The Fundamental Theorem of q-Clan Geometry

S. E. Payne

1 Introduction

Let q be any prime power, $F = GF(q)$. A *q-clan* is a set $\mathcal{C} = \{A_t : t \in F\}$ of q, 2×2 matrices over F such that their pairwise differences are all anisotropic, i.e., for distinct $s, t \in F$, $(a,b)(A_s - A_t)\begin{pmatrix} a \\ b \end{pmatrix} = 0$ has only the trivial solution $a = b = 0$. Starting with a q-clan $\mathcal{C}$, there are at least the following geometries associated with $\mathcal{C}$ in a canonical way (cf. [18]): a generalized quadrangle $GQ(\mathcal{C})$ with parameters (q^2, q); a flock $\mathcal{F}(\mathcal{C})$ of a quadratic cone in $PG(3,q)$; a line spread $S(\mathcal{C})$ of $PG(3,q)$; a translation plane $T(\mathcal{C})$ of dimension at most 2 over its kernel. Starting with a natural definition of equivalence for q-clans, the Fundamental Theorem of q-clan geometry (F.T.) interprets the equivalence of q-clans $\mathcal{C}_1$ and $\mathcal{C}_2$ as an isomorphism between $G(\mathcal{C}_1)$ and $G(\mathcal{C}_2)$, where $G(\mathcal{C}_i)$ is any of the geometries (mentioned above) associated with $\mathcal{C}_i$. The F.T. was first recognized in its present form in [1], but it was stated there in detail only for $q = 2^e$, and the proof was claimed to be only a slightly revised version of the proof given in [14] of an important special case of the F.T. However, the proof in [14] starts off by assuming a fairly technical result from [11] where it is embedded in a more general theory.

Consequently we feel that the F.T. deserves its own self-contained treatment. What we offer here is a detailed proof that q-clans $\mathcal{C}_1$ and $\mathcal{C}_2$ are equivalent if and only if the generalized quadrangles $GQ(\mathcal{C}_1)$ and $GQ(\mathcal{C}_2)$ are isomorphic in a canonical way. Using a result from J. W. P. Hirschfeld [5] this is interpreted as projective equivalence of the flocks $\mathcal{F}(\mathcal{C}_1)$ and $\mathcal{F}(\mathcal{C}_2)$. The reader is then directed to H. Gevaert and N. Johnson [4] for the connection with spreads and translation planes.

The F.T. occupies sections two, three, and four. In section five we generalize the material in [1] dealing with shifts, flips and scales to all prime powers q and give a complete proof that two lines through the point (∞) of $GQ(\mathcal{C})$ are in the same orbit of the collineation group of $GQ(\mathcal{C})$ if and only if the flocks assigned to them are projectively equivalent. This latter result was already observed in [14] using results from [11]. We feel the present treatment is a much more direct and satisfying approach to the algebraic version of derivation of flocks introduced in [2] for q odd. In the sixth section these ideas are applied to the Cohen-Ganey-Barriga flock to fill in a gap in [12].

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2 The q-clan geometries

As above, q is any prime power and $F = GF(q)$. If $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ and $B = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$ are any 2×2 matrices over F, we write $A \equiv B$ to mean that $x = r$, $w = u$, and $y+z = s+t$. So $A \equiv B$ if and only if $\alpha A \alpha^T = \alpha B \alpha^T$ for all $\alpha \in F^2$. If $\mathcal{C} = \{A_t : t \in F\}$ is a q-clan, the matrices of $\mathcal{C}$ are used to define quadratic forms, so an $A_t \in \mathcal{C}$ may be replaced by A_t' whenever $A_t \equiv A_t'$ without effectively changing $\mathcal{C}$. If q is odd we adjust each $A_t \in \mathcal{C}$ to be symmetric, say $A_t = \begin{pmatrix} x_t & y_t/2 \\ y_t/2 & z_t \end{pmatrix}$, $t \in F$. If $q = 2^e$, we adjust each $A_t \in \mathcal{C}$ to be upper triangular, say $A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$. So for both q even and q odd we also have $A_t \equiv \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$, where $X : t \mapsto x_t$, $Y : t \mapsto y_t$ and $Z : t \mapsto z_t$ are three functions from F to F. It is easy to check that if $\mathcal{C} = \{A_t \equiv \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ is a q-clan, then X and Z are always one-to-one, and for $q = 2^e$, Y is also one-to-one. Moreover, if $0 \neq u \in F$, $\sigma \in Aut(F)$, $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,q)$, and if x, y, and z are any fixed elements of F, replacing each $A_t \in \mathcal{C}$ with $A_t' \equiv \mu B A_t^\sigma B^T + \begin{pmatrix} x & y \\ 0 & z \end{pmatrix}$ gives a new q-clan $\mathcal{C}' = \{A_t' : t \in F\}$. This suggests the following definition of equivalence for q-clans.

Let $\mathcal{C} = \{A_t : t \in F\}$ and $\mathcal{C}' = \{A_t' : t \in F\}$ be q-clans. We say $\mathcal{C}$ and $\mathcal{C}'$ are *equivalent* and write $\mathcal{C} \sim \mathcal{C}'$ provided there exist the following: $0 \neq \mu \in F$, $B \in GL(2,q)$, $\sigma \in \mathrm{Aut}(F)$, M a 2×2 matrix over F, and a permutation $\pi : t \mapsto \bar{t}$ on F such that the following holds.

$$A_{\bar{t}}' \equiv \mu B A_t^\sigma B^T + M, \text{ for all } t \in F. \tag{1}$$

This will turn out to be exactly the correct definition of equivalence for the F.T. And if we put $\mu = 1$, $B = I$, $\sigma = id$, $M = -A_0$, so $A_t' \equiv A_t - A_0$, then $\mathcal{C}'$ is a q-clan equivalent to $\mathcal{C}$ and having $A_0' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence from now on, *without loss of generality we assume that each q-clan contains the zero matrix.* For one of the known q-clans with q odd, this is inconvenient, but see S. E. Payne and J. A. Thas [17] for the appropriate adjustments in that case. Also see pp. 213–214 of [16].

Let $K = \{(x_0, x_1, x_2, x_3) \in PG(3,q) : x_1^2 = x_0 x_2\}$. So K is a cone with vertex $V = (0,0,0,1)$. Given a q-clan $\mathcal{C}$, for each $A_t \in \mathcal{C}$ define the plane $\pi_t = [x_t, y_t, z_t, 1]^T$ not containing the vertex V, and put $C_t = \pi_t \cap K$. Then $\mathcal{F}(\mathcal{C}) = \{C_t : t \in F\}$ is a *flock* of K, i.e., a partition of $K \backslash \{V\}$ into disjoint conics, precisely because $\mathcal{C}$ is a q-clan (cf. J. A. Thas [18]). For $t \in F$, π_t is referred to as a *plane of the flock.* The following is an immediate consequence of Corollary 7, p. 143, of J. W. P. Hirschfeld [5].

Lemma 1 *Let* $\mathcal{C} = \{A_t \equiv \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ *and* $\mathcal{C}' = \{A'_t \equiv \begin{pmatrix} x'_t & y'_t \\ 0 & z'_t \end{pmatrix} : t \in F\}$ *be two (not necessarily distinct) q-clans (with* $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A'_0$*) with corresponding flocks* $\mathcal{F}(\mathcal{C})$ *and* $\mathcal{F}(\mathcal{C}')$*. Then* $\mathcal{F}(\mathcal{C})$ *and* $\mathcal{F}(\mathcal{C}')$ *are projectively equivalent if and only if there exist the following:*

(i) $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,q)$

(ii) $0 \neq \lambda \in F$

(iii) $\sigma \in \ Aut(F)$

(iv) $\pi : F \to F : t \mapsto \bar{t}$*, a permutation, such that the following condition is satisfied:*

(2)
$$\begin{bmatrix} x'_{\bar{t}} \\ y'_{\bar{t}} \\ z'_{\bar{t}} \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda a^2 & \lambda ab & \lambda b^2 & x'_{\bar{0}} \\ 2\lambda ac & \lambda(ad+bc) & 2\lambda bd & y'_{\bar{0}} \\ \lambda c^2 & \lambda cd & \lambda d^2 & z'_{\bar{0}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t^\sigma \\ y_t^\sigma \\ z_t^\sigma \\ 1 \end{bmatrix}.$$

For q odd, so the matrices of $\mathcal{C}$ and $\mathcal{C}'$ are symmetric, Eq. (2) is equivalent to

(3)
$$A'_{\bar{t}} = \lambda B A_t^\sigma B^T + A'_{\bar{0}}.$$

For $q = 2^e$, so the matrices of $\mathcal{C}$ and $\mathcal{C}'$ are upper triangular, Eq. (2) is equivalent to

(4)
$$A'_{\bar{t}} \equiv \lambda B A_t^\sigma B^T + A'_{\bar{0}}.$$ (Actually this holds for all q.) ∎

Clearly Lemma 1 is just a very explicit way of saying that $\mathcal{C} \sim \mathcal{C}'$ if and only if $\mathcal{F}(\mathcal{C})$ and $\mathcal{F}(\mathcal{C}')$ are projectively equivalent!

Let $G = \{(\alpha, c, \beta) \in F^2 \times F \times F^2 : \alpha, \beta \in F^2, \ c \in F\}$. Define a binary operation on G by

(5)
$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta').$$

This makes G into a group of order q^5 with $(\alpha, c, \beta)^{-1} = (-\alpha, \alpha\beta^T - c, -\beta)$.

Starting with a q-clan $\mathcal{C}$, put $K_t = A_t + A_t^T$ for all $A_t \in \mathcal{C}$. Then we may define the following subgroups of G.

$$A(\infty) = \{(\overline{0}, 0, \beta) \in G : \beta \in F^2\}, \ A^*(\infty) = \{(\overline{0}, c, \beta) \in G : c \in F, \beta \in F^2\}$$

$$A(t) = \{(\alpha, \alpha A_t \alpha^T, \alpha K_t) \in G : \alpha \in F^2\}, \ A^*(t) = \{(\alpha, c, \alpha K_t) \in G : \alpha \in F^2, c \in F\},$$

for $t \in F$. Then $\mathcal{J}(\mathcal{C}) = \{A(t) : t \in \tilde{F} = F \cup \{\infty\}\}$ is a 4-gonal family for G precisely because $\mathcal{C}$ is a q-clan (cf. [8], [10]). This means that

K1. $A(s)A(t) \cap A(u) = \{(\overline{0}, 0, \overline{0})\}$ for distinct $s, t, u \in \tilde{F}$.

K2. $A^*(s) \cap A(t) = \{(\overline{0}, 0, \overline{0})\}$ for distinct $s, t \in \tilde{F}$.

Using the 4-gonal family $\mathcal{J}(\mathcal{C})$ for G there is a standard construction of a generalized quadrangle $GQ(\mathcal{C})$ of order (q^2, q) (cf. [7], [16]).

The points of $GQ(\mathcal{C})$ are of three types:

(i) The elements $g = (\alpha, c, \beta)$ of G,

(ii) Cosets $A^*(t)g$, $t \in \tilde{F}$, $g \in G$,

(iii) The symbol (∞).

The lines of $GQ(\mathcal{C})$ are of two types:

(a) Cosets $A(t)g$, $t \in \tilde{F}$, $g \in G$,

(b) Symbols $[A(t)]$, $t \in \tilde{F}$.

Incidence in $GQ(\mathcal{C})$ is defined by: The point (∞) is on the $q+1$ lines $[A(t)]$ of type (b). The point $A^*(t)g$ is on the line $[A(t)]$ and on the q lines of type (a) contained in $A^*(t)g$. The point g of type (i) is on the $q+1$ lines $A(t)g$ of type (a) that contain it. There are no other incidences.

The resulting point-line geometry $GQ(\mathcal{C})$ is a GQ of order (q^2, q) precisely because $\mathcal{C}$ is a q-clan (cf. [8], [10], [16]). All the GQ considered in this work will be *nonclassical* and derived from a q-clan (so the corresponding flock is nonlinear, cf. [18]). Hence the point (∞) is the unique point fixed by all collineations of $GQ(\mathcal{C})$ (cf. [15]). Moreover, right multiplication by elements of G induces a group of collineations of $GQ(\mathcal{C})$ acting regularly on those points of $GQ(\mathcal{C})$ not collinear with (∞), and fixing each line through (∞). Hence to determine the full collineation group $\mathcal{G}$ of $GQ(\mathcal{C})$ it suffices to determine the subgroup $\mathcal{G}_0$ fixing $(\overline{0}, 0, \overline{0})$ (and, of course, fixing (∞)).

For later computations it will be convenient to note collinearities in $GQ(\mathcal{C})$. For $t \in \tilde{F}$, $p \overset{t}{\sim} r$ means that points p and r lie on a line that is a coset of $A(t)$, with the appropriate adjustment if $r = A^*(t)g$.

(6)
(i) $(\alpha, c, \beta) \overset{\infty}{\sim} (\alpha, c + \gamma\alpha^T, \beta + \gamma)$
(ii) $(\alpha, c, \beta) \overset{\infty}{\sim} A^*(\infty)(\alpha, 0, \overline{0})$
(iii) $(\alpha, c, \beta) \overset{t}{\sim} (\alpha + \gamma, c + \gamma A_t \gamma^T + \gamma K_t \alpha^T, \beta + \gamma K_t)$, $t \in F$, $\gamma \in F^2$.
(iv) $(\alpha, c, \beta) \overset{t}{\sim} A^*(t)(\overline{0}, 0, \beta - \alpha K_t)$, $t \in F$.

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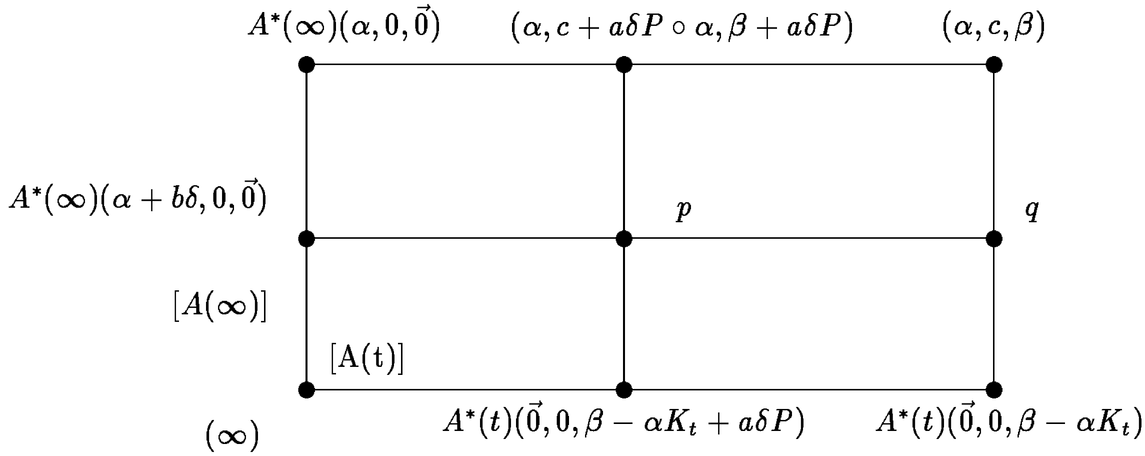
3 Property (G) and some projective planes

From now on $\mathcal{C} = \{A_t : t \in F\}$ will be a fixed q-clan normalized so that $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and each A_t is symmetric or upper triangular according as q is odd or even. $GQ(\mathcal{C})$ is the GQ of order (q^2, q) constructed using $\mathcal{C}$ as in section 2. The notion of Property (G) was introduced in [11], where it was shown that $GQ(\mathcal{C})$ has Property (G) at (∞). This means that there are many $(q+1)\times(q+1)$ grids containing the point (∞). The proof in [14] assumed this result and more from [11]. But here we want to give a self-contained, explicit treatment, so these grids are exhibited in Figs. 1 and 2, where a and b vary independently over the elements of F, and $\delta \neq (0,0)$ is arbitrary but fixed in F^2. Also from now on $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\alpha \circ \beta = \alpha\beta^T$, so $\alpha P \alpha^T = \alpha \circ \alpha P^T = \alpha P \circ \alpha = 0$.



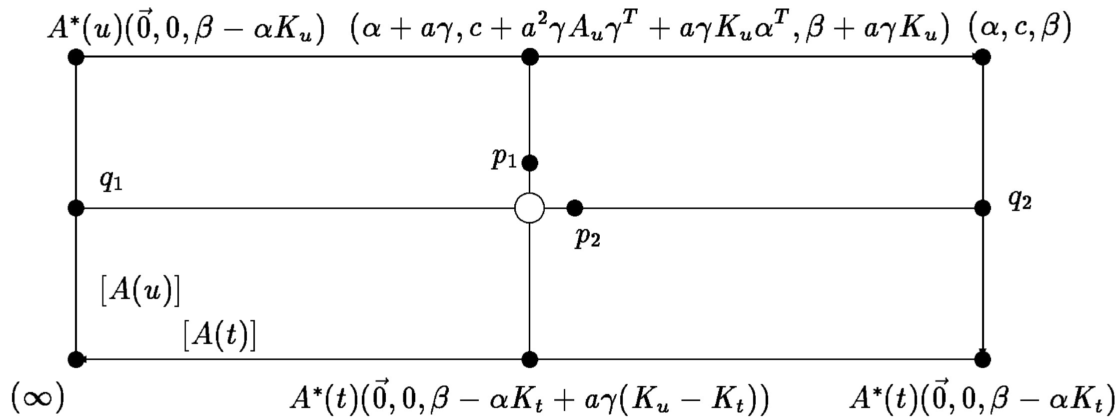
$$p = (\alpha + b\delta, c + \delta(aP + bK_t)\alpha^T + b^2\delta A_t \delta^T, \beta + \delta(aP + bK_t))$$

$$q = (\alpha + b\delta, c + b^2\delta A_t\delta^T + b\delta K_t\alpha^T, b\delta K_t + \beta)$$

Figure 1: Grid containing $[A(\infty)]$ and $[A(t)]$

The grid lines of Fig. 1 that meet $[A(\infty)]$ at points other than (∞) meet $[A(\infty)]$ at points of the form $A^*(\infty)(\alpha + b\delta, 0, \overline{0})$, $b \in F$. This set of points is clearly independent of t. Moreover, if we coordinatize the point $A^*(\infty)(\alpha, 0, \overline{0})$ with $(\alpha, 1) \in PG(2, q)$, then the set of points of the form $A^*(\infty)(\alpha + b\delta, 0, \overline{0})$ is coordinatized by the line $[\delta P, -\alpha \circ \delta P]^T$ in $PG(2, q)$. So the points of $[A^*(\infty)]$ other than (∞) are the affine points of a projective plane $\pi(\infty)$ isomorphic to $PG(2, q)$ with infinite line $[0, 0, 1]^T$, and whose finite lines are the sets of points meeting the lines of a grid containing (∞).

Similarly, the grid lines meeting $[A(t)]$ at points $\neq (\infty)$ meet it at the points $A^*(t)(\overline{0}, 0, \beta - \alpha K_t + a\delta P)$, $a \in F$. And the points of $[A(t)]$ different from (∞) are the affine points of a plane $\pi(t)$ isomorphic to $PG(2,q)$. Coordinatize $A^*(t)(\overline{0},0,\beta)$ as $(\beta,1) \in PG(2,q)$. Then the affine line of points of the form $A^*(t)(\overline{0},0,\beta-\alpha K_t+a\delta P)$, $a \in F$, is coordinatized by the line $[\delta,(\alpha K_t-\beta)\circ\delta]^T$, and the infinite line is $[0,0,1]^T$. We claim that all grids containing (∞) and meeting $[A(t)]$ determine the same affine lines, and hence exactly the same projective plane $\pi(t)$. This follows from Fig. 2, which shows that all affine lines of $\pi(t)$ have the form $[\delta,(\alpha K_t-\beta)\circ\delta]^T$ for some $\delta \neq (0,0)$, $\delta \in F^2$. This is independent of $u \in \tilde{F}\backslash\{t\}$, so a unique plane $\pi(t)$ is constructed.



$$q_1 = A^*(u)(\vec{0},0,\beta-\alpha K_u+b\gamma(K_t-K_u))$$

$$q_2 = (\alpha+b\delta, c+b^2\delta A_t\delta^T+b\delta K_t\alpha^T, \beta+b\delta K_t)$$

Figure 2: Grid containing $[A(u)]$ and $[A(t)]$

The points $p_1=(\alpha+a\gamma+b\delta, c+a^2\gamma A_u\gamma^T+a\gamma K_u\alpha^T+b^2\delta A_t\delta^T+b\delta K_t(\alpha+a\gamma)^T, \beta+a\gamma K_u+b\delta K_t)$ and $p_2=(\alpha+b\delta+a\gamma, c+b^2\delta A_t\delta^T+b\delta K_t\alpha^T+a^2\gamma A_n\gamma^T+a\gamma K_u(\alpha+b\delta)^T, \beta+b\delta K_t+a\gamma K_u)$ are the same point if and only if $\gamma(K_t-K_u)\delta^T=0$. So if δ is fixed with $(0,0)\neq\delta\in F^2$, put $\gamma=\delta(K_t-K_u)P$ to obtain a $(q+1)\times(q+1)$ grid. And the set of points on $[A(t)]$ different from (∞) and on the grid lines is the set of points of the form $A^*(t)(\overline{0},0,\beta-\alpha K_t+a\gamma(K_u-K_t))$, which is coordinatized by the line $[\gamma(K_u-K_t)P,(\alpha K_t-\beta)\circ(\gamma(K_u-K_t)P)]^T$. Putting $\delta=\gamma(K_u-K_t)P$ here shows that lines have the general form claimed just preceding Fig. 2.

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4 The fundamental theorem

Let $\mathcal{C}=\{A_t : t\in F\}$ and $\mathcal{C}'$ be (not necessarily distinct) q-clans normalized as indicated at the beginning of section 3. Let $\theta : GQ(\mathcal{C})\to GQ(\mathcal{C}')$ be an isomorphism for which $\theta : (\overline{0},0,\overline{0}) \mapsto (\overline{0},0,\overline{0})$, $[A(\infty)] \mapsto [A'(\infty)]$, $(\infty)\mapsto(\infty)$. Our goal in this section is to determine as precisely as possible the exact form of θ. Clearly θ is completely determined as a permutation of the elements of G. And there must be a permutation $\pi : t \mapsto \bar{t}$ on F for which $\theta : [A(t)] \mapsto [A(\bar{t})]$. Moreover, θ must map grids to grids. Therefore θ must map $\pi(\infty)$ to $\pi'(\infty)$ and $\pi(t)$ to $\pi'(\bar{t})$, always mapping infinite points to infinite points and infinite lines to infinite lines. This means that there is a $B \in GL(2,q)$ and a $\sigma \in Aut(F)$, and for each $t \in F$ a $D_t \in GL(2,q)$ and a $\sigma_t \in Aut(F)$, for which

$$(7) \qquad \begin{array}{ll} \text{(i)} & A^*(\infty)(\alpha, 0, \overline{0}) \overset{\theta}{\mapsto} (A')^*(\infty)(\alpha^\sigma B, 0, \overline{0}) \\ \text{(ii)} & A^*(t)(\overline{0}, 0, \beta) \overset{\theta}{\mapsto} (A')^*(\bar{t})(\overline{0}, 0, \beta^{\sigma_t} B_t) \end{array}$$

Since $(\alpha, c, \beta) \overset{\infty}{\sim} A^*(\infty)(\alpha, 0, \overline{0})$, clearly $(\alpha, c, \beta)^\theta \overset{\infty}{\sim} (A')^*(\infty)(\alpha^\sigma B, 0, \overline{0})$. This implies that as a permutation of the elements of G, θ must have the form

$$(8) \qquad \theta : (\alpha, c, \beta) \mapsto (\alpha^\sigma B, (\alpha, c, \beta)^{\theta_2}, (\alpha, c, \beta)^{\theta_3})$$

for some functions $\theta_2 : F^5 \to F$ and $\theta_3 : F^5 \to F^2$ that we proceed to determine.

Since $(\alpha, c, \beta) \overset{t}{\sim} A^*(t)(\overline{0}, 0, \beta - \alpha K_t)$ for each $t \in F$, $(\alpha^\sigma B, (\alpha, c, \beta)^{\theta_2}, (\alpha, c, \beta)^{\theta_3}) \overset{\bar{t}}{\sim} (A')^*(\bar{t})(\overline{0}, 0, (\beta - \alpha K_t)^{\sigma_t} D_t)$. This implies $(\alpha, c, \beta)^{\theta_3} - \alpha^\sigma B K'_{\bar{t}} = (\beta - \alpha K_t)^{\sigma_t} D_t$, or

$$(9) \qquad (\alpha, c, \beta)^{\theta_3} = (\beta - \alpha K_t)^{\sigma_t} D_t + \alpha^\sigma B K'_{\bar{t}} \ \ \text{(independent of } c\text{)}.$$

With $\alpha = \overline{0}$, $(\overline{0}, c, \beta)^{\theta_3} = \beta^{\sigma_t} D_t = \beta^{\sigma_0} D_0$ for all $t \in F$. Put $\beta = (1, 0)$; then $\beta = (0, 1)$, to force $D_t = D_0 = D$ for all $t \in F$. And D is invertible, so $\sigma_t = \sigma_0$ for all $t \in F$.

So now Eq. (9) implies

$$(10) \qquad (\alpha, c, \beta)^{\theta_3} = (\beta - \alpha K_t)^{\sigma_0} D + \alpha^\sigma B K'_{\bar{t}} \ \ (\forall t \in F; \forall \alpha, \beta \in F^2).$$

Then for distinct $s, t \in F$, Eq. (10) implies $(\beta - \alpha K_t)^{\sigma_0} D + \alpha^\sigma B K'_{\bar{t}} = (\beta - \alpha K_s)^{\sigma_0} D + \alpha^\sigma B K'_{\bar{s}}$, which may be rewritten as

$$(11) \qquad [\alpha(K_s - K_t)]^{\sigma_0} D = \alpha^\sigma B (K'_{\bar{s}} - K'_{\bar{t}}).$$

Put $\alpha = (1, 0)$; then $\alpha = (0, 1)$ to obtain

$$(12) \qquad (K_s - K_t)^{\sigma_0} D = B (K'_{\bar{s}} - K'_{\bar{t}}).$$

Put Eq. (12) into Eq. (11) to get

$$(13) \qquad \alpha^{\sigma_0} (K_s - K_t)^{\sigma_0} D = \alpha^{\sigma_0} B (K'_{\bar{s}} - K'_{\bar{t}}) = \alpha^\sigma B (K'_{\bar{s}} - K'_{\bar{t}}).$$

Since $K_s - K_t$, D, B, $K'_{\bar{s}} - K'_{\bar{t}}$ are all invertible,

$$(14) \qquad \sigma_0 = \sigma \text{ and } D = (K_s - K_t)^{-\sigma} B (K'_{\bar{s}} - K'_{\bar{t}}).$$

At this point θ has the following appearance:

$$(15) \qquad \theta : (\alpha, c, \beta) \mapsto$$
$$(\alpha^\sigma B, (\alpha, c, \beta)^{\theta_2}, (\beta - \alpha K_t)^\sigma (K_s - K_t)^{-\sigma} B (K'_{\bar{s}} - K'_{\bar{t}}) + \alpha^\sigma B K'_{\bar{t}}).$$

Now consider Fig. 1 specialized so that $(\alpha, c, \beta) = (\overline{0}, 0, \overline{0})$, $\gamma = b\delta = a\delta$.

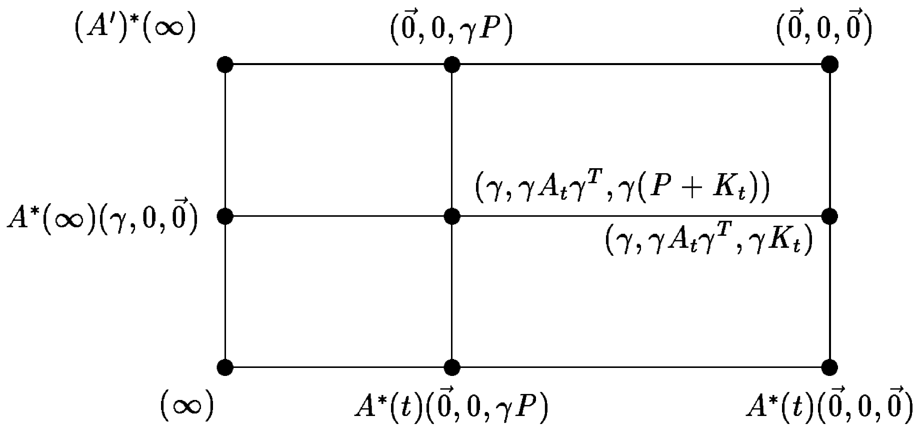

Figure 3: Special Case of Fig. 1

Finally, construct in Fig. 4 the image under θ of Fig. 3, using especially Eq. (7).

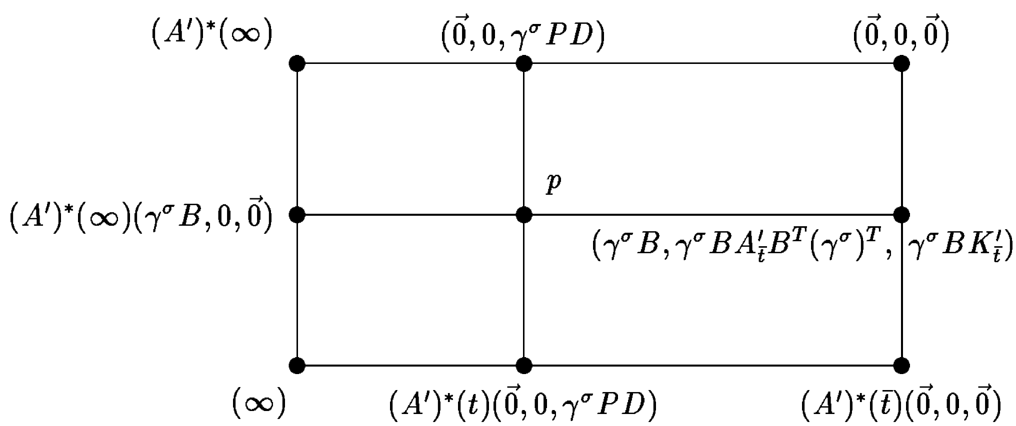

Figure 4: Image Under θ of Fig. 3

Consider the impact of the existence of the center point p. The fact that p lies on the horizontal line forces p to have the form $p=(\gamma^\sigma B,\gamma^\sigma BA'_{\bar t}B^T(\gamma^\sigma)^T+\delta\circ\gamma^\sigma B,\gamma^\sigma BK'_{\bar t}+\delta)$ for some $\delta\neq(0,0)$ (by Eq. (6(i)). Then the fact that p lies on the vertical line forces p to have the form $p=(\gamma^\sigma B,\gamma^\sigma BA'_{\bar t}B^T(\gamma^\sigma)^T,\gamma^\sigma PD+\gamma^\sigma BK'_{\bar t})$. Hence $\delta=\gamma^\sigma PD$ *and* $\gamma^\sigma PD\circ\gamma^\sigma B=0$. This is for all $\gamma\neq(0,0)$. Hence $DB^T=\mu I$ for some μ with $0\neq\mu\in F$. Set $\lambda=\mu^{-1}$, so

(16) $$B=\lambda^{-1}D^{-T}\ \text{ for some nonzero } \lambda\in F.$$

Using this in Eq. (14) we obtain

(17) $$(K_s-K_t)^\sigma=\lambda^{-1}D^{-T}(K'_{\bar s}-K'_{\bar t})D^{-1}\ \text{ for all } s,t\in F.$$

Then $s=0$ in Eq. (17) gives

(18) $$\lambda D^TK_t^\sigma D=K'_{\bar t}-K'_{\bar 0}.$$

So now Eq. (15) may be rewritten as

$$\theta : (\alpha, c, \beta) \mapsto (\lambda^{-1}\alpha^\sigma D^{-T}, (\alpha, c, \beta)^{\theta_2}, \beta^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}}). \tag{19}$$

Putting $\gamma = -\alpha$ in Eq. (6)(iii) we obtain $p_1 = (\alpha, c, \beta) \overset{t}{\sim} p_2 = (\overline{0}, c - \alpha A_t \alpha^T, \beta - \alpha K_t)$, so that $p_1^\theta \overset{\bar{t}}{\sim} p_2^\theta$, where by Eq. (19) $p_1^\theta = (\lambda^{-1}\alpha^\sigma D^{-T}, (\alpha, c, \beta)^{\theta_2}, \beta^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}})$ and $p_2^\theta = (\overline{0}, (\overline{0}, c - \alpha A_t \alpha^T, \beta - \alpha K_t)^{\theta_2}, (\beta - \alpha K_t)^\sigma D)$. Since $(\alpha, c, \beta)^{-1} = (-\alpha, \alpha \circ \beta - c, -\beta)$, $p_1^\theta \circ (p_2^\theta)^{-1} = (\lambda^{-1}\alpha^\sigma D^{-T}, (\alpha, c, \beta)^{\theta_2}, \beta^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}}) \circ (\overline{0}, -(\overline{0}, c - \alpha A_t \alpha^T, \beta - \alpha K_t)^{\theta_2}, -(\beta - \alpha K_t)^\alpha D) = (\lambda^{-1}\alpha^\sigma D^{-T}, (\alpha, c, \beta)^{\theta_2} - (\overline{0}, c - \alpha A_t \alpha^T, \beta - \alpha K_t)^{\theta_2}, \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}} + \alpha^\sigma K_t^\sigma D)$ must be in $A'(\bar{t})$. Eq. (18) implies that the third coordinate is o.k., but from the middle coordinate we have

$$(\alpha, c, \beta)^{\theta_2} = (\overline{0}, c - \alpha A_t \alpha^T, \beta - \alpha K_t)^{\theta_2} + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\bar{t}} D^{-1} (\alpha^\sigma)^T. \tag{20}$$

Put $t = 0$ to obtain

$$(\alpha, c, \beta)^{\theta_2} = (\overline{0}, c, \beta)^{\theta_2} + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1} (\alpha^\sigma)^T. \tag{21}$$

Now put $\gamma = -\beta$ in Eq. (6)(i) to obtain $p_1 = (\alpha, c, \beta) \overset{\infty}{\sim} p_2 = (\alpha, c - \beta\alpha^T, \overline{0})$, so $p_1^\theta \overset{\infty}{\approx} p_2^\theta$ where by Eq. (19) $p_1^\theta = (\lambda^{-1}\alpha^\sigma D^{-T}, (\alpha, c, \beta)^{\theta_2}, \beta^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}})$ and $p_2^\theta = (\lambda^{-1}\alpha^\sigma D^{-T}, (\alpha, c - \beta\alpha^T, \overline{0})^{\theta_2}, \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}})$. So $p_1^\theta \circ (p_2^\theta)^{-1} = (\lambda^{-1}\alpha^\sigma D^{-T}, (\alpha, c, \beta)^{\theta_2}, \beta^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}}) \circ (-\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-2}\alpha^\sigma D^{-T} K'_{\overline{0}} D^{-1}(\alpha^\sigma)^T - (\alpha, c - \beta\alpha^T, \overline{0})^{\theta_2}, -\lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}}) = (\overline{0}, (\alpha, c, \beta)^{\theta_2} - (\alpha, c - \beta\alpha^T, \overline{0})^{\theta_2} - \lambda^{-1}\beta^\sigma(\alpha^\sigma)^T, \beta^\sigma D)$ must be in $A'(\infty)$. Hence

$$(\alpha, c, \beta)^{\theta_2} = (\alpha, c - \beta\alpha^T, \overline{0})^{\theta_2} + \lambda^{-1}\alpha^\sigma(\beta^\sigma)^T. \tag{22}$$

Set $\alpha = \overline{0}$, so $(\overline{0}, c, \beta)^{\theta_2} = (\overline{0}, c, \overline{0})^{\theta_2}$, which is independent of β. Now use this in Eq. (21).

$$(\alpha, c, \beta)^{\theta_2} = (\overline{0}, c, \overline{0})^{\theta_2} + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1} (\alpha^\sigma)^T. \tag{23}$$

Then

$$\begin{aligned}
(\overline{0}, c, 0)^{\theta_2} + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1}(\alpha^\sigma)^T &\overset{(23)}{=} (\alpha, c, \beta)^{\theta_2} \\
&\overset{(22)}{=} (\alpha, c - \beta\alpha^T, \overline{0})^{\theta_2} + \lambda^{-1}\alpha^\sigma(\beta^\sigma)^T \\
&\overset{(23)}{=} (\overline{0}, c - \beta\alpha^T, \overline{0})^{\theta_2} + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1}(\alpha^\sigma)^T + \lambda^{-1}\alpha^\sigma(\beta^\sigma)^T.
\end{aligned}$$

Put $c = \alpha\beta^T$ in these last equations to obtain $(\overline{0}, \alpha\beta^T, \overline{0})^{\theta_2} = \lambda^{-1}\alpha^\sigma(\beta^\sigma)^T$, where we used the fact that $(\overline{0}, 0, \overline{0})^\theta = (\overline{0}, 0, \overline{0})$ implies $(\overline{0}, 0, \overline{0})^{\theta_2} = 0$. Hence

$$(\overline{0}, c, \overline{0})^{\theta_2} = \lambda^{-1} c^\sigma. \tag{24}$$

So Eq. (23) becomes

$$(\alpha, c, \beta)^{\theta_2} = \lambda^{-1} c^\sigma + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1}(\alpha^\sigma)^T. \tag{25}$$

Now use Eq. (25) in Eq. (19).

(26) $\theta : (\alpha, c, \beta) \mapsto$
$$(\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-1}c^\sigma + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1}(\alpha^\sigma)^T, \beta^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}}).$$

Note: We have now shown that θ is an automorphism of G.

Conversely, let $0 \neq \lambda \in F$, $\sigma \in \mathrm{Aut}(F)$, $D \in GL(2,q)$, and $\overline{0} \in F$. Then it is routine to check that θ defined by Eq. (26) is an automorphism of G which maps $A(\infty)$ to $A'(\infty)$. So θ is an isomorphism from $GQ(\mathcal{C})$ to $GQ(\mathcal{C}')$ if and only if θ maps $A(t)$ to $A(\overline{t})$ for all $t \in F$, for some permutation $\pi : t \mapsto \overline{t}$. Consider the effect of θ on $A(t)$: $\theta : (\alpha, \alpha A_t \alpha^T, \alpha K_t) \mapsto (\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-1}\alpha^\sigma A_t^\sigma (\alpha^\sigma)^T + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1}(\alpha^\sigma)^T, \alpha^\sigma K_t^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{0}})$, which must be in $A'(\overline{t})$. From the middle coordinates,

$$\lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{t}} D^{-1}(\alpha^\sigma)^T = \lambda^{-1}\alpha^\sigma A_t^\sigma (\alpha^\sigma)^T + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\overline{0}} D^{-1}(\alpha^\sigma)^T.$$

This can be rewritten as $(\alpha^\sigma D^{-T})[\lambda D^T A_t^\sigma D + A'_{\overline{0}} - A'_{\overline{t}}](\alpha^\sigma D^{-T})^T$ for all $\alpha \in F^2$, $t \in F$, which is equivalent to

(27) $$\lambda D^T A_t^\sigma D \equiv A'_{\overline{t}} - A'_{\overline{0}} \text{ for all } t \in F.$$

From the third coordinates we obtain $\alpha^\sigma(K_t^\sigma D + \lambda^{-1} D^{-T} K_{\overline{0}}) = \lambda^{-1}\alpha^\sigma D^{-T} K'_{\overline{t}}$. As this holds for all $\alpha \in F^2$, $t \in F$, multiplying on the left by λD^T, we obtain

(28) $$\lambda D^T K_t^\sigma D = K'_{\overline{t}} - K'_{\overline{0}}.$$

We claim that the condition in Eq. (27) implies that of Eq. (28). First suppose q is odd. If K and M are symmetric matrices, then $K \equiv M$ if and only if $K = M$. So from Eq. (27), $\lambda D^T A_t^\sigma D = A'_{\overline{t}} - A'_{\overline{0}}$. Add this to its transpose to get Eq. (28). Now suppose Eq. (27) holds with $q = 2^e$. Let $D^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\Delta = \det(D)$. If K and M are upper triangular matrices, then $K \equiv M$ if and only if $K = M$. So from Eq. (27), $\begin{pmatrix} x'_{\overline{t}} - x'_{\overline{0}} & y'_{\overline{t}} - y'_{\overline{0}} \\ 0 & z'_{\overline{t}} - z'_{\overline{0}} \end{pmatrix} = A'_{\overline{t}} - A'_{\overline{0}} \equiv \lambda D^T A_t^\sigma D \equiv \lambda \begin{pmatrix} a^2 x_t^\sigma + aby_t^\sigma + b^2 z_t^\sigma & \Delta y_t^\sigma \\ 0 & c^2 x_t^\sigma + cdy_t^\sigma + d^2 z_t^\sigma \end{pmatrix}$. So in particular, $y'_{\overline{t}} - y'_{\overline{0}} = \lambda \Delta y_t^\sigma$. But Eq. (28) says $(y'_{\overline{t}} - y'_{\overline{0}})P = K'_{\overline{t}} - K'_{\overline{0}} = \lambda D^T K_t^\sigma D = \lambda y_t^\sigma D^T P D = \lambda y_t^\sigma \Delta P$, which holds if and only if $y'_{\overline{t}} - y'_{\overline{0}} = \lambda \Delta y_t^\sigma$. So for all q Eq. (27) does imply Eq. (28).

This completes the proof (in the hard direction) of the following fundamental lemma.

Lemma 2 *Let $\mathcal{C} = \{A_t : t \in F\}$ and $\mathcal{C}' = \{A'_t : t \in F\}$ be (not necessarily distinct) normalized q-clans (i.e., the matrices are symmetric if q is odd, upper triangular if $q = 2^e$, and $A_0 = A'_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$). Let $\theta : GQ(\mathcal{C}) \to GQ(\mathcal{C}')$ be an isomorphism for which $\theta : (\infty) \mapsto (\infty)$, $[A(\infty)] \mapsto [A'(\infty)]$, $(\overline{0}, 0, \overline{0}) \mapsto (\overline{0}, 0, \overline{0})$. Then there exist the following:*

(i) $\sigma \in Aut(F)$

(ii) $D \in GL(2, q)$

(iii) $0 \neq \lambda \in F$

(iv) $\pi : F \to F : t \mapsto \bar{t}$, *a permutation on* F, *for which*

(v) $A'_{\bar{t}} \equiv \lambda D^T A_t^\sigma D + A'_{\bar{0}}$ *for all* $t \in F$.

Moreover, θ *is (induced by) an automorphism of* G *given by*

$$(29) \quad \theta = \theta(\sigma, D, \lambda, \pi) : (\alpha, c, \beta) \mapsto \\ (\lambda^{-1}\alpha^\sigma D^{-T}, \lambda^{-1}c^\sigma + \lambda^{-2}\alpha^\sigma D^{-T} A'_{\bar{0}} D^{-1}(\alpha^\sigma)^T, \beta^\sigma D + \lambda^{-1}\alpha^\sigma D^{-T} K'_{\bar{0}}).$$

Conversely, given σ, D, λ, *and* π *satisfying condition (v),* θ *defined by Eq. (29) gives such an isomorphism.* ■

Frequently it is convenient to change the parameters in Lemma 2. Put $\mu = \lambda^{-1}$ and $B = \lambda^{-1}D^{-T}$, so $D = \mu B^{-T}$. Then collect the results of Lemmas 1 and 2 into the following major result.

The Fundamental Theorem of q-clan Geometry

For any prime power q, *let* $\mathcal{C} = \{A_t \equiv \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ *and* $\mathcal{C}' = \{A'_t \equiv \begin{pmatrix} x'_t & y'_t \\ 0 & z'_t \end{pmatrix} : t \in F\}$ *be two (not necessarily distinct) normalized* q*-clans. Then the following are equivalent:*

(i) $\mathcal{C} \sim \mathcal{C}'$.

(ii) The flocks $\mathcal{F}(\mathcal{C})$ *and* $\mathcal{F}(\mathcal{C}')$ *are projectively equivalent.*

(iii) $GQ(\mathcal{C})$ *and* $GQ(\mathcal{C}')$ *are isomorphic by an isomorphism mapping* (∞) *to* (∞), $[A(\infty)]$ *to* $[A'(\infty)]$, *and* $(\bar{0}, 0, \bar{0})$ *to* $(\bar{0}, 0, \bar{0})$.

Moreover, these three conditions hold if and only if there exist $\sigma \in Aut(F)$, $B \in GL(2, q)$, $0 \neq \mu \in F$, *and a permutation* $\pi : t \mapsto \bar{t}$ *on* F *for which*

(iv) $A'_{\bar{t}} \equiv \mu B^{-1} A_t^\sigma B^{-T} + A'_{\bar{0}}$ *for all* $t \in F$.

The corresponding isomorphism $\theta : GQ(\mathcal{C}) \to GQ(\mathcal{C}')$ *is given by*

(v) $\theta = \theta(\mu, B, \sigma, \pi) : (\alpha, c, \beta) \mapsto (\alpha^\sigma B, \mu c^\sigma + \alpha^\sigma B A'_{\bar{0}} B^T (\alpha^\sigma)^T, \mu\beta^\sigma B^{-T} + \alpha^\sigma B K'_{\bar{0}})$.

If $B^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *the corresponding isomorphism* $T_\theta : \mathcal{F}(\mathcal{C}) \to \mathcal{F}(\mathcal{C}')$ *is defined for planes of the flocks by*

$$(vi)\ T_\theta : \begin{bmatrix} x_t \\ y_t \\ z_t \\ 1 \end{bmatrix} \mapsto \begin{bmatrix} x'_{\bar t} \\ y'_{\bar t} \\ z'_{\bar t} \\ 1 \end{bmatrix} = \begin{bmatrix} \mu a^2 & \mu ab & \mu b^2 & x'_{\bar 0} \\ 2\mu ac & \mu(ad+bc) & 2\mu bd & y'_{\bar 0} \\ \mu c^2 & \mu cd & \mu d^2 & z'_{\bar 0} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_t^\sigma \\ y_t^\sigma \\ z_t^\sigma \\ 1 \end{bmatrix}. \blacksquare$$

Before leaving this section we consider the F.T. a bit more with the additional hypothesis that $\mathcal{C} = \mathcal{C}'$, or more specifically, that $A'_t = A_t$, $t \in F$. So $\mathcal{C}$ is a normalized q-clan, non-classical so that the group $\mathcal{G}_0$ of collineations of $GQ(\mathcal{C})$ fixing $(\overline{0}, 0, \overline{0})$ also fixes (∞). Label the following subgroups of $\mathcal{G}_0$.

$$(30)\quad \begin{array}{rcl} \text{(i)}\ \mathcal{H} &=& \{\theta \in \mathcal{G}_0 : [A(\infty)]^\theta = [A(\infty)]\}, \\ \text{(ii)}\ \mathcal{M} &=& \{\theta \in \mathcal{H} : [A(0)]^\theta = [A(0)]\}, \\ \text{(iii)}\ \mathcal{N} &=& \{\theta = \theta_a \in \mathcal{M} | \theta_a : (\alpha, c, \beta) \mapsto (a\alpha, a^2 c, a\beta),\ 0 \neq a \in F\}. \end{array}$$

The following observation is a trivial consequence of the F.T.

Proposition 1 *$\mathcal{M}$ always contains the following subgroup:*

$$\mathcal{M}^* = \{\theta = \theta(a^2, aI, \sigma, \pi) : 0 \neq a \in F,\ \sigma \in\ Aut(F)\ \textit{with}\ A_{t^\sigma} = A_t^\sigma\ \textit{for}\ t \in F\}.$$

Clearly $\theta(a^2, aI, \sigma, \pi)$ satisfies $\pi : t \mapsto \bar{t} = t^\sigma$. $\blacksquare$

Now let $\theta_i = \theta(\mu_i, B_i, \sigma_i, \pi_i)$, $i = 1, 2$. We want to compute $\theta = \theta_1 \circ \theta_2$, where the notation means do θ_1 first. Then

$$\begin{array}{rcl} (\alpha, c, \beta)^{\theta_1 \circ \theta_2} &=& (\alpha^{\sigma_1} B_1, \mu_1 c^{\sigma_1} + \alpha^{\sigma_1} B_1 A_{0^{\pi_1}} B_1^T (\alpha^{\sigma_1})^T, \mu_1 \beta^{\sigma_1} B_1^{-T} + \alpha^{\sigma_1} B_1 K_{0^{\pi_1}})^{\theta_2} \\ &=& ((\alpha^{\sigma_1} B_1)^{\sigma_2} B_2, \mu_2 [\mu_1 c^{\sigma_1} + \alpha^{\sigma_1} B_1 A_{0^{\pi_1}} B_1^T (\alpha^{\sigma_1})^T]^{\sigma_2} \\ && + (\alpha^{\sigma_1} B_1)^{\sigma_2} B_2 A_{0^{\pi_2}} B_2^T ((\alpha^{\sigma_1} B_1)^{\sigma_2})^T, \mu_2 (\mu_1 \beta^{\sigma_1} B_1^{-T} \\ && + \alpha^{\sigma_1} B_1 K_{0^{\pi_1}})^{\sigma_2} B_2^{-T} + (\alpha^{\sigma_1} B_1)^{\sigma_2} B_2 K_{0^{\pi_2}}) \\ &=& (\alpha^{\sigma_1 \circ \sigma_2} B_1^{\sigma_2} B_2, \mu_1^{\sigma_2} \mu_2 C^{\sigma_1 \circ \sigma_2} + \mu_2 \alpha^{\sigma_1 \circ \sigma_2} B_1^{\sigma_2} A_{0^{\pi_1}}^{\sigma_2} (B_1^{\sigma_2})^T (\alpha^{\sigma_1 \circ \sigma_2})^T \\ && + \alpha^{\sigma_1 \circ \sigma_2} B_1^{\sigma_2} B_2 A_{0^{\pi_2}} (B_1^{\sigma_2} B_1)^T (\alpha^{\sigma_1 \circ \sigma_2})^T, \mu_1^{\sigma_2} \mu_2 \beta^{\sigma_1 \circ \sigma_2} (B_1^{\sigma_2} B_2)^{-T} \\ && + \mu_2 \alpha^{\sigma_1 \circ \sigma_2} B_1^{\sigma_2} K_{0^{\pi_1}}^{\sigma_2} B_2^{-T} + \alpha^{\sigma_1 \circ \sigma_2} B_1^{\sigma_2} B_2 K_{0^{\pi_2}}). \end{array}$$

Since $\theta = \theta_1 \circ \theta_2 = \theta(\mu, B, \sigma, \pi)$ for some μ, B, σ, and π according to the F.T., we can read off $\mu = \mu_1^{\sigma_2} \mu_2$, $B = B_1^{\sigma_2} B_2$, $\sigma = \sigma_1 \circ \sigma_2$. So we have at least

$$(31)\qquad \theta(\mu_1, B_1, \sigma_1, \pi_1) \circ \theta(\mu_2, B_2, \sigma_2, \pi_2) = \theta(\mu_1^{\sigma_2} \mu_2, B_1^{\sigma_2} B_2, \sigma_1 \circ \sigma_2, \pi),$$

where we say a little more about π. We have now the fact that

$$\begin{array}{l} \theta_1 \circ \theta_2 : (\alpha, c, \beta) \mapsto \\ \quad (\alpha^\sigma B, \mu c^\sigma + \mu_2 \alpha^\sigma B_1^{\sigma_2} A_{0^{\pi_1}}^{\sigma_2} (B_1^{\sigma_2})^T (\alpha^\sigma)^T + \alpha^\sigma B A_{0^{\pi_2}} B^T (\alpha^\sigma)^T, \\ \quad \mu \beta^\sigma B^{-T} + \mu_2 \alpha^\sigma B_1^{\sigma_2} K_{0^{\pi_1}}^{\sigma_2} B_2^{-T} + \alpha^\sigma B K_{0^{\pi_2}}), \end{array}$$

where this image must equal $(\alpha^\sigma B, \mu c^\sigma + \alpha^\sigma B A_{0^\pi} B^T (\alpha^\sigma)^T, \mu\beta^\sigma B^{-T} + \alpha^\sigma B K_{0^\pi})$. This is equivalent to

$$(32)\qquad \begin{aligned} &\text{(i)} && \alpha^\sigma[\mu_2 B_1^{\sigma_2} A_{0^{\pi_1}}^{\sigma_2}(B_1^{\sigma_2})^T + B_1^{\sigma_2} B_2 A_{0^{\pi_2}}(B_2)^T(B_1^{\sigma_2})^T](\alpha^\sigma)^T \\ &&& = \alpha^\sigma(B_1^{\sigma_2} B_2 A_{0^\pi} B_2^T (B_1^{\sigma_2})^T)(\alpha^\sigma)^T \\ &\text{(ii)} && B_1^{\sigma_2} B_2 K_{0^\pi} = \mu_2 B_1^{\sigma_2} K_{0^{\pi_1}}^{\sigma_2} B_2^{-T} + B_1^{\sigma_2} B_2 K_{0^{\pi_2}}. \end{aligned}$$

Since $[A(0)] \overset{\theta_1}{\mapsto} [A(0^{\pi_1})] \overset{\theta_2}{\mapsto} [A((0^{\pi_1})^{\pi_2})]$, at least $0^\pi = 0^{\pi_1 \circ \pi_2}$. So by hypothesis $\mu_2 B_2^{-1} A_{0^{\pi_1}}^{\sigma_2} B_2^{-T} + A_{0^{\pi_2}} + A_{(0^{\pi_1})^\pi} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, since $\theta_2 \in \mathcal{G}_0$, and also $\mu_2 B_2^{-1} K_{0^{\pi_1}}^{\sigma_2} B_2^{-T} = K_{0^{\pi_2}} + K_{(0^{\pi_1})^{\pi_2}}$. The first relationship verifies Eq. (32)(i); the latter equality verifies Eq. (32)(ii). In fact, this gives a direct proof that the composition in Eq. (31) is correct. But π is essentially controlled by σ, B, μ, and 0^π according to the F.T. (iv), which may be written out as follows.

$$(33)\qquad \begin{aligned} x_{t^\pi} &= \mu(a^2 x_t^\sigma + ab y_t^\sigma + b^2 z_t^\sigma) + x_{0^\pi} \\ y_{t^\pi} &= \mu(2ac x_t^\sigma + (ad+bc) y_t^\sigma + 2bd z_t^\sigma) + y_{0^\pi} \\ z_{t^\pi} &= \mu(c^2 x_t^\sigma + cd y_t^\sigma + d^2 z_t^\sigma) + z_{0^\pi} \end{aligned}$$

Since $t \mapsto x_t$ and $t \mapsto z_t$ must be permutations, t^π is determined (though not conveniently!) by σ, B, μ, and 0^π. This suggests that we make the following additional normalizing convention: If q is odd, *we assume that the members of* $\mathcal{C}$ *are indexed so that* $x_t = t$ *for all* $t \in F$. And if $q = 2^e$, so that $t \mapsto y_t$ is also a permutation, *we assume that the members of* $\mathcal{C}$ *are indexed so that* $y_t = t$ *for all* $t \in F$. This makes it possible to be more explicit with the description of π.

$$(34)\qquad \text{For } \theta(\sigma, B, \mu, \pi) \in \mathcal{H} \text{ with } B^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$\begin{aligned} &\text{(i)} && t^\pi = \mu(a^2 t^\sigma + ab y_t^\sigma + b^2 z_t^\sigma) + 0^\pi, \text{ if } q \text{ is odd.} \\ &\text{(ii)} && t^\pi = \mu(ad+bc) t^\sigma + 0^\pi, \text{ if } q \text{ is even.} \end{aligned}$$

Read Theorem 1(iv) as equality with q odd, and apply $\theta_1 \circ \theta_2$ with π as in Eq. (31)

$$(35)\qquad \begin{aligned} A_{t^\pi} &= \mu_2 B_2^{-1}(\mu_1 B_1^{-1} A_t^{\sigma_1} B_1^{-T} + A_{0^{\pi_1}})^{\sigma_2} B_2^{-T} + A_{0^{\pi_2}} \\ &= \mu_1^{\sigma_1}\mu_2((B_1^{\sigma_2} B_2)^{-1} A_t^{\sigma_1 \circ \sigma_2}(B_1^{\sigma_2} B_2)^{-T} + \mu_2 B_2^{-1} A_{0^{\pi_1}}^{\sigma_2} B_2^{-T} + A_{0^{\pi_2}}. \end{aligned}$$

And here $\mu_2 B_2^{-1} A_{0^{\pi_1}}^{\sigma_2} B_2^{-T} + A_{0^{\pi_2}} = A_{0^{\pi_1 \circ \pi_2}}$, confirming again (put $t = 0$ in Eq. (35)) that $0^\pi = 0^{\pi_1 \circ \pi_2}$.

A relationship similar to Eq. (35) holds for q even, and it follows that (use Theorem 1(vi)):

Proposition 2 *$T : \theta \mapsto T_\theta$ is a homomorphism from $\mathcal{H}$ onto the subgroup of $P\Gamma L(4,q)$ leaving invariant the cone K and the flock $\mathcal{F}(\mathcal{C})$. The kernel of T is $\mathcal{N}$.* ■

We define $\mathcal{N}$ to be the *kernel* of $GQ(\mathcal{C})$ (as was done in [1] for q even). This definition of kernel may not be as satisfactory as that for translation generalized quadrangles (TGQ) studied in [16], but the following proposition shows that for many q-clans $\mathcal{C}$ the kernel of $GQ(\mathcal{C})$ plays a role similar to one played by the multiplicative group of the kernel of a TGQ.

Proposition 3 *Let $\mathcal{K}$ be the group of collineations of $GQ(\mathcal{C})$ fixing $(\overline{0}, 0, \overline{0})$ and (∞) linewise. Clearly $\mathcal{K} \leq \mathcal{H}$, so the elements of $\mathcal{K}$ are described by the F.T. Since $\mathcal{C}$ is assumed to be non-classical, we can say the following about $\mathcal{K}$.*

(i) If $q = 2^e$, then $\mathcal{K} = \mathcal{N}$. (This was first proved in [1].)

(ii) If q is odd and $\mathcal{K}^* = \{\theta(\mu, B, \sigma, \pi) \in \mathcal{K} : \sigma = id\}$, then $\mathcal{K}^* = \mathcal{N}$, except for the following q-clan discovered by W. M. Kantor [8]. For arbitrary odd q, let $id \neq \tau \in Aut(F)$ and $m = \not\square \in F$. Then put $\mathcal{C} = \{A_t = \begin{pmatrix} t & 0 \\ 0 & -mt^\tau \end{pmatrix} : t \in F\}$. In this case $\mathcal{K} = \mathcal{K}^*$ if $\tau^2 \neq id$, and $[\mathcal{K} : \mathcal{N}] = 2$. If $\tau^2 = id \neq \tau$, then $[\mathcal{K} : \mathcal{K}^*] = [\mathcal{K}^* : \mathcal{N}] = 2$.

Proof: Let $\theta \in \mathcal{K}$. By the F.T., $\theta = \theta(\mu, B, \sigma, id)$ for $B^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, q)$, etc. Since $t^\pi = t$ by hypothesis, Eq. (33) implies

$$(36) \qquad \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \mu \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \begin{pmatrix} x_t^\sigma \\ y_t^\sigma \\ z_t^\sigma \end{pmatrix} \quad \text{for all } t \in F.$$

If $q = 2^e$, $y_t = \mu(ad+bc)y_t^\sigma$ for all $t \in F$, so $\sigma = id$. If q is odd, we assume $\sigma = id$, i.e., $\theta \in \mathcal{K}^*$. So Eq. (36) may be rewritten as

$$(37) \qquad \left(I - \mu \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} \right) \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for all } t \in F.$$

We claim that this forces $I = \mu \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$. For if not, then there are elements $w_0, w_1, w_2 \in F$, not all zero, for which the point $(w_0, w_1, w_2, 0) \in PG(3, q)$ lies on all the planes $\pi_t = [x_t, y_t, z_t, 1]^T$ of $\mathcal{F}(\mathcal{C})$. By two results of J. A. Thas [17], we know that since $\mathcal{F}(\mathcal{C})$ is not linear, q must be odd and $\mathcal{C}$ must be equivalent to the q-clan given in part (ii) of the theorem. And in that case the results claimed in part (ii) are proved in S. E. Payne [12] (cf. Section IV, Example 3). But it is also easy to check directly that $\mu = 1$, $a = 1$, $d = -1$, $b = c = 0$, gives a $\theta(\mu, B, \sigma, id) \in \mathcal{K}^* \backslash \mathcal{N}$ with $\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then $\theta : (\alpha, c, \beta) \mapsto \left(\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, c, \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$. ∎

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5 Recoordinatization: shifts, flips and scales

Any automorphism of G replaces one 4-gonal family for G with another. But in general we cannot expect that a 4-gonal family $\mathcal{J}(\mathcal{C})$ arising from a q-clan $\mathcal{C}$ will be replaced with one for which a q-clan is clearly present. In this section we study three basic automorphisms of G that even replace normalized q-clans with normalized q-clans. One goal is to develop a procedure to use the F.T. to describe all elements of $\mathcal{G}_0$. A second goal is to assign to each line through (∞) in some $GQ(\mathcal{C})$ an equivalence class of flocks (or q-clans) in such a way that two lines through (∞) are in the same $\mathcal{G}_0$-orbit if and only if their flocks (or q-clans) are equivalent. This provides a very concrete algebraic approach to results obtained in [2], [14], and [15], with the case for $q = 2^e$ worked out in [1] for a representation of G that works only in characteristic 2.

Throughout this section we assume that $\mathcal{C}$ is a normalized q-clan with the specific indexing given in Section 4. Our first observation is just a special corollary of the F.T.

Observation 1 *Let $B \in GL(2,q)$ and $\pi : F \to F : t \mapsto \bar{t}$ a permutation. For each $A_t \in \mathcal{C}$ set $A'_{\bar{t}} = B^{-1}A_tB^{-T}$. Then $\mathcal{C}' = \{A'_{\bar{t}} : t \in F\}$ is a q-clan equivalent to $\mathcal{C}$, and $\theta : GQ(\mathcal{C}) \to GQ(\mathcal{C}') : (\alpha, c, \beta) \mapsto (\alpha B, c, \beta B^{-T})$ is an isomorphism.* ■

Lemma 3 Shift by s. *Fix $s \in F$. Define $\tau_s : G \to G : (\alpha, c, \beta) \mapsto (\alpha, c - \alpha A_s \alpha^T, \beta - \alpha K_s)$. For $A_t \in \mathcal{C}$, put $A^{\tau_s}_{\bar{t}} = A_t - A_s$, so $A^{\tau_s}_{x} = A_{x+s} - A_s$. Then $\mathcal{C}^{\tau_s} = \{A^{\tau_s}_t : t \in F\}$ is a normalized q-clan equivalent to $\mathcal{C}$, and $\tau_s = \theta(1, I, id, \pi : t \mapsto t - s) : GQ(\mathcal{C}) \to GQ(\mathcal{C}^{\tau_s})$ is an isomorphism.*

Proof: This is immediate from the F.T. For q odd, $x_t = t$, so $x^{q_s}_t = (t+s) - s = t$. For q even, $y_t = t$, so $y^{\tau_s}_t = (t+s) - s = t$. Hence $\mathcal{C}^{\tau_s}$ is normalized and $\overline{0} = -s$. ■

Lemma 4 Scale by a. *Let $0 \neq a \in F$. Define $\sigma_a : G \to G : (\alpha, c, \beta) \mapsto (\alpha, ac, a\beta)$. For $A_t \in \mathcal{C}$, put $A^{\sigma_a}_{\bar{t}} = aA_t$, so $A^{\sigma_a}_{at} = aA_t$, or $A^{\sigma_a}_t = aA_{t/a}$. Then $\mathcal{C}^{\sigma_a} = \{A^{\sigma_a}_t : t \in F\}$ is a normalized q-clan equivalent to $\mathcal{C}$, and $\sigma_a = \theta(a, I, id, \pi : t \mapsto at) : GQ(\mathcal{C}) \to GQ(\mathcal{C}^{\sigma_a})$ is an isomorphism.* ■

Before considering the third type of automorphism of G, we collect some rather trivial results that are helpful in doing computations.

Lemma 5 *Put $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and for $B \in GL(2,q)$, put $\Delta = \det(B)$.*

(i) $P^T = P^{-1} = -P$.

(ii) $P^TBP = \Delta B^{-T} = PBP^T \equiv \Delta B^{-1}$.

(iii) $B^TPB = \Delta P$; $B^TP^TB = \Delta P^T$.

(iv) *If q is odd and A is nonsingular and symmetric with $K = A + A^T = 2A$, then $P^TK^{-1}AKP = (\det(K))^{-1}A$.*

(v) *If* $q = 2^e$ *and* A *is upper triangular with* $K = A + A^T = (\det(K))^{1/2}P$ *nonsingular, then* $P^T K^{-1} A K P = (\det(K))^{-1} A$.

Lemma 6 The flip. *Define* $\varphi : G \to G : (\alpha, c, \beta) \mapsto (\beta P, c - \alpha\beta^T, \alpha P^T)$. *Then* φ *is an automorphism of* G *that interchanges* $A(0)$ *and* $A(\infty)$. *It maps a normalized q-clan* $\mathcal{C} = \{A_t : t \in F\}$ *to a q-clan* $\mathcal{C}^\varphi = \{A^\varphi_{\bar{t}} = -P^T K_t^{-1} A_t K_t^{-1} P = -(\det(K_t))^{-1} A_t : t \in F\}$. *The permutation* $\pi : t \mapsto \bar{t}$ *is chosen so that* $\mathcal{C}^\varphi$ *will have a normalized indexing. If* q *is even with* $y_t = t$, *clearly* $\pi : t \mapsto \bar{t} = t^{-1}$ *for all* $t \in \tilde{F}$. *For* q *odd with* $x_t = t$, *the permutation* $\pi : t \mapsto \bar{t}$ *depends on the functions* $t \mapsto y_t$ *and* $t \mapsto z_t$. *Explicitly:* $\pi : t \mapsto \bar{t} = t/(4x_t z_t - y_t^2)$.

Proof: It is routine to check that φ is an automorphism of G interchanging $A(0)$ and $A(\infty)$. So $A^\varphi(0) = A(\infty)$, $A^\varphi(\infty) = A(0)$. And

$$\begin{aligned}
&\varphi : (\alpha, \alpha A_t \alpha^T, \alpha K_t) \mapsto (\alpha K_t P, \alpha A_t \alpha^T - \alpha K_t \alpha^T, \alpha P^T) \\
&= (\alpha K_t P, \alpha K_t P(-P^T K_t^{-1} A_t K_t^{-1} P)(\alpha K_t P)^T, \alpha K_t P(P^T K_t^{-1} P^T)) \\
&= (\gamma, \gamma(-(\det(K_t))^{-1} A_t)\gamma^T, -(\det(K_t))^{-1}\gamma K_t), \text{ for } \gamma = \alpha K_t P,\ 0 \neq t \in F. \blacksquare
\end{aligned}$$

Shifting by s and scaling by a are automorphisms of G that map normalized q-clans to equivalent normalized q-clans, and affect on $\tilde{F}$ (the index set for the lines through (∞)) the permutations $t \mapsto \bar{t} = t - s$ and $t \mapsto \bar{t} = at$, respectively. So the group generated by these permutations stabilizes ∞. On the other hand, the flip φ affects a permutation π on $\tilde{F}$ that interchanges 0 and ∞, and for $q = 2^e$ $\pi : t \mapsto t^{-1}$ for all $t \in \tilde{F}$. In general for q odd, π is determined by $t \mapsto y_t$ and $t \mapsto z_t$, and $\mathcal{C}$ and $\mathcal{C}^\varphi$ are not equivalent!

For $s \in F$ define the *shift-flip* i_s by

(38) (i) $i_s = \tau_s \circ \varphi$, $s \in F$.

(ii) $i_s : (\alpha, c, \beta) \mapsto ((\beta - \alpha K_s)P, c - \alpha\beta^T + \alpha A_s \alpha^T, \alpha P^T)$

(iii) $i_s^{-1} : (\alpha, c, \beta) \mapsto (\beta P, c - \alpha\beta^T + (\beta P)A_s(\beta P)^T, \alpha P^T + \beta P K_s)$.

Compute the effect of i_s on an element of $A(t)$. $i_s : (\alpha, \alpha A_t \alpha^T, \alpha K_t) \mapsto (\alpha(K_t - K_s)P, -\alpha(A_t - A_s)^T, \alpha P^T)$. Put $\gamma = \alpha(K_t - K_s)P$, so for $t \neq s$, $\alpha = \gamma P(K_t - K_s)^{-1}$. Then the image is $(\gamma, \gamma(-P^T(K_t - K_s)^{-1}(A_t - A_s)(K_t - K_s)^{-1}P)\gamma^T, -\gamma P^T(K_t - K_s)^{-1}P) = (\gamma, \gamma(-(\det(K_t - K_s))^{-1}(A_t - A_s))\gamma^T, \gamma(-(\det(K_t - K_s))^{-1})(K_t - K_s))$. So we note that

(39) (i) $i_s : \mathcal{C} \to \mathcal{C}^{i_s} : A_t \mapsto A^{i_s}_{\bar{t}} = -(\det(K_t - K_s))^{-1}(A_t - A_s)$,
$\bar{t} = -(t - s)(\det(K_t - K_s))^{-1}$.

(ii) If $q = 2^e$, $\bar{t} = (t + s)^{-1}$.

Also define $i_\infty = id : G \to G$. Then to each line of $GQ(\mathcal{C})$ through (∞) we are going to assign an equivalence class of q-clans and its associated class of flocks. Start with a normalized q-clan $\mathcal{C}$. For each $s \in \tilde{F}$, applying i_s to G yields a normalized q-clan $\mathcal{C}^{i_s}$. We *assign to the line* $[A(s)]$ *of* $GQ(\mathcal{C})$ *the class of q-clans equivalent to* $\mathcal{C}^{i_s}$, and also *the class of flocks projectively equivalent to* $\mathcal{F}(\mathcal{C}^{i_s})$. The following application of the F.T. makes this assignment natural and useful.

Theorem 1 *Let $\mathcal{C}$ be a normalized q-clan. Then for arbitrary $s, t \in \tilde{F}$, there is a collineation (an automorphism) of $GQ(\mathcal{C})$ mapping $[A(s)]$ to $[A(t)]$ if and only if $\mathcal{F}(\mathcal{C}^{i_s})$ and $\mathcal{F}(\mathcal{C}^{i_t})$ are projectively equivalent (i.e., if and only if $\mathcal{C}^{i_s} \sim \mathcal{C}^{i_t}$).*

Proof: Since τ_s always effects the permutation $t \mapsto t - s$ and φ always interchanges 0 and ∞, the shift-flip i_s always maps s to ∞, i.e., $i_s : GQ(\mathcal{C}) \to GQ(\mathcal{C}^{i_s}) : [A(s)] \mapsto [A^{i_s}(\infty)]$. Let θ be an automorphism of $GQ(\mathcal{C})$ mapping $[A(s)]$ to $[A(t)]$. Without loss of generality we may assume that θ fixes $(\overline{0}, 0, \overline{0})$ (and of course (∞)). Then $\overline{\theta} = i_s^{-1} \circ \theta \circ i_t : GQ(\mathcal{C}^{i_s}) \to GQ(\mathcal{C}^{i_t}) : [A^{i_s}(\infty)] \mapsto [A^{i_t}(\infty)]$. Since clearly $\overline{\theta}$: $(\infty) \mapsto (\infty)$ and $(\overline{0}, 0, \overline{0}) \mapsto (\overline{0}, 0, \overline{0})$, by the F.T. $\mathcal{F}(\mathcal{C}^{i_s})$ and $\mathcal{F}(\mathcal{C}^{i_t})$ are projectively equivalent. Conversely, given that $\mathcal{F}(\mathcal{C}^{i_s})$ and $\mathcal{F}(\mathcal{C}^{i_t})$ are projectively equivalent, there is an appropriate $\overline{\theta} : GQ(\mathcal{C}^{i_s}) \to GQ(\mathcal{C}^{i_t}) : [A^{i_s}(\infty)] \mapsto [A^{i_t}(\infty)]$. Hence we may put $\theta = i_s \circ \overline{\theta} \circ i_t^{-1} : GQ(\mathcal{C}) \to GQ(\mathcal{C}) : [A(s)] \mapsto [A(t)]$. ∎

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6 The Cohen-Ganley-Barriga flock

Let $q \equiv \pm 2 (\text{mod} 5)$. Put $\mathcal{C} = \{A_t \equiv \begin{pmatrix} t & 5t^3 \\ 0 & 5t^5 \end{pmatrix} : t \in F\}$. Then $\mathcal{C}$ is a q-clan discovered by W. M. Kantor [8] when q is odd and by S. E. Payne [10] when q is even. It is Example 4 in [12]. In the latter paper all collineations fixing $[A(\infty)]$ are determined. They must all fix $[A(0)]$ also. And there is a collineation (essentially the flip) interchanging $[A(t)]$ and $[A(t^{-1})]$ for all $t \in \tilde{F}$. This means that for q even, since the flip fixes only $[A(1)]$, the lines $[A(\infty)]$ and $[A(1)]$ must be in different orbits. The group fixing $[A(\infty)]$ and $[A(0)]$ is transitive on all lines $[A(t)]$, $0 \neq t \in F$. So for all q there are at most two orbits on the lines through (∞). For $q = 2^e$ there are clearly exactly two orbits, but the situation for q odd is not yet clear.

In [12] the group leaving the set $\{[A(\infty)], [A(0)]\}$ invariant was claimed to be the complete group of collineations. And it is! But for q odd there seems to be no easy proof. In this section we give a rather elementary proof that fills this gap.

From now on q is odd, $q \equiv \pm 2 (\text{mod} 5)$, and $q \geq 7$. And we adjust the matrices of $\mathcal{C}$ to be symmetric, i.e., $A_t = \begin{pmatrix} t & 5t^3/2 \\ 5t^3/2 & 5t^5 \end{pmatrix}$, $t \in F$. In N. Johnson [6] the gap is filled in for those q for which $(q+1)/2$ is not a prime power—using rather deep group theory. In G. Lunardon [9] a proof is given that depends on less sophisticated group theory. His proof gives the result we want for $q > 7$, but it still relies heavily on the theory of groups acting on projective planes. Our proof relies on the F.T. for q-clan geometries but is otherwise quite elementary.

The flock associated with $\mathcal{C}$ consists of the conics which are the intersections with the cone $K : x_1^2 = x_0 x_2$ in $PG(3, q)$ of the planes in $\mathcal{F}'(\mathcal{C}) = \{\pi_t = [t, 5t^3, 5t^5, 1]^T : t \in F\}$. Our first step is to analyze the intersections of four (then three) planes of $\mathcal{F}'(\mathcal{C})$. Let s, t, u, and v be distinct elements of F. The intersection $\pi_s \cap \pi_t \cap \pi_u \cap \pi_v$ is the left null space of the matrix P_1, where (we use P_1^T since we prefer row reduction to column reduction)

$$(40)\qquad P_1^T = \begin{pmatrix} s & 5s^3 & 5s^5 & 1 \\ t & 5t^3 & 5t^5 & 1 \\ u & 5u^3 & 5u^5 & 1 \\ v & 5v^3 & 5v^5 & 1 \end{pmatrix}.$$

Straightforward row reduction of P_1^T (including factoring out the nonzero factors $t-s$, $u-s$, $v-s$, $u-t$, $v-t$, $v-u$) produces the following matrix, where the missing terms are now irrelevant.

$$(41)\quad P_2^T = \begin{pmatrix} 0 & -- & -- & 1 \\ 1 & -- & -- & 0 \\ 0 & u+t & u^3+u^2t+ut^2+t^3-s(v^2+vu+vt)-s^2v & 0 \\ 0 & 1 & v^2+vu+u^2+(v+u)t+t^2+s(v+u+t)+s^2 & 0 \end{pmatrix}.$$

This matrix has rank 3 or 4. It has rank 3 if and only if the lower middle 2×2 determinant equals 0. This is if and only if

$$(42)\qquad 0 = s^2(t+u+v)+s(t+u+v)^2+(u+v)(t+u)(t+v).$$

We can now draw the following conclusions.

(43) (i) $\pi_s \cap \pi_t \cap \pi_u \cap \pi_v$ is never a line.
(ii) If $s=0$, then $\pi_0 \cap \pi_t \cap \pi_u \cap \pi_v$ is a point if and only if t^2, u^2, and v^2 are not distinct.

Using the F.T. and our knowledge (from [12]) of the collineation group of $GQ(\mathcal{C})$, we know there is a collineation group of $PG(3,q)$ leaving K invariant, leaving $\mathcal{F}(\mathcal{C})$ invariant, fixing π_0, and transitive on the π_t with $0\neq t\in F$. So suppose $\pi_0\cap\pi_t\cap\pi_u\cap\pi_v$ is a point with $t^2=u^2$. Without loss of generality we may then assume $t=1$ and $u=-1$. It is easy to check that

(44) (i) $\pi_0\cap\pi_1\cap\pi_{-1} = \langle(5,-1,0,0),(0,-1,1,0)\rangle$, and
(ii) $\pi_0\cap\pi_1\cap\pi_{-1}\cap\pi_v = (5v^2,-(v^2+1),1,0)$
$= v^2(5,-1,0,0)+(0,-1,1,0)$.

Now suppose t, u, and v are distinct and nonzero. If $\pi_t\cap\pi_u\cap\pi_v$ were a line, then $\pi_0\cap\pi_t\cap\pi_u\cap\pi_v$ would be a point. So we may assume $t=1$, $u=-1$. To check the size of $\pi_1\cap\pi_{-1}\cap\pi_v$, row reduce the matrix

$$(45)\qquad P_3^T = \begin{pmatrix} 1 & 5 & 5 & 1 \\ -1 & -5 & -5 & 1 \\ v & 5v^3 & 5v^5 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 5 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & v^2+1 & 0 \end{pmatrix}.$$

The latter matrix has rank 3. Hence we may conclude

(46) For distinct, nonzero $t,u,v\in F$, $\pi_t\cap\pi_u\cap\pi_v$ is always a point.

Now suppose that t and u are nonzero with $t^2 \neq u^2$. If $\pi_0 \cap \pi_t \cap \pi_u$ were a line, then for any $v \in F\backslash\{0, t, u\}$, $\pi_0 \cap \pi_t \cap \pi_u \cap \pi_v$ would be a point, so either $v = -t$ or $v = -u$. This would force F to have only five elements, but $q \equiv \pm 2(\mathrm{mod} 5)$. Hence we may conclude the following:

(47) For distinct $s, t, u \in F$, $\pi_s \cap \pi_t \cap \pi_u$ is a line if and only if $\{s, t, u\} = \{0, x, -x\}$ for some nonzero $x \in F$.

In particular, for any $s \in F$, there are $t, u \in F$ with s, t, and u distinct, such that $\pi_s \cap \pi_t \cap \pi_u$ is a line.

Now suppose there is some collineation θ of $GQ(\mathcal{C})$ moving $[A(\infty)]$ to $[A(1)]$. The shift-flip i_1 ("derivation" in the terminology of [2]) maps the q-clan $\mathcal{C}$ to a q-clan $\mathcal{C}^{i_1}$ whose general matrix is $A^{i_1}_{\bar{t}} = \hat{A}_{\bar{t}} = -(\det(K_t - K_1))^{-1}(A_t - A_1)$, for $t \neq 1$, and $A^{i_1}_{\bar{1}} = \hat{A}_{\bar{1}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

$$\text{(48)} \qquad \begin{aligned} \det(K_t - K_1) &= -5(t-1)^2(t^4 + 6t^3 + 11t^2 + 6t + 1) \\ &= -5(t-1)(t^5 + 5t^4 + 5t^3 - 5t^2 - 5t - 1). \end{aligned}$$

So we may compute

$$\text{(49)} \qquad \hat{A}_{\bar{t}} = (t^5 + 5t^4 + 5t^3 - 5t^2 - 5t - 1)^{-1} \begin{pmatrix} 5^{-1} & (t^2+t+1)/2 \\ (t^2+t+1)/2 & t^4+t^3+t^2+t+1 \end{pmatrix}.$$

The corresponding plane $\hat{\pi}_{\bar{t}}$ of $\mathcal{F}(\mathcal{C}^{i_1})$ is then:

(50) (i) $\hat{\pi}_{\bar{1}} = [0, 0, 0, 1]^T$, and for $t \neq 1$,

(ii) $\hat{\pi}_{\bar{t}} = [\frac{1}{5}, t^2 + t + 1, t^4 + t^3 + t^2 + t + 1, t^5 + 5t^4 + 5t^3 - 5t^2 - 5t - 1]^T$.

(51) $\hat{\pi}_{\bar{1}} \cap \hat{\pi}_{\bar{t}} \cap \hat{\pi}_{\bar{u}}$ is a line if and only if $\{1, t, u\} = \{1, 0, -1\}$.

Proof: Row reduce

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{5} & t^2+t+1 & t^4+t^3+t^2+t+1 & -- \\ \frac{1}{5} & u^2+u+1 & u^4+u^3+u^2+u+1 & -- \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 5^{-1} & t^2+t+1 & t^4+t^3+t^2+t+1 & 0 \\ 0 & u^2-t^2+u-t & u^4-t^4+u^3-t^3+u^2-t^2+u-t & 0 \end{pmatrix} \rightarrow$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 5^{-1} & t^2+t+1 & t^4+t^3+t^2+t+1 & 0 \\ 0 & u+t+1 & (u+t+1)^3+ut & 0 \end{pmatrix}.$$

This latter matrix has rank 2 if and only if $u + t + 1 = 0 = ut$. ∎

(52) $\hat{\pi}_{\overline{0}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line if and only if $\{0, t, u\} = \{0, 1, -1\}$ or $t^2 = u^2 = -1$.

Proof: If $t = 1$ or $u = 1$, the conclusion follows by Eq. (51). So suppose $t \neq 1 \neq u$ and $ut \neq 0$. Then $\hat{\pi}_{\overline{0}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line if and only if 2 equals the rank of the matrix (row reduce)

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -1 \\ 5^{-1} & t^2+t+1 & t^4+t^3+t^2+t+1 & t^5+5t^4+5t^3-5t^2-5t-1 \\ 5^{-1} & u^2+u+1 & u^4+u^3+u^2+u+1 & u^5+5u^4+5u^3-5u^2-5u-1 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -1 \\ 0 & t^2+t & t^4+t^3+t^2+t & t^5+5t^4+5t^3-5t^2-5t \\ 0 & u^2+u & u^4+u^3+u^2+u & u^5+5u^4+5u^3-5u^2-5u \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -1 \\ 0 & t+1 & t^3+t^2+t+1 & t^4+5t^3+5t^2-5t-5 \\ 0 & u+1 & u^3+u^2+u+1 & u^4+5u^3+5u^2-5u-5 \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -1 \\ 0 & t+1 & t^3+t^2+t+1 & t^4+5t^3+5t^2-5t-5 \\ 0 & u-t & \begin{matrix} u^3-t^3+u^2-t^2 \\ +u-t \end{matrix} & \begin{matrix} u^4-t^4 \\ +5(u^3-t^3+u^2-t^2-(u-t)) \end{matrix} \end{pmatrix} \rightarrow$$

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -1 \\ 0 & t+1 & t^3+t^2+t+1 & t^4+5t^3+5t^2-5t-5 \\ 0 & 1 & \begin{matrix} u^2+ut+t^2 \\ +u+t+1 \end{matrix} & \begin{matrix} u^3+u^2t+ut^2+t^3 \\ +5(u^2+ut+t^2+u+t-1) \end{matrix} \end{pmatrix} = M.$$

$$\Delta_{23} = \det \begin{pmatrix} t+1 & t^3+t^2+t+1 \\ 1 & u^2+ut+t^2+u+t+1 \end{pmatrix} = (t+1)(u+1)(u+t).$$

The third form of the matrix given above is symmetric in t and u. Putting $t = -1$ in it gives $\begin{pmatrix} 5^{-1} & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & u+1 & u^3+u^2+u+1 & -- \end{pmatrix}$, which has rank 2 if and only if $u = -1$, an impossibility. So if $\hat{\pi}_{\overline{0}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line it must be that $u + t = 0$. Put $u = -t$ in the matrix labeled M above. This gives

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -1 \\ 0 & t+1 & t^3+t^2+t+1 & t^4+5t^3+5t^2-5t-5 \\ 0 & 1 & t^2+1 & 5(t^2-1) \end{pmatrix}.$$

So finally $\hat{\pi}_{\overline{0}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line if and only if $u = -t$ and

$$0 = \Delta_{34} = \det \begin{pmatrix} t^3+t^2+t+1 & t^4+5(t^3+t^2-t-1) \\ t^2+1 & 5(t^2-1) \end{pmatrix} = -t^4(t^2+1).$$

This completes a proof of Eq. (52). ∎

(53) $\hat{\pi}_{\overline{-1}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line if and only if
$\{-1, t, u\} = \{-1, 0, 1\}$ or $u = -t$ and $t^4 + t^2 + 1 = 0$.

Proof: By the earlier results we may assume $\{t, u\} \cap \{-1, 0, 1\} = \emptyset$. So $\hat{\pi}_{\overline{-1}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line if and only if 2 = rank of

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -2 \\ 5^{-1} & t^2+t+1 & t^4+t^3+t^2+t+1 & t^5+5(t^4+t^3-t^2-t)-1 \\ 5^{-1} & u^2+u+1 & u^4+u^3+u^2+u+1 & u^5+5(u^4+u^3-u^2-u)-1 \end{pmatrix},$$

which row reduces to

$$\begin{pmatrix} 5^{-1} & 1 & 1 & -2 \\ 0 & t & t^3+t & t^4+t^3+t^2+t+1+5(t^3-t) \\ 0 & u & u^3+u & u^4+u^3+u^2+u+1+5(u^3-u) \end{pmatrix}.$$

Here $\Delta_{23} = \det \begin{pmatrix} t & t^3+t \\ u & u^3+u \end{pmatrix} = tu(u^2-t^2) = 0$ if and only if $u = -t \neq 0$. Then with $u = -t$,

$$\Delta_{24} = \det \begin{pmatrix} t & t^4+t^3+t^2+t+1+5(t^3-t) \\ -t & t^4-t^3+t^2-t+1-5(t^3-t) \end{pmatrix} = 2t(t^4+t^2+1) = 0$$

if and only if $t^4 + t^2 + 1 = 0$. ■

For planes of the original flock, π_0 has the property that whenever $\pi_s \cap \pi_t \cap \pi_u$ is a line, π_0 must be one of π_s, π_t, or π_u. And π_0 belongs to $(q-1)/2$ triples $\{\pi_0, \pi_t, \pi_{-t}\}$ whose intersection is a line. We now observe that the derived flock $\mathcal{F}(\mathcal{C}^{i_1})$ has no special plane that plays this role. $\hat{\pi}_{\overline{0}} \cap \hat{\pi}_{\overline{1}} \cap \hat{\pi}_{\overline{-1}}$ is a line, and by Eq. (51) $\hat{\pi}_{\overline{1}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line only for this choice $\{\overline{0}, \overline{-1}\} = \{\overline{t}, \overline{u}\}$. So if there were a special plane it would have to be $\hat{\pi}_{\overline{0}}$ or $\hat{\pi}_{\overline{-1}}$. Eq. (52) says $\hat{\pi}_{\overline{0}} \cap \hat{\pi}_{\overline{t}} \cap \hat{\pi}_{\overline{u}}$ is a line if and only if $\{t, u\} = \{1, -1\}$ or $t^2 = u^2 = -1$. So $\hat{\pi}_{\overline{0}}$ belongs to one or two triples whose intersection is a line.

So consider $\hat{\pi}_{\overline{-1}}$. By Eq. (53) it belongs to at most 3 triples whose intersection is a line. But $(q-1)/2 > 3$ if and only if $q > 7$. And for $q = 7$ derivation actually yields the original flock (cf. [3]). This completes a proof of the desired result.

Theorem 2 *If $\mathcal{C}$ is the q-clan for which $\mathcal{F}(\mathcal{C})$ is the Cohen-Ganley-Barriga flock and $q > 7$, then $\mathcal{C} \not\sim \mathcal{C}^{i_1}$, and $[A(\infty)]$ and $[A(1)]$ are in distinct $\mathcal{G}_0$-orbits on $GQ(\mathcal{C})$.* ■

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Address of Author:
S. E. Payne
Dept. Mathematics
Campus Box 170
CU-Denver
P.O. Box 173364
Denver, Colorado 80217-3364
Email: SPAYNE@CARBON.DENVER.COLORADO.EDU