proof follows from Theorems 3.7 and 3.9 in Chapter I of [15] and Theorem 3.1 in [10]. ∎

Given the more general assumptions on A, Γ_D, and Γ_N, the upper bound in (2.23) is immediate, so our task is again to establish the lower bound in (2.23). We first gather some tools. The next two lemmas are technical, but essential to what follows.

LEMMA 2.5. *Let $\Omega \subset \Re^3$ such that the boundary, $\partial\Omega$, is piecewise $C^{1,1}$. If $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$ and $\mathbf{n} \times \mathbf{v} = \mathbf{0}$ on $\partial\Omega$, then $\mathbf{n} \cdot (\nabla\times\mathbf{v}) = 0$ on $\partial\Omega$.*

Proof. We offer only a heuristic proof. Details can be found in [21]. First, since $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$, then $\mathbf{n} \times \mathbf{v}$ is well defined on $\partial\Omega$. Also, $\nabla\times\mathbf{v} \in H(\text{div};\Omega)$ so $\mathbf{n} \cdot (\nabla\times\mathbf{v})$ is well defined on $\partial\Omega$. Now $\mathbf{n} \times \mathbf{v} = \mathbf{0}$ on $\partial\Omega$ implies $\mathbf{v}$ is normal to $\partial\Omega$. Assume that $\mathbf{v} \in \mathcal{D}(\overline{\Omega})$ and let $\mathbf{x} \in \partial\Omega$. Consider the definition

$$\nabla\times\mathbf{v}(\mathbf{x}) = \lim_{\Delta\mathcal{V}\to 0} \frac{1}{\Delta\mathcal{V}} \int_{\partial\mathcal{V}} \hat{\mathbf{n}} \times \mathbf{v},$$

where the limit is taken over any convenient neighborhood $\mathcal{V}$ of $\mathbf{x}$ with volume $\Delta\mathcal{V}$ and surface normal $\hat{\mathbf{n}}$. For example, since $\partial\Omega$ is piecewise $C^{1,1}$, we may choose cubes with two sides tangent to the boundary, on which $\hat{\mathbf{n}} = \mathbf{n}$, which yields $\mathbf{n} \cdot (\hat{\mathbf{n}} \times \mathbf{v}) = \mathbf{v} \cdot (\hat{\mathbf{n}} \times \mathbf{n}) = \mathbf{0}$; on the other cube sides, we have the limiting property $\mathbf{n} \cdot (\hat{\mathbf{n}} \times \mathbf{v}) \frac{\Delta\mathcal{A}}{\Delta\mathcal{V}} \to 0$ since $\mathbf{v}$ is normal to $\partial\Omega$ at $\mathbf{x}$. The result for $\mathbf{v} \in H(\mathbf{curl}\,;\Omega)$ follows by continuity. ∎

LEMMA 2.6. *Let $\Omega \subset \Re^3$ and let Γ be a simple, closed, piecewise $C^{1,1}$ surface in Ω. Let $p \in H^1(\Omega)$ and $\boldsymbol{\psi} \in H^1(\Omega)^3$. Then*

$$\int_{\Gamma} ((\mathbf{n} \cdot \nabla\times\boldsymbol{\psi})p + (\mathbf{n} \times \boldsymbol{\psi}) \cdot \nabla p) = 0. \tag{2.50}$$

Proof. Again, we offer only a heuristic proof. Details can be found in [21]. Let

Proof. See Theorem 2.9 in Chapter I of [15]. ■

Next, as in the two-dimensional proof, we provide a result that allows us to declare that a vector in $\mathbf{W}$ is $\mathbf{0}$.

LEMMA 2.7. *Let A be uniformly symmetric positive definite on $\Omega \subset \Re^3$, which satisfies* **A1** *and* **A2**. *Let $\mathbf{z} \in \mathbf{W}$ satisfy*

$$\begin{array}{lrcl} i) & \nabla^*\mathbf{z} & = & 0, \ \ in\ \Omega, \\ ii) & \nabla\times A^{-1}\mathbf{z} & = & 0, \ \ in\ \Omega, \\ iii) & \int_{\Gamma_i}\mathbf{n}\cdot\mathbf{z} & = & 0, \ \ for\ i \in D. \end{array} \tag{2.56}$$

Then $\mathbf{z} = \mathbf{0}$.

Proof. If $\Gamma_0 \subseteq \Gamma_N$, then $\mathbf{n}\cdot\mathbf{z} = 0$ on Γ_0 and we let $\widehat{N} = N \cup \{0\}$, $\widehat{D} = D$. If $\Gamma_0 \subseteq \Gamma_D$, let $\widehat{N} = N$, $\widehat{D} = D \cup \{0\}$. In either case, since $\mathbf{z} \in \mathbf{W}$, then assumptions (2.56) i) and iii) and Theorem 2.4 imply there exists $\boldsymbol{\psi} \in H^1(\Omega)^3$ such that $\mathbf{z} = \nabla\times\boldsymbol{\psi}$. Assumptions (2.56) ii) and Theorem 2.5 imply there exists $p \in H^1(\Omega)$ such that $A^{-1}\mathbf{z} = \nabla p$. Assumption (2.56) i) and Green's formula then yield

$$\begin{aligned} 0 &= (\nabla^*\mathbf{z},\ p) \\ &= (\mathbf{z},\ \nabla p) - \int_\Gamma (\mathbf{n}\cdot\mathbf{z})p \\ &= (A^{-1}\mathbf{z},\ \mathbf{z}) - \int_\Gamma (\mathbf{n}\cdot\mathbf{z})p. \end{aligned}$$

Since $\mathbf{z} \in \mathbf{W}$ implies $\mathbf{n}\cdot\mathbf{z} = 0$ on Γ_N, we then have

$$\begin{aligned} (A^{-1}\mathbf{z},\ \mathbf{z}) &= \sum_{i\in\widehat{D}} \int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p + \sum_{i\in\widehat{N}} \int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p \\ &= \sum_{i\in\widehat{D}} \int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p. \end{aligned}$$

Likewise, $\mathbf{z} \in \mathbf{W}$ implies $\mathbf{n}\times A^{-1}\mathbf{z} = \mathbf{0}$ on Γ_D, which by Lemma 2.6 yields

$$\begin{aligned} \int_{\Gamma_i} (\mathbf{n}\cdot\mathbf{z})p &= \int_{\Gamma_i} (\mathbf{n}\cdot\nabla\times\boldsymbol{\psi})p = -\int_{\Gamma_i} (\mathbf{n}\times\boldsymbol{\psi})\cdot\nabla p \\ &= \int_{\Gamma_i} (\mathbf{n}\times\nabla p)\cdot\boldsymbol{\psi} = \int_{\Gamma_i} (\mathbf{n}\times A^{-1}\mathbf{z})\cdot\boldsymbol{\psi} = 0, \end{aligned}$$

for $i \in \widehat{D}$. Thus,

$$(A^{-1}\mathbf{z},\ \mathbf{z}) = 0.$$

Since A is uniformly symmetric positive definite, it follows that $\mathbf{z} = \mathbf{0}$. ■

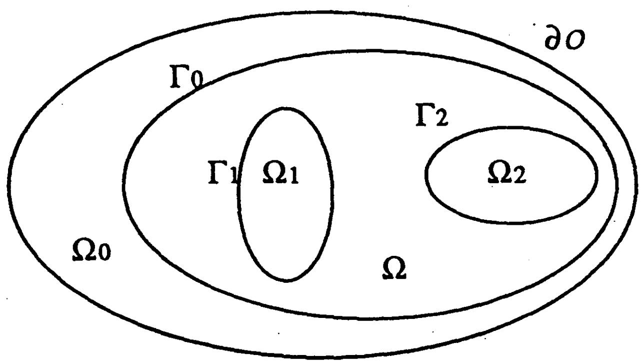

FIG. 2.

Since $\nabla^* \hat{\mathbf{w}} \in L^2(\Omega_i)$ and $\mathbf{n} \cdot \hat{\mathbf{w}}$ is continuous across each Γ_i for $i \in \widehat{D}$, then $\hat{\mathbf{w}} \in H(\text{div}; \widehat{\Omega})$ and $\nabla^* \hat{\mathbf{w}} = 0$ in $\widehat{\Omega}$. In either case 1) or 2), $\int_{\Gamma_i} \mathbf{n} \cdot \hat{\mathbf{w}} = 0$ on each piece of $\partial \widehat{\Omega}$ by assumption, so Theorem 2.4c implies that there exists a unique $\widehat{\boldsymbol{\psi}} \in H(\mathbf{curl}; \widehat{\Omega})$ such that

$$\hat{\mathbf{w}} = \nabla \times \widehat{\boldsymbol{\psi}}, \quad \nabla^* \widehat{\boldsymbol{\psi}} = 0, \quad \mathbf{n} \cdot \widehat{\boldsymbol{\psi}} = 0 \quad \text{on } \partial \widehat{\Omega}, \tag{2.60}$$

and if Γ_0 is $C^{1,1}$, then $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{\Omega})^3$, by Theorem 2.4d.

We now construct $\boldsymbol{\varphi} \in H(\mathbf{curl}; \widehat{\Omega})$ such that $\nabla^* \boldsymbol{\varphi} = 0$, $\nabla \times \boldsymbol{\varphi} = \mathbf{0}$ in $\widehat{\Omega}$, and $\mathbf{n} \times \boldsymbol{\varphi}$ matches $\mathbf{n} \times \widehat{\boldsymbol{\psi}}$ on Γ_D. Notice that $\nabla \times \widehat{\boldsymbol{\psi}} = \mathbf{0}$ in Ω_i for $i \in \widehat{D}$. By Theorem 2.5, $\widehat{\boldsymbol{\psi}} = \nabla q_i$ in Ω_i, where by (2.60) q_i satisfies

$$\begin{cases} \nabla^* \nabla q_i = 0, & \text{in } \Omega_i, \\ \mathbf{n} \cdot \nabla q_i = \mathbf{n} \cdot \widehat{\boldsymbol{\psi}}, & \text{on } \Gamma_i, \end{cases} \tag{2.61}$$

for $i \in D$, which is a compatible Neumann problem in $H^1(\Omega_i)/\Re$.

If Γ_0 is $C^{1,1}$, then $\widehat{\boldsymbol{\psi}} \in H^1(\widehat{\Omega})^3$, which implies $\widehat{\boldsymbol{\psi}} \in H^1(\Omega_i)^3$. Together with the fact that Γ_i is $C^{1,1}$, we conclude that $\mathbf{n} \cdot \widehat{\boldsymbol{\psi}} \in H^{1/2}(\Gamma_i)$. Thus, (2.61) is $H^2(\Omega_i)$ regular and

$$\|q_i\|_{2,\Omega_i} \le M_i \|\mathbf{n} \cdot \widehat{\boldsymbol{\psi}}\|_{1/2,\Gamma_i}, \tag{2.62}$$

where M_i is a constant depending only on Ω_i.

For case 2), we need to define the additional function q_0 that satisfies

$$\begin{cases} \nabla^* \nabla q_0 = 0, & \text{in } \Omega_0, \\ \mathbf{n} \cdot \nabla q_0 = \mathbf{n} \cdot \widehat{\boldsymbol{\psi}}, & \text{on } \Gamma_0 \cup \partial \mathcal{O}, \end{cases} \tag{2.63}$$

which is a compatible Neumann problem in $H^1(\Omega_0)/\Re$. If Γ_0 is $C^{1,1}$, then the same argument as above shows that(2.63) is $H^2(\Omega_0)$ regular and

$$\|q_0\|_{2,\Omega_0} \le M_0 \|\mathbf{n} \cdot \widehat{\boldsymbol{\psi}}\|_{1/2,\Gamma_0}, \tag{2.64}$$

It is easy to see that φ is divergence and curl free in Ω, i.e.,

$$\nabla^*\varphi = 0 \quad \text{and} \quad \nabla\times\varphi = \mathbf{0} \quad \text{in } \Omega. \tag{2.72}$$

Now the α_i are determined by solving the matrix equation

$$\sum_{j\in\widehat{D}} \left(\int_{\Gamma_i} \mathbf{n}\cdot\nabla p_j \right) \alpha_j = \int_{\Gamma_i} \mathbf{n}\cdot(\widehat{\psi} - \nabla q), \tag{2.73}$$

for $i \in \widehat{D}$. $\widehat{\psi} - \nabla q$ and each ∇p_j are divergence free and their normal components vanish on Γ_N, so the integrals of their normal components over the entire boundary are zero. This implies that each column of the system sums to zero, as does the right-hand side. Hence, it is a consistent but singular system, with a null space that contains constant solutions, which yield a zero of ∇. Now if we delete any column and corresponding row, setting the corresponding α_i to zero, then Lemma 2.7 with $A = I$ implies that this reduced system is nonsingular: the difference of any two functions arising from (2.70) and (2.73) satisfies the hypotheses of Lemma 2.7, so that difference is zero; this implies that any solution of the reduced system must be unique; since the system is square, it must be nonsingular.

Since the left-hand side of (2.73) depends only on Ω (see (2.69)), then there is a constant C_4 depending only on Ω such that

$$\begin{aligned} \max_{i\in\widehat{D}} |\alpha_i| &\leq C_4 \sum_{i\in\widehat{D}} \left| \int_{\Gamma_i} \mathbf{n}\cdot(\widehat{\psi} - \nabla q) \right| \leq C_4 \sum_{i\in\widehat{D}} \|\mathbf{n}\cdot(\widehat{\psi} - \nabla q)\|_{-1/2,\Gamma_i} \\ &\leq C_4 \|\widehat{\psi} - \nabla q\|_{H(\mathrm{div};\Omega)} = C_4 \|\widehat{\psi} - \nabla q\|_{0,\Omega}. \end{aligned} \tag{2.74}$$

We now define

$$\psi = \widehat{\psi} - \varphi. \tag{2.75}$$

To see that ψ satisfies (2.58), note first that (2.60) and (2.72) imply

$$\nabla^*\psi = 0 \quad \text{and} \quad \nabla\times\psi = \mathbf{w} \quad \text{in } \Omega.$$

Eqn. (2.71) implies that

$$\int_{\Gamma_i} \mathbf{n}\cdot\psi = 0$$

for $i \in \widehat{D}$. Finally, note that (2.60), (2.65), (2.68), and (2.70) yield

$$\mathbf{n}\cdot\psi = \mathbf{n}\cdot\widehat{\psi} - \mathbf{n}\cdot\varphi = 0$$

on Γ_N, while (2.70), (2.68), and (2.66) yield

$$\mathbf{n}\times\psi = \mathbf{n}\times\widehat{\psi} - \mathbf{n}\times\nabla q = \mathbf{0}$$

on Γ_D. Thus, $\mathbf{w}$ satisfies the hypothesis (2.57) of Theorem 2.6, so there exists $\boldsymbol{\psi} \in H^1(\Omega)^3$ such that

$$\mathbf{w} = \nabla\times\boldsymbol{\psi}, \quad \nabla^*\boldsymbol{\psi} = 0, \quad \mathbf{n}\cdot\boldsymbol{\psi} = 0 \text{ on } \Gamma_N,$$

$$\mathbf{n}\times\boldsymbol{\psi} = \mathbf{0} \text{ on } \Gamma_D, \quad \textstyle\int_{\Gamma_i} \mathbf{n}\cdot\boldsymbol{\psi} = 0 \text{ for } i \in D. \tag{2.80}$$

Moreover,

$$\|\boldsymbol{\psi}\|_{1,\Omega} \le C_1\|\mathbf{w}\|_{0,\Omega} = C_1\|\nabla\times A^{-1}\mathbf{v}\|_{0,\Omega}. \tag{2.81}$$

Next, consider $A^{-1}\mathbf{v} - \boldsymbol{\psi}$, which is in $H(\mathbf{curl}\,;\Omega)$. Since $\nabla\times(A^{-1}\mathbf{v} - \boldsymbol{\psi}) = \mathbf{0}$, then Theorem 2.5 implies that $A^{-1}\mathbf{v} - \boldsymbol{\psi} = \nabla p$ for some $p \in H^1(\Omega)$. We first construct p, then bound it. Let p_0 satisfy

$$\begin{cases} \nabla^* A\nabla p_0 = \nabla^*\mathbf{v} - \nabla^* A\boldsymbol{\psi}, & \text{in } \Omega, \\ p_0 = 0, & \text{on } \Gamma_D, \\ \mathbf{n}\cdot A\nabla p_0 = -\mathbf{n}\cdot A\boldsymbol{\psi}, & \text{on } \Gamma_N. \end{cases} \tag{2.82}$$

By hypothesis, problem (2.82) is $H^2(\Omega)$ regular and, using (2.81), we have

$$\begin{aligned} \|p_0\|_{2,\Omega} &\le C_2(\|\nabla^*\mathbf{v}\|_{0,\Omega} + \|\nabla^* A\boldsymbol{\psi}\|_{0,\Omega} + \|\mathbf{n}\cdot A\boldsymbol{\psi}\|_{1/2,\Gamma_N}) \\ &\le C_2\,\|\nabla^*\mathbf{v}\|_{0,\Omega} + C_3\|\boldsymbol{\psi}\|_{1,\Omega} \\ &\le C_2\,\|\nabla^*\mathbf{v}\|_{0,\Omega} + C_4\|\nabla\times A^{-1}\mathbf{v}\|_{0,\Omega}. \end{aligned} \tag{2.83}$$

As in the proof of Theorem 2.6, consider the functions p_i for $i \in D$ that satisfy

$$\begin{cases} \nabla^* A\nabla p_i = 0, & \text{in } \Omega, \\ p_i = 1, & \text{on } \Gamma_i, \\ p_i = 0, & \text{on } \Gamma_D\setminus\Gamma_i, \\ \mathbf{n}\cdot A\nabla p_i = 0, & \text{on } \Gamma_N. \end{cases} \tag{2.84}$$

Clearly, (2.84) is $H^2(\Omega)$ regular and

$$\|p_i\|_{2,\Omega} \le N_i \tag{2.85}$$

for some constants N_i depending only on Ω and A.

Now set

$$p = p_0 + \sum_{i\in D} \alpha_i p_i, \tag{2.86}$$

and choose the α_i so that

$$\int_{\Gamma_i} \mathbf{n}\cdot(\mathbf{v} - A\nabla p - A\boldsymbol{\psi}) = 0 \text{ for } i \in D. \tag{2.87}$$

for some constants $C_8 - C_{10}$ and C depending only on Ω and A. This proves the theorem for smooth Γ_0.

The proof when Γ_0 is a convex polyhedron is analogous to the proof for this case offered in Theorem 3.9, Chapter I of [15]. There, a sequence of subregions $\Omega_j \subseteq \Omega$ with $C^{1,1}$ boundaries is constructed to converge outward to Ω. Using the fact that the result holds on each Ω_j, the result is shown to hold on Ω. ■

In the remainder of this paper we will assume that the conclusion of Theorem 2.2 holds.

3. Finite Element Approximation. To discretize the least-squares variational form (2.14), let T_h be a regular triangulation of Ω with elements of size $O(h)$ satisfying the inverse assumption (see [14]). (Here and henceforth, the use of standard two-dimensional finite element terminology should be taken loosely to include the general case. Thus, for example, the respective terms *triangles* and *edges* refer to *tetrahedrons* and *surfaces* in three dimensions.) Assume we are given two finite element approximation subspaces

$$\mathbf{W}_h \subset \mathbf{W} \quad \text{and} \quad V_h \subset V$$

defined on the triangulation T_h. Then the finite element approximation to (2.14) is to find $(\mathbf{u}_h, p_h) \in \mathbf{W}_h \times V_h$ such that

$$\mathcal{F}(\mathbf{u}_h, p_h; \mathbf{v}, q) = f(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{W}_h \times V_h. \tag{3.1}$$

For simplicity, we only consider continuous piecewise linear finite element spaces, i.e.,

$$V_h = \{q \in C^0(\Omega) \,:\, q|_K \in P_1(K) \quad \forall K \in T_h,\ q \in V\}$$

and

$$\mathbf{W}_h = \{\mathbf{v} \in C^0(\Omega)^n \,:\, v_l|_K \in P_1(K) \quad \forall K \in T_h,\ \mathbf{v} \in \mathbf{W}\},$$

where $P_1(K)$ is the space of polynomials of degree at most one. Extension of the following results to higher-order finite element approximation spaces is straightforward. (See [10] for the more general case and for the proofs of both theorems of this section.)

THEOREM 3.1. *Assume that the solution,* $(\mathbf{u}, p)$, *of* (2.14) *is in* $H^{1+\alpha}(\Omega)^{n+1}$ *for some* $\alpha \in [0, 1]$, *and let* $(\mathbf{u}_h, p_h) \in \mathbf{W}_h \times V_h$ *be the solution of* (3.1). *Then*

$$\|p - p_h\|_{1,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \le C\, h^\alpha \left(\|p\|_{1+\alpha,\Omega} + \|\mathbf{u}\|_{1+\alpha,\Omega}\right), \tag{3.2}$$

where the constant C *does not depend on* h, p, *or* $\mathbf{u}$.

Proof. The proof is a direct result of Theorem 2.2, Céa's Lemma, and interpolation properties of piecewise linear functions (c.f. [14]). ■

the mesh parameter h and the number of levels J. We will also use subscripts like j in place of the more cumbersome h_j so that V_j is used in place of V_{h_j}, for example.) Without loss of generality, we may assume more specifically that

$$\gamma_j = 2, \quad j = 0, 1, ..., J-1.$$

For each $j = 0, 1, ..., J$, we associate the triangulation T_j with the continuous piecewise linear finite element space $\mathbf{W}_j \times V_j$. It is easy to verify that the family of spaces $\{\mathbf{W}_j \times V_j\}$ is nested, i.e.,

$$\mathbf{W}_0 \times V_0 \subset \mathbf{W}_1 \times V_1 \subset \cdots \subset \mathbf{W}_J \times V_J = \mathbf{W}_h \times V_h.$$

Our multigrid algorithms are described in terms of certain auxiliary operators. Fixing $j \in \{0, 1, ..., J\}$, define the level j operator $\mathcal{F}_j : \mathbf{W}_j \times V_j \longrightarrow \mathbf{W}_j \times V_j$ by

$$(\mathcal{F}_j(\mathbf{u}, p); \mathbf{v}, q) = \mathcal{F}(\mathbf{u}, p; \mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{W}_j \times V_j,$$

where the inner product $(\cdot; \cdot)$ is defined by

$$(\mathbf{u}, p; \mathbf{v}, q) = (\mathbf{u}, \mathbf{v})_{0,\Omega} + (p, q)_{0,\Omega}.$$

Also, define the respective elliptic and L^2 projection operators $\mathcal{P}_j$, $\mathcal{Q}_j : \mathbf{W}_h \times V_h \longrightarrow \mathbf{W}_j \times V_j$ by

$$\mathcal{F}(\mathcal{P}_j(\mathbf{u}, p); \mathbf{v}, q) = \mathcal{F}(\mathbf{u}, p; \mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{W}_j \times V_j$$

and

$$(\mathcal{Q}_j(\mathbf{u}, p); \mathbf{v}, q) = (\mathbf{u}, p; \mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{W}_j \times V_j.$$

It is easy to verify that

(4.1) $$\mathcal{Q}_j \mathcal{F}_J = \mathcal{F}_j \mathcal{P}_j.$$

Let $\mathcal{R}_j : \mathbf{W}_j \times V_j \longrightarrow \mathbf{W}_j \times V_j$ be the (*Richardson*) smoothing operator defined as follows: $\mathcal{R}_0 = \mathcal{F}_0^{-1}$; for $j > 0$, then $\mathcal{R}_j = \frac{1}{\lambda_j}\mathcal{I}$, where $\mathcal{I}$ is the identity operator and λ_j is the spectral radius of $\mathcal{F}_j$. Note from Remark 3.1 that $\lambda_j = O(h_j^{-2})$. We restrict ourselves to this smoother for simplicity; extension of the following to the more general case is straightforward (see [19] and [3] for more detail).

Problem (3.1) can be rewritten as

(4.2) $$\mathcal{F}_J(\mathbf{u}_J, p_J) = F_J,$$

where F_J is the right-hand side vector. The *additive* multigrid preconditioner is defined by

(4.3) $$\mathcal{G}_J = \sum_{j=0}^{J} \mathcal{R}_j \mathcal{Q}_j.$$

where the $\mathcal{F}$*-norm* is defined by

$$\|\cdot\|_{\mathcal{F}} = \sqrt{\mathcal{F}(\cdot\,;\,\cdot)}.$$

REMARK 4.1. Algorithm 4.1 *is a conventional* $V(1, 1)$*-cycle multigrid algorithm applied to problem* (4.2). *More general multigrid algorithms, including* W*-cycles, may be defined* (see [17] *and* [19]), *but we restrict ourselves to the above algorithm for concreteness and because the results are most interesting for this case. These results do immediately extend to* W*-cycles, although such extensions do not give stronger results* (see [5]). *Moreover, our optimal* V*-cycle theory can be used in the usual way* (cf. [19]) *to establish optimal* total *complexity of Full-Multigrid V-cycles (FMV). This is straightforward, but nevertheless important: while the* V*-cycle is viewed as an iterative method that obtains optimal algebraic convergence* factors, *an FMV algorithm based on such cycles is essentially a direct method that achieves overall accuracy to the level of discretization error at a total cost proportional to the number of unknowns.*

5. Convergence Analysis. In this section, we analyze the additive multigrid algorithm by estimating the condition number of $\mathcal{G}_J\mathcal{F}_J$ and the multiplicative multigrid algorithm by estimating $\|\mathcal{E}_J\|_{\mathcal{F}}$. We first provide estimates that deteriorate linearly with the number of levels in the multigrid algorithms. Then, with additional regularity, we establish optimal estimates that are uniform with respect to the number of levels. For simplicity, we base the results here on recent work cited below.

Letting

$$\begin{aligned}\mathcal{G}(\mathbf{u}, p; \mathbf{v}, q) &= (\mathbf{u}, p; \mathbf{v}, q) + (\nabla^*\mathbf{u}, \nabla^*\mathbf{v})_{0,\Omega} \\ &\quad + (\mathbf{curl}\,(A^{-1}\mathbf{u}), \mathbf{curl}\,(A^{-1}\mathbf{v}))_{0,\Omega} + (\nabla p, \nabla q)_{0,\Omega}\end{aligned}$$

and

$$(\mathbf{u}, p; \mathbf{v}, q)_{1,\Omega} = (\mathbf{u}, p; \mathbf{v}, q) + (\nabla\mathbf{u}, \nabla\mathbf{v})_{0,\Omega} + (\nabla p, \nabla q)_{0,\Omega}, \tag{5.1}$$

by Theorems 2.1 and 2.2 we know that the bilinear forms $\mathcal{F}(\cdot\,;\,\cdot)$ and $\mathcal{G}(\cdot\,;\,\cdot)$ and the inner product $(\cdot\,;\,\cdot)_{1,\Omega}$ are uniformly equivalent on $\mathbf{W}_h \times V_h$ with respect to h and J, i.e., there exist positive constants C_i ($i = 0, 1, 2, 3$) such that, for any $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$,

$$\begin{aligned}C_0(\mathbf{v}, q; \mathbf{v}, q)_{1,\Omega} &\le C_1\,\mathcal{G}(\mathbf{v}, q; \mathbf{v}, q) \le \mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) \\ &\le C_2\,\mathcal{G}(\mathbf{v}, q; \mathbf{v}, q) \le C_3(\mathbf{v}, q; \mathbf{v}, q)_{1,\Omega}.\end{aligned} \tag{5.2}$$

Standard approximation arguments imply that, for any $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$,

$$\|(\mathcal{Q}_j - \mathcal{Q}_{j-1})(\mathbf{v}, q)\|^2_{0,\Omega} \le C\,\lambda_j^{-1}\mathcal{F}(\mathbf{v}, q; \mathbf{v}, q), \quad \text{for } j = 1, 2, ..., J, \tag{5.3}$$

and

$$\mathcal{F}(\mathcal{Q}_j(\mathbf{v}, q); \mathcal{Q}_j(\mathbf{v}, q)) \le C\,\mathcal{F}(\mathbf{v}, q; \mathbf{v}, q), \quad \text{for } j = 0, 1, ..., J. \tag{5.4}$$

We next show that the operator $\mathcal{T}_j$ is "small" when applied to functions in $\mathbf{W}_i \times V_i$ with $i \leq j$. The proof is similar to that of Lemma 4.2 in [4]; see also [23].

LEMMA 5.2. *For any fixed $\beta \in [0, \frac{1}{2})$, we have*

$$\mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathbf{v}, q) \leq C \left(\frac{h_j}{h_i}\right)^{2\beta} \mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) \tag{5.8}$$

for any $(\mathbf{v}, q) \in \mathbf{W}_i \times V_i$ with $i \leq j$.

Proof. First note that (see Lemma 4.3 of [4]) for any $\beta \in [0, \frac{1}{2})$, $\varphi \in H^1(\Omega)^{n+1}$, and $\psi \in H^{1+\beta}(\Omega)^{n+1}$, we have

$$\left| \int_\Omega \frac{\partial \varphi_s}{\partial x_k} \frac{\partial \psi_t}{\partial x_l} \, dx \right| \leq C \left(\eta^{-1} \|\varphi_s\|_{0,\Omega} + \eta^{\frac{\beta}{1-\beta}} \|\varphi_s\|_{1,\Omega} \right) \|\psi_t\|_{1+\beta,\Omega}$$

for $1 \leq s, t \leq n+1$, $\quad 1 \leq k, l \leq n$, and any $\eta > 0$. (It is important to note that continuous piecewise linear functions are in $H^{1+\beta}(\Omega)$ so that $\|\psi_t\|_{1+\beta,\Omega} < \infty$; see [6].) Hence,

$$|\mathcal{F}(\varphi; \psi)| \leq C \left(\eta^{-1} \|\varphi\|_{0,\Omega}^2 + \eta^{\frac{\beta}{1-\beta}} \|\varphi\|_{1,\Omega}^2 \right)^{\frac{1}{2}} \|\psi\|_{1+\beta,\Omega}. \tag{5.9}$$

Analogous to the proof of Lemma 4.2 in [4], we use the vector inverse inequalities

$$\begin{aligned} \|\varphi\|_{1,\Omega} &\leq C h_j^{-1} \|\varphi\|_{0,\Omega}, \quad \forall \varphi \in \mathbf{W}_j \times V_j, \\ \|\psi\|_{1+\beta,\Omega} &\leq C h_i^{-\beta} \|\psi\|_{1,\Omega}, \quad \forall \psi \in \mathbf{W}_i \times V_i, \end{aligned} \tag{5.10}$$

for any $\beta \in [0, \frac{1}{2})$, which follow immediately from the inverse inequalities for scalar functions; see [6]. From the definitions of $\mathcal{F}_j$ and $\mathcal{T}_j$, we have

$$\mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathbf{v}, q) = \frac{1}{\lambda_j} \mathcal{F}(\mathcal{F}_j(\mathbf{v}, q); \mathbf{v}, q) = \frac{1}{\lambda_j} (\mathcal{F}_j(\mathbf{v}, q); \mathcal{F}_j(\mathbf{v}, q)). \tag{5.11}$$

But (5.9) and (5.10) imply

$$\begin{aligned} \|\mathcal{F}_j(\mathbf{v}, q)\|_{0,\Omega} &= \sup_{(\mathbf{w},p) \in \mathbf{W}_j \times V_j} \frac{|((\mathbf{w}, p); \mathcal{F}_j(\mathbf{v}, q))|}{\|(\mathbf{w}, p)\|_{0,\Omega}} = \sup_{(\mathbf{w},p) \in \mathbf{W}_j \times V_j} \frac{|\mathcal{F}((\mathbf{w}, p); (\mathbf{v}, q))|}{\|(\mathbf{w}, p)\|_{0,\Omega}} \\ &\leq C(\eta^{-1} + C \eta^{\frac{\beta}{1-\beta}} h_j^{-2})^{1/2} h_i^{-\beta} \|(\mathbf{v}, q)\|_{1,\Omega}. \end{aligned} \tag{5.12}$$

Setting $\eta = h_j^{2-2\beta}$ and using (5.12) in (5.11) yields (5.8). ∎

LEMMA 5.3. *For any $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$,*

$$\mathcal{F}(\mathcal{T}_0(\mathbf{v}, q); \mathcal{T}_0(\mathbf{v}, q)) = \mathcal{F}(\mathcal{T}_0(\mathbf{v}, q); \mathbf{v}, q) \tag{5.13}$$

and

$$\mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathcal{T}_j(\mathbf{v}, q)) \leq \mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathbf{v}, q), \quad \textit{for } j = 1, 2, \ldots, J. \tag{5.14}$$

Proof. To prove inequality (5.16), we first establish

$$\sum_{j=0}^{J}(T_j v,\, v) \le \max\{1, \omega\tilde{C}_0^2\}|\tilde{\mathcal{E}}|^2 \sum_{j=0}^{J}(T_j E_{j-1} v,\, E_{j-1} v), \quad \forall\, v \in \mathcal{H} \tag{5.17}$$

with $E_{j-1} = (I - T_{j-1})(I - T_{j-2}) \cdots (I - T_0)$ and $E_{-1} = I$, by an argument similar to that for Lemma 2.2 in [8]. For any $j \in \{0, 1, ..., J\}$ and $v \in \mathcal{H}$, let $b_j = (T_j E_{j-1} v,\, E_{j-1} v)^{\frac{1}{2}} \ge 0$. Note that

$$E_{j-1} = E_{j-2} - T_{j-1} E_{j-2}, \tag{5.18}$$

which leads to the identity

$$I = E_{j-1} + \sum_{i=0}^{j-1} T_i E_{i-1}. \tag{5.19}$$

Using this identity, the Cauchy-Schwarz inequality in the inner product $(T_j \cdot,\, \cdot)$, and our assumptions, we then have

$$\begin{aligned}
(T_j v,\, v) &= (T_j v,\, E_{j-1} v) + \sum_{i=0}^{j-1}(T_j v,\, T_i E_{i-1} v) \\
&\le (T_j v,\, v)^{\frac{1}{2}} \left((T_j E_{j-1} v,\, E_{j-1} v)^{\frac{1}{2}} + \sum_{i=0}^{j-1}(T_j T_i E_{i-1} v,\, T_i E_{i-1} v)^{\frac{1}{2}} \right) \\
&\le (T_j v,\, v)^{\frac{1}{2}} \left(b_j + \sum_{i=0}^{j-1} \tilde{C}_0 \varepsilon^{j-i} (T_i E_{i-1} v,\, T_i E_{i-1} v)^{\frac{1}{2}} \right) \\
&\le (T_j v,\, v)^{\frac{1}{2}} \left(b_j + \tilde{C}_0 \omega^{\frac{1}{2}} \sum_{i=0}^{j-1} \varepsilon^{j-i} b_i \right) \\
&\le (T_j v,\, v)^{\frac{1}{2}} \max\{1,\, \tilde{C}_0 \omega^{\frac{1}{2}}\} \sum_{i=0}^{j} \varepsilon^{j-i} b_i.
\end{aligned}$$

Eqn. (5.17) now follows from canceling $(T_j v,\, v)^{\frac{1}{2}}$ from both sides of the above inequality, squaring, and summing, so that

$$\begin{aligned}
\sum_{j=0}^{J}(T_j v,\, v) &\le \max\{1,\, \omega\tilde{C}_0^2\} \sum_{j=0}^{J} \left(\sum_{i=0}^{j} \varepsilon^{j-i} b_i \right)^2 \\
&\le \max\{1,\, \omega\tilde{C}_0^2\} |\tilde{\mathcal{E}}|^2 \sum_{j=0}^{J} b_j^2.
\end{aligned}$$

We next establish the inequality

$$\sum_{j=0}^{J}(T_j E_{j-1} v,\, E_{j-1} v) \le \frac{1}{2 - \omega} \left(\|v\|^2 - \|Ev\|^2 \right), \tag{5.20}$$

THEOREM 5.4. *(Additive Multigrid Algorithm) Assume that* (5.23) *holds. Then, for any* $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$,

$$C_1\mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) \le \sum_{j=0}^{J} \mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathbf{v}, q) \le C_2\mathcal{F}(\mathbf{v}, q; \mathbf{v}, q), \tag{5.24}$$

where C_1 *and* C_2 *are independent of* J.

Proof. The upper bound follows from Theorem 5.1. To prove the lower bound, consider (5.22) with $(\mathbf{v}, q)$ replaced by $(\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q)$ and let $(\mathbf{u}_j, p_j)$ be the best approximation to $(\mathbf{u}, p)$ from $\mathbf{W}_j \times V_j$ in the $H^1(\Omega)^{n+1}$ norm. Then (2.22), Theorem 3.1, and (5.23) imply that

$$\begin{aligned}(\mathbf{f}; (\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q)) &= \mathcal{F}(\mathbf{u}, p; (\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q)) = \mathcal{F}((\mathbf{u}, p) - (\mathbf{u}_j, p_j); (\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q)) \\ &\le C\,\|(\mathbf{u}, p) - (\mathbf{u}_j, p_j)\|_{1,\Omega}\|(\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q)\|_{1,\Omega} \\ &\le C\,h_j^{\alpha}\|(\mathbf{u}, p)\|_{1+\alpha,\Omega}\mathcal{F}^{\frac{1}{2}}((\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q); (\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q)) \\ &\le C\,h_j^{\alpha}\|\mathbf{f}\|_{\alpha-1,\Omega}\mathcal{F}^{\frac{1}{2}}((\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q); \mathbf{v}, q),\end{aligned}$$

which implies, for any $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$, that

$$\|(\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q)\|^2_{1-\alpha,\Omega} \le C\,\lambda_j^{-\alpha}\mathcal{F}((\mathcal{I}-\mathcal{P}_j)(\mathbf{v}, q); \mathbf{v}, q). \tag{5.25}$$

By virtue of (5.3), (5.4), and (5.25), a proof similar to that for Lemma 4.1 in [4] shows that

$$\mathcal{F}(\mathcal{Q}_0(\mathbf{v}, q); \mathcal{Q}_0(\mathbf{v}, q)) + \sum_{j=1}^{J}\lambda_j\|(\mathcal{Q}_j - \mathcal{Q}_{j-1})(\mathbf{v}, q)\|^2_{0,\Omega} \le C\,\mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) \tag{5.26}$$

for any $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$. Eqn. (5.24) now follows from an argument analogous to that for Lemma 5.1: the first inequality in (5.6), together with (5.7), yields

$$\begin{aligned}\mathcal{F}(\mathbf{v}, q; \mathbf{v}, q) &\le \|\mathcal{T}_0(\mathbf{v}, q)\|_{\mathcal{F}}\,\|\mathcal{Q}_0(\mathbf{v}, q)\|_{\mathcal{F}} + \sum_{j=1}^{J}\lambda_j\|\mathcal{T}_j(\mathbf{v}, q)\|_{0,\Omega}\|(\mathcal{Q}_j - \mathcal{Q}_{j-1})(\mathbf{v}, q)\|_{0,\Omega} \\ &\le C\left(\sum_{j=0}^{J}\mathcal{F}(\mathcal{T}_j(\mathbf{v}, q); \mathbf{v}, q)\right)^{\frac{1}{2}} \cdot \\ &\qquad \left(\mathcal{F}(\mathcal{Q}_0(\mathbf{v}, q); \mathcal{Q}_0(\mathbf{v}, q)) + \sum_{j=1}^{J}\lambda_j\|(\mathcal{Q}_j - \mathcal{Q}_{j-1})(\mathbf{v}, q)\|^2_{0,\Omega}\right)^{\frac{1}{2}};\end{aligned}$$

which with (5.26), dividing by $\mathcal{F}^{\frac{1}{2}}(\mathbf{v}, q; \mathbf{v}, q)$, and squaring yields the result. ∎

Finally, Theorems 5.2 and 5.4 imply

THEOREM 5.5. *(Multiplicative Multigrid Algorithm) Assume that* (5.23) *holds. Then, for any* $(\mathbf{v}, q) \in \mathbf{W}_h \times V_h$,

$$\mathcal{F}(\mathcal{E}_J(\mathbf{v}, q); \mathcal{E}_J(\mathbf{v}, q)) \le \gamma\,\mathcal{F}(\mathbf{v}, q; \mathbf{v}, q), \tag{5.27}$$

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