A Tensor Product Action on q-Clan Generalized Quadrangles with $q = 2^e$

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Abstract

The Fundamental Theorem of q-clan geometry implies (among other things) that all automorphisms of the generalized quadrangle $GQ(\mathcal{C})$ of order (q^2, q) associated with a q-clan $\mathcal{C}$ which fix a special pair of points are automorphisms of the elation group of the quadrangle. When $q = 2^e$, with a slight modification of the usual representation, we describe these automorphisms in terms of tensor products of pairs of matrices in $GL(2, q)$. The resulting efficiency in computation allows a simplified description of the automorphisms of $GQ(\mathcal{C})$. We apply the general theory to give an improved description of the induced stabilizers of the ovals in $PG(2, q)$ that are associated with the new Subiaco q-clans introduced and studied in [2], [7], [1] and [8].

1 Introduction

Let q be any prime power and set $F = GF(q)$. A q-**clan** is a set

$\mathcal{C} = \left\{ A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F \right\}$ of q, 2×2 matrices over F, indexed by the elements of F with the property that for distinct s, $t \in F$, the matrix $A_s - A_t$ is **anisotropic**, i.e. $\alpha(A_s - A_t)\alpha^T = 0$ iff $\alpha = (0, 0)$. Associated with any q-clan is a rather large collection of special kinds of geometrical structures including generalized quadrangles (GQ) with parameters (q^2, q), flocks of a quadratic cone, spreads of $PG(3, q)$, and translation planes of dimension at most 2 over their kernel (cf. [3], [5], [11]). A slight generalization of Theorem IV.1 of [9]

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says that with natural definitions of equivalence for the various geometric structures involved, two q-clans are equivalent iff their corresponding geometries of the same type are equivalent. This theorem was named the Fundamental Theorem of q-Clan Geometry in [1] because of its similarity in spirit and form to the Fundamental Theorem of Projective Geometry.

For a given q-clan $\mathcal{C}$, the associated generalized quadrangle $GQ(\mathcal{C})$ of order (q^2, q) is constructed as a group coset geometry starting with a 4-gonal family $\mathcal{J}(\mathcal{C})$ of subgroups of a standard group $\mathcal{K}$ of order q^5 (see Section 2). The Fundamental Theorem says (among other things) that each automorphism of $GQ(\mathcal{C})$ fixing the points usually labeled (∞) and $(\underline{0}, 0, \underline{0})$ must be induced by an automorphism of $\mathcal{K}$ having a rather special form. The present essay grew out of the observation that when $q = 2^e$ these automorphisms may be described completely in terms of a field automorphism combined with multiplication by the tensor product of two 2×2 matrices over F. Sections 2 through 5 develop this tensor product action so that collineations of $GQ(\mathcal{C})$ can be studied very efficiently. When $q = 2^e$, $GQ(\mathcal{C})$ has subquadrangles of order q associated with ovals in $PG(2, q)$. And in fact the matrix tensor product mentioned above amounts to a tensor product of the group action on the lines through the point (∞) with the group action on the subquadrangles. In Section 6 we consider the collineations induced on $PG(2, q)$ by the automorphism group of $GQ(\mathcal{C})$ that stabilize one of the associated ovals. In Section 7 the general machinery is applied to the Subiaco q-clans. This material extends only slightly the work in [7], [1], and [8]. But it strongly suggests further lines of investigation, and gives a helpful perspective to many of the results in the papers just cited.

The original inspiration for using the tensor product action derived from remarks made by Dr. Tim Penttila, and we thank him for the many insightful and encouraging conversations that accompanied the development of the ideas in this paper.

2 The Underlying Group

For q an arbitrary prime power, let $F = GF(q)$ and put $P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The standard group $\mathcal{K}$ used for q-clan geometry (at least for the generalized quadrangles) is

$$\mathcal{K} = \{(\alpha, c, \beta) \in F^2 \times F \times F^2 : \alpha, \beta \in F^2, c \in F\}, \tag{1}$$

with binary operation

$$(\alpha, c, \beta) \cdot (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta(\alpha')^T, \beta + \beta'). \tag{2}$$

The map $\psi^{\otimes} : (\alpha, c, \beta) \mapsto ((\alpha, \beta P), c) \in F^2 \times F^2 \times F$ is an isomorphism from $\mathcal{K}$ to a group $G^{\otimes}$ given by

$$G^{\otimes} = \{((\alpha, \beta), c) \in F^2 \times F^2 \times F : \alpha, \beta \in F^2, c \in F\}, \tag{3}$$

with binary operation

$$((\alpha,\beta),c)\cdot((\alpha',\beta'),c') = ((\alpha+\alpha',\beta+\beta'),c+c'+\beta\circ\alpha'), \tag{4}$$

where

$$(\alpha,\beta)\mapsto\alpha\circ\beta = -\alpha P\beta^T \tag{5}$$

is a nonsingular, alternating bilinear form. In particular, $\alpha\circ\beta=0$ iff $\{\alpha,\beta\}$ is F-dependent

In the published literature on q-clan geometry with $q=2^e$ there is a third representation of the underlying group. In this representation, which here we denote by G, the typical element is denoted (α,c,β) and the binary operation is $(\alpha,c,\beta)\circ(\alpha',c',\beta') = (\alpha+\alpha',c+c'+\sqrt{\beta\circ\alpha'},\beta+\beta')$. However, in this article we prefer to write $((\alpha,\beta),c)$ in place of (α,c,β). Then

$$\psi : G^{\otimes}\to G : ((\alpha,\beta),c)\mapsto((\alpha,\beta),c^{\frac{1}{2}}) \tag{6}$$

is an isomorphism from $G^{\otimes}$ to the group G with binary operation

$$((\alpha,\beta),c)\circ((\alpha',\beta'),c') = ((\alpha+\alpha',\beta+\beta'),c+c'+\sqrt{\beta\circ\alpha'}). \tag{7}$$

This latter representation only works when $q=2^e$.

For the remainder of this paper we assume that $q=2^e$.

There are two families of subgroups of $G^{\otimes}$, each subgroup having order q^3 and being elementary abelian, that play important roles in q-clan geometry.

$$\text{For }\ \underline{0}\neq\gamma\in F^2, \mathcal{L}_\gamma = \{(\gamma\otimes\alpha,c)\in G^{\otimes} : \alpha\in F^2, c\in F\}, \tag{8}$$

$$\text{For }\ \underline{0}\neq\alpha\in F^2, \mathcal{R}_\alpha = \{(\gamma\otimes\alpha,c)\in G^{\otimes} : \gamma\in F^2, c\in F\}. \tag{9}$$

For nonzero α, $\gamma\in F^2$, $\mathcal{L}_\gamma=\mathcal{L}_\alpha$ (resp., $\mathcal{R}_\gamma=\mathcal{R}_\alpha$) iff $\{\alpha,\gamma\}$ is F-dependent. Hence we may think of the $\mathcal{L}_\gamma$ (resp., $\mathcal{R}_\alpha$) as indexed by the points of $PG(1,q)$. The scalar multiplication $d((\gamma\otimes\alpha),c) = (d\gamma\otimes\alpha,d^2c) = (\gamma\otimes d\alpha,d^2c)$ makes each $\mathcal{L}_\gamma$ and $\mathcal{R}_\alpha$ into a 3-dimensional vector space over F. So there are associated projective planes $\bar{L}_\gamma\cong\bar{R}_\alpha\cong PG(2,q)$. Clearly the center $Z=\{((\underline{0},\underline{0}),c)\in G^{\otimes} : c\in F\}$ of $G^{\otimes}$ is contained in each L_γ and R_α.

Let $\sigma\in\mathrm{Aut}(F)$, $0\neq x\in F$, and let H and S be 4×4 matrices over F. Define $\theta(\sigma,x,H,S) : G^{\otimes}\to G^{\otimes}$ by

$$\theta(\sigma,x,H,S) : ((\alpha,\beta),c)\mapsto((\alpha^\sigma,\beta^\sigma)H, xc^\sigma+(\alpha^\sigma,\beta^\sigma)S(\alpha^\sigma,\beta^\sigma)^T). \tag{10}$$

A routine computation shows that

$$\theta(\sigma_1,x_1,H_1,S_1)\cdot\theta(\sigma_2,x_2,H_2,S_2) = \theta(\sigma_1\cdot\sigma_2, x_1^{\sigma_2}x_2, H_1^{\sigma_2}H_2, x_2S_1^{\sigma_2}+H_1^{\sigma_2}S_2(H_1^{\sigma_2})^T). \tag{11}$$

For a given $\theta = \theta(\sigma, x, H, S)$, we may write H and S in block form, with 2×2 blocks. Without loss of generality we may assume that S is in block-upper triangular form. Say $H = \begin{pmatrix} A & K \\ B & M \end{pmatrix}$, $S = \begin{pmatrix} C & D \\ O & E \end{pmatrix}$. Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ means $a = x$, $d = w$, and $b + c = y + z$. Clearly C (resp., E) may be replaced with any matrix $C' \equiv C$ (resp., $E' \equiv E$), so we usually take C and E to be upper triangular. Then a computation similar to that used in the proof of 10.5.2 of [10] shows that θ is an automorphism of $G^{\otimes}$ if and only if H is nonsingular and

$$C + C^T = KPA^T \tag{12}$$

$$D = KPB^T \tag{13}$$

$$D = APM^T + xP \tag{14}$$

$$E + E^T = MPB^T \tag{15}$$

For each automorphism of $G^{\otimes}$ of interest here there will be matrices $\bar{A} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$ and $\bar{B}$, both in $GL(2, q)$, for which $A = d\bar{B}$, $B = c\bar{B}$, $K = b\bar{B}$, $M = a\bar{B}$, i.e., $H = \bar{A} \otimes \bar{B}$. Put $\mu = \det(\bar{A})$, $\Delta = \det(\bar{B})$. Then we may rewrite Eq. 10 as

$$\begin{gathered} \theta = \theta(\sigma, x, \bar{A} \otimes \bar{B}, \begin{pmatrix} C & D \\ O & E \end{pmatrix}) : ((\alpha, \beta), c) \mapsto \\ ((\alpha^\sigma, \beta^\sigma)(\bar{A} \otimes \bar{B}), xc^\sigma + \alpha^\sigma C(\alpha^\sigma)^T + \alpha^\sigma D(\beta^\sigma)^T + \beta^\sigma E(\beta^\sigma)^T). \end{gathered} \tag{16}$$

And the conditions of Eqs. 12 through 15 are equivalent to the following.

$$C + C^T = \Delta bdP \tag{17}$$

$$D = \Delta bcP \tag{18}$$

$$x = \mu\Delta = \det(\bar{A} \otimes \bar{B}) \tag{19}$$

$$E + E^T = \Delta acP. \tag{20}$$

This means that the symbols x and D may be suppressed in the notation for θ. So temporarily we write

$$\begin{gathered} \theta = \theta(\sigma, \bar{A} \otimes \bar{B}, C, E) : ((\alpha, \beta), c) \mapsto \\ ((\alpha^\sigma, \beta^\sigma)(\bar{A} \otimes \bar{B}),\ \mu\Delta c^\sigma + \alpha^\sigma C(\alpha^\sigma)^T + \Delta bc(\alpha^\sigma \circ \beta^\sigma) + \beta^\sigma E(\beta^\sigma)^T), \end{gathered} \tag{21}$$

where $\mu = \det(\bar{A})$, $\Delta = \det(\bar{B})$, $C + C^T = \Delta bdP$ and $E + E^T = \Delta acP$.

NOTE:

$$\theta(\sigma, \bar{A} \otimes \bar{B}, C, E) : \mathcal{L}_\gamma \to \mathcal{L}_{\gamma^\sigma \bar{A}}, \tag{22}$$

$$\theta(\sigma, \bar{A} \otimes \bar{B}, C, E) : \mathcal{R}_\alpha \to \mathcal{R}_{\alpha^\sigma \bar{B}}. \tag{23}$$

Identify elements of $\tilde{F} = F \cup \{\infty\}$ with points of $PG(1,q)$ in the following way. For $t \in F$, write $\gamma_t = (1,t)$, and $\gamma_\infty = (0,1)$. So for $\underline{0} \neq \gamma = (a,b)$, $\gamma \equiv \gamma_{b/a}$. For $t \in \tilde{F}$, we then write $\mathcal{L}_t = \mathcal{L}_{\gamma_t}$ and $\mathcal{R}_t = \mathcal{R}_{\gamma_t}$. If $\bar{A} = \begin{pmatrix} a_4 & a_2 \\ a_3 & a_1 \end{pmatrix}$ and $\bar{B} = \begin{pmatrix} b_4 & b_2 \\ b_3 & b_1 \end{pmatrix}$, then we have

$$\theta(\sigma, \bar{A} \otimes \bar{B}, C, E) : \mathcal{L}_t \to \mathcal{L}_{\bar{t}}, \text{where } \bar{t} = \frac{a_1 t^\sigma + a_2}{a_3 t^\sigma + a_4},\ t \in \tilde{F}. \tag{24}$$

$$\theta(\sigma, \bar{A} \otimes \bar{B}, C, E) : \mathcal{R}_t \to \mathcal{R}_{\bar{t}}, \text{where } \bar{t} = \frac{b_1 t^\sigma + b_2}{b_3 t^\sigma + b_4},\ t \in \tilde{F}. \tag{25}$$

3 q-Clans

A q-**clan** is a set $\mathcal{C} = \{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ of q distinct (upper triangular) 2×2 matrices over F for which all pairwise differences $A_t - A_s$, $t \neq s$, are anisotropic. $\mathcal{C}$ is τ-**normalized** for some $\tau \in \text{Aut}(F)$ provided $A_t = \begin{pmatrix} x_t & t^\tau \\ 0 & z_t \end{pmatrix}$ for all $t \in F$, and $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. If $\mathcal{C}$ is τ-normalized for $\tau = id$, we usually just say that $\mathcal{C}$ is **normalized**. Also put $A_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but put $\gamma_{y_\infty} = (0,1)$.

When the q-clan $\mathcal{C} = \{A_t \in F\}$ is understood to be given, we write $g_t(\alpha) = \alpha A_t \alpha^T$, for $t \in \tilde{F}$, $\alpha \in F^2$. Then the following is easily checked.

$$g_t(\alpha + \beta) \quad = \quad g_t(\alpha) + g_t(\beta) + y_t(\alpha \circ \beta). \tag{26}$$

If $\mathcal{C}' = \{A'_t : t \in F\}$ is a second (not necessarily distinct) q-clan, we write $g'_t(\alpha) = \alpha A'_t \alpha^T$.

Let $\mathcal{C} = \{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ be a given q-clan. Then we may define subgroups of $G^\otimes$ as follows.

$$\text{For } t \in \tilde{F},\ A(t) = \{(\gamma_{y_t} \otimes \alpha, g_t(\alpha)) \in G^\otimes : \alpha \in F^2\}. \tag{27}$$

It follows that $A(t)$ is a subgroup of $\mathcal{L}_{y_t}$ having order q^2. And as usual, $\mathcal{J}(\mathcal{C}) = \{A(t) : t \in \tilde{F}\}$ is a 4-gonal family for $G^\otimes$ with $\mathcal{L}_{y_t}$ being the **tangent space** of $\mathcal{J}(\mathcal{C})$ at $A(t)$. (See [1] for details using the isomorphic group G with only superficially different notation.) The notation here has been chosen so that if $\mathcal{C}$ is normalized (with $\tau = id$), then for $\gamma = (a,b) \in PG(1,q)$,

$$A(\gamma) = A(b/a) = \{(\gamma \otimes \alpha, \alpha A_{b/a} \alpha^T) : \alpha \in F^2\}. \tag{28}$$

For some $\tau \in \text{Aut}(F)$, let $\mathcal{C} = \{A_t : t \in \tilde{F}\}$ and $\mathcal{C}' = \{A'_t : t \in \tilde{F}\}$ be two (not necessarily distinct) τ^{-1}-normalized q-clans. Let $\theta = \theta(\sigma, \bar{A} \otimes \bar{B}, C, E)$ be an automorphism of $G^\otimes$ as in Eq.(10). We want to determine **necessary** conditions on C and E for θ to map the members of $\mathcal{J}(\mathcal{C})$ to the members of $\mathcal{J}(\mathcal{C}')$. We know by Eq. 24 that $\theta : \mathcal{L}_{y_t} \to \mathcal{L}_{\bar{y}_t}$, where $\bar{y}_t = \dfrac{a_1 y_t^\sigma + a_2}{a_3 y_t^\sigma + a_4}$. So putting $y_t = t^{\tau^{-1}}$, we have $\bar{y}_t = \left(\dfrac{a_1^\tau t^\sigma + a_2^\tau}{a_3^\tau t^\sigma + a_4^\tau}\right)^{\tau^{-1}}$.

Since $A(t) \leq \mathcal{L}_{y_t}$ and $A'(t) \leq \mathcal{L}_{y'_t}$, if θ does map the members of $\mathcal{J}(\mathcal{C})$ to those of $\mathcal{J}(\mathcal{C}')$, it must map as follows.

$$\theta : A(t) \to A'(t'), \text{where } t' = \frac{a_1^\tau t^\sigma + a_2^\tau}{a_3^\tau t^\sigma + a_4^\tau}, \text{since } y_t = t^{\tau^{-1}} = y'_t. \tag{29}$$

First consider $t = 0$, so $t' = 0' = (a_2/a_4)^\tau$. (By convention $\infty^\tau = \infty$.) So consider the image of $A(0)$. Here $\theta(\sigma, \bar{A} \otimes \bar{B}, C, E) : ((1,0) \otimes \alpha, 0) \mapsto$
$((1,0)\bar{A} \otimes \alpha^\sigma \bar{B}, (\alpha^\sigma, \underline{0}) \begin{pmatrix} C & \Delta a_2 a_3 P \\ 0 & E \end{pmatrix} (\alpha^\sigma, \underline{0})^T) = ((a_4, a_2) \otimes \alpha^\sigma \bar{B}, \alpha^\sigma C (\alpha^\sigma)^T)$, which must be in $A'((a_2/a_4)^\tau)$. If $a_4 \neq 0$, put $s = a_2/a_4 \in F$, so $(a_4, a_2) \equiv (1, s) = \gamma_s$. And $((a_4, a_2) \otimes \alpha^\sigma \bar{B}$, $\alpha^\sigma C(\alpha^\sigma)^T) = ((1,s) \otimes a_4 \alpha^\sigma \bar{B},\ \alpha^\sigma C(\alpha^\sigma)^T)$ is in $\mathcal{L}_s$, which contains $A'(s^\tau)$. Then $((1,s) \otimes a_4 \alpha^\sigma \bar{B}, \alpha^\sigma C(\alpha^\sigma)^T) = (\gamma_{s^\tau} \otimes a_4 \alpha^\sigma \bar{B}, \alpha^\sigma C(\alpha^\sigma)^T)$ is in $A'(s^\tau)$ iff $\alpha^\sigma C(\alpha^\sigma)^T = g'_{s^\tau}(a_4 \alpha^\sigma \bar{B}) = a_4^2 \alpha^\sigma \bar{B} A'_{s^\tau} \bar{B}^T (\alpha^\sigma)^T$. This holds for all $\alpha \in F^2$ iff $C \equiv a_4^2 \bar{B} A'_{s^\tau} \bar{B}^T$. Now suppose $a_4 = 0$, so $s = \infty$. Then $((0, a_2) \otimes \alpha^\sigma \bar{B}, \alpha^\sigma C(\alpha^\sigma)^T) \in A'(\infty) = A'(\infty^\tau)$ iff $\alpha^\sigma C(\alpha^\sigma)^T = 0$. This holds for all $\alpha \in F^2$ iff $C \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. But $C + C^T = \Delta a_2 a_4 P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and we take C to be upper triangular. Hence $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = a_4^2 \bar{B} A'_{\infty^\tau} \bar{B}^T$, and in all cases we have

$$C \equiv a_4^2 \bar{B} A'_{(a_2/a_4)^\tau} \bar{B}^T. \tag{30}$$

Now consider $t = \infty$, so $t' = \infty' = (a_1/a_3)^\tau$. Arguing exactly as in the previous case, we find

$$E \equiv a_3^2 \bar{B} A'_{(a_1/a_3)^\tau} \bar{B}^T. \tag{31}$$

This completes a proof of the following theorem (where we use $\bar{B} P \bar{B}^T = \Delta P$).

Theorem 3.1 *Let* $\sigma \in Aut(F)$, $\bar{A} = \begin{pmatrix} a_4 & a_2 \\ a_3 & a_1 \end{pmatrix} \in GL(2,q)$, $\bar{B} \in GL(2,q)$, $\mu = det(\bar{A})$, $\Delta = det(\bar{B})$. *Suppose that* $\theta = \theta(\sigma, \bar{A} \otimes \bar{B}, C, E)$ *as in Eq.(10) is an automorphism of* $G^\otimes$ *mapping the 4-gonal family* $\mathcal{J}(\mathcal{C})$ *to the 4-gonal family* $\mathcal{J}(\mathcal{C}')$, *where both* $\mathcal{C}$ *and* $\mathcal{C}'$ *are* τ^{-1}*-normalized. Then the following hold:*

i. $\begin{pmatrix} C & D \\ 0 & E \end{pmatrix} = (I \otimes \bar{B}) \begin{pmatrix} a_4^2 A'_{(a_2/a_4)^\tau} & a_2 a_3 P \\ 0 & a_3^2 A'_{(a_1/a_3)^\tau} \end{pmatrix} (I \otimes \bar{B})^T.$

ii. $\theta(\sigma, \bar{A} \otimes \bar{B}) : ((\alpha, \beta), c) \mapsto ((\alpha^\sigma, \beta^\sigma)(\bar{A} \otimes \bar{B}),\ \mu\Delta c^\sigma +$ $a_4^2 \alpha^\sigma (\bar{B} A'_{(a_2/a_4)^\tau} \bar{B}^T)(\alpha^\sigma)^T + \Delta a_2 a_3 (\alpha^\sigma \circ \beta^\sigma) + a_3^2 \beta^\sigma (\bar{B} A'_{(a_1/a_3)^\tau} \bar{B}^T)(\beta^\sigma)^T)$. *And the permutation* $A(t) \mapsto A'(t')$ *is given by*

iii. $t' = \dfrac{a_1^\tau t^\sigma + a_2^\tau}{a_3^\tau t^\sigma + a_4^\tau}$.

It may seem that the field automorphism τ^{-1} just unnecessarily complicates matters. But $\tau = id$ is the easiest automorphism to compute with in the general case, and $\tau = 2$ is the standard choice for the Subiaco q-clans that we study in Section 7.

4 The Fundamental Theorem

Let $\mathcal{C} = \{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ and $\mathcal{C}' = \{A'_t = \begin{pmatrix} x'_t & y'_t \\ 0 & z'_t \end{pmatrix} : t \in F\}$ be two q-clans with $A_0 = A'_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Let $GQ(\mathcal{C})$, $GQ(\mathcal{C}')$ be their associated GQ of order (q^2, q) (cf.[5],[1]. A full statement of the Fundamental Theorem includes the assertion that if $\theta : GQ(\mathcal{C}) \to GQ(\mathcal{C}')$ is an somorphism mapping (∞), $(\underline{0}, 0, \underline{0}) \in GQ(\mathcal{C})$, to (∞), $(\underline{0}, 0, \underline{0}) \in GQ(\mathcal{C}')$, respectively, then θ is induced by an automorphism (also called θ) of $G^\otimes$ mapping the 4-gonal family $\mathcal{J}(\mathcal{C})$ to $\mathcal{J}(\mathcal{C}')$. For our purposes here it suffices to give that part of the theorem that characterizes such automorphisms of $G^\otimes$ mapping $A(\infty)$ to $A'(\infty)$.

Theorem 4.1 *The Fundamental Theorem (Partial Version!) There is an automorphism* $\theta : G^\otimes \to G^\otimes$ *mapping* $\mathcal{J}(\mathcal{C})$ *to* $\mathcal{J}(\mathcal{C}')$, *and mapping* $A(\infty)$ *to* $A'(\infty)$, *for each 4-tuple* $(\lambda, B, \sigma, \pi)$, $0 \neq \lambda \in F$, $B \in GL(2, q)$, $\sigma \in Aut\ (F)$, $\pi : F \to F : t \mapsto \bar{t}$ *a permutation, for which*

(i) $A'_{\bar{t}} \equiv \lambda B^{-1} A_t^\sigma B^{-T} + A'_{\bar{0}}$ *for all* $t \in F$.

The corresponding automorphism $\theta : G^\otimes \to G^\otimes$ *is given by (put* $\Delta = det(B)$*):*

(ii) $\theta = \theta(\sigma, \begin{pmatrix} 1 & y'_{\bar{0}} \\ 0 & \lambda/\Delta \end{pmatrix} \otimes B) : ((\alpha, \beta), c) \mapsto ((\alpha^\sigma, \beta^\sigma)[\begin{pmatrix} 1 & y'_{\bar{0}} \\ 0 & \lambda/\Delta \end{pmatrix} \otimes B], \lambda c^\sigma + g'_{\bar{0}}(\alpha^\sigma B))$.

(iii) $\pi : t \mapsto \bar{t}$ *satisfies* $y'_{\bar{t}} = (\lambda/\Delta) y_t^\sigma + y'_{\bar{0}}$.

Conversely, all such automorphisms of G must be of this form.

For $0 \neq a \in F$, $\theta_a = \theta(id, I \otimes aI) : ((\alpha, \beta), c) \mapsto ((a\alpha, a\beta), a^2 c)$ is an automorphism of $G^\otimes$ that leaves invariant each $A(t)$ for each $t \in \tilde{F}$ and for each q-clan $\mathcal{C}$.

$\mathcal{N} = \{\theta_a : 0 \neq a \in F\}$ is the q-**clan kernel**, or just the **kernel** (cf. II.3 and II.4 of [8].

For any θ as in Theorem 3.1, Eq.(6) has a particularly simple form.

$$\theta(\sigma_1, A_1 \otimes B_1) \cdot \theta(\sigma_2, A_2 \otimes B_2) = \theta(\sigma_1 \cdot \sigma_2, A_1^{\sigma_2} A_2 \otimes B_1^{\sigma_2} B_2). \tag{32}$$

So $\theta(\sigma, A\otimes B)\cdot\theta(id, I\otimes aI) = \theta(\sigma, A\otimes aB)$, and $\theta(id, I\otimes aI)\cdot\theta(\sigma, A\otimes B) = \theta(\sigma, A\otimes a^\sigma B)$. Hence $\theta(\sigma, A\otimes B)$ commutes with $\theta(id, I\otimes aI)$ for all nonzero $a\in F$ iff $\sigma = id$.

We say that $\theta = \theta(\sigma, A\otimes B)$ as in Theorem 3.1 is **linear** provided $\sigma = id$, and is **special** provided $\Delta = \det(B) = 1$. Clearly the product of two linear (resp. special) automorphisms θ is again linear (resp., special).

For any θ as in Theorem 3.1,

$$\theta(\sigma, A\otimes B) = \theta(\sigma, I\otimes I)\cdot\theta(id, A\otimes\Delta^{-\frac{1}{2}}B)\cdot\theta(id, I\otimes\Delta^{\frac{1}{2}}I), \tag{33}$$

where $\theta_\sigma = \theta(\sigma, I\otimes I) : ((\alpha,\beta),c)\mapsto((\alpha^\sigma,\beta^\sigma),c^\sigma)$, $\theta(id, A\otimes\Delta^{-\frac{1}{2}}B)$ is special linear, and $\theta(id, I\otimes\Delta^{\frac{1}{2}}I)$ is in the kernel. To study all automorphisms of $G^\otimes$ mapping $\mathcal{J}(\mathcal{C})$ to $\mathcal{J}(\mathcal{C}')$ it clearly suffices to study just those that are special. This suggests that in characteristic 2 we should use a variation on the notion of equivalence of q-clans. In the conditions of Theorem 4.1 replace λ with $\mu = \lambda/\Delta$ and B^{-1} with $\Delta^{\frac{1}{2}}B^{-1}$. So $\det(\Delta^{\frac{1}{2}}B^{-1}) = 1$. Then we obtain a variation of (part of) the Fundamental Theorem in the following simpler form ($q = 2^e$ only!).

Theorem 4.2 *The Fundamental Theorem (partial version for $q = 2^e$). For q-clans $\mathcal{C}$ and $\mathcal{C}'$ as in Theorem 4.1, there is an automorphism θ of $G^\otimes$ mapping $\mathcal{J}(\mathcal{C})$ to $\mathcal{J}(\mathcal{C}')$ and $A(\infty)$ to $A'(\infty)$ for each 4-tuple $(\mu, B, \sigma, \pi)\in F^\bullet\times SL(2,q)\times Aut(F)\times\mathcal{S}_F$ for which*

(i) $A'_{\bar t}\equiv \mu B^{-1}A_t^\sigma B^{-T} + A'_{\bar 0}$ *for all* $t\in F$.

The associated automorphism $\theta : G^\otimes\to G^\otimes$ *is given by*

(ii) $\theta = \theta(\sigma, \begin{pmatrix}1 & y'_{\bar 0}\\ 0 & \mu\end{pmatrix}\otimes B) : ((\alpha,\beta),c)\mapsto((\alpha^\sigma,\beta^\sigma)[\begin{pmatrix}1 & y'_{\bar 0}\\ 0 & \mu\end{pmatrix}\otimes B], \mu c^\sigma + g'_{\bar 0}(\alpha^\sigma B))$.

(iii) $\pi : t\mapsto\bar t$ *satisfies* $y'_{\bar t} = \mu y_t^\sigma + y'_{\bar 0}$.

5 The Shift-Flip in $G^\otimes$

To begin with let $\mathcal{C} = \{A_t = \begin{pmatrix}x_t & y_t\\ 0 & z_t\end{pmatrix} : t\in F\}$ be a fixed q-clan with $A_0 = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} = A_\infty$. Recall the shift-flip of [1], but for the group $G^\otimes$.

$$\textbf{Shift by} s : \tau_s = \theta(id, \begin{pmatrix}1 & y_s\\ 0 & 1\end{pmatrix}\otimes I) : ((\alpha,\beta),c)\mapsto((\alpha, y_s\alpha+\beta), c+\alpha A_s\alpha^T). \tag{34}$$

$$\textbf{Flip} : \phi = \theta(id, P\otimes I) : ((\alpha,\beta),c)\mapsto((\beta,\alpha), c+\alpha\circ\beta). \tag{35}$$

$$\textbf{Shift-Flip} : i_s = \tau_s\cdot\phi = \theta(id, \begin{pmatrix}y_s & 1\\ 1 & 0\end{pmatrix}\otimes I) : ((\alpha,\beta),c)\mapsto((y_s\alpha+\beta,\alpha), c+\alpha A_s\alpha^T+\alpha\circ\beta). \tag{36}$$

Also define $i_\infty = id : G^\otimes \to G^\otimes$.

Each of the maps defined in Eqs. 34, 35 and 36 is an automophism of $G^\otimes$ that replaces the 4-gonal family $\mathcal{J}(\mathcal{C})$ with some 4-gonal family $\mathcal{J}'$. We want to compute a q-clan $\mathcal{C}^{i_s}$ so that i_s replaces $\mathcal{J}(\mathcal{C})$ with $\mathcal{J}(\mathcal{C}^{i_s})$.

Consider the image under i_s, $s \in F$, of the typical element of $A(t)$. Here $i_s : (\gamma_{y_t} \otimes \alpha, g_t(\alpha)) \mapsto ((\gamma_{y_t} \otimes \alpha)[\begin{pmatrix} y_s & 1 \\ 1 & 0 \end{pmatrix} \otimes I], g_t(\alpha) + \alpha A_s \alpha^T)$
$= (\gamma_{y_t} \begin{pmatrix} y_s & 1 \\ 1 & 0 \end{pmatrix} \otimes \alpha, g_t(\alpha) + g_s(\alpha)) = ((y_s + y_t, 1) \otimes \alpha, g_t(\alpha) + g_s(\alpha))$
$= ((1, \{y_s + y_t\}^{-1}) \otimes (y_s + y_t)\alpha, g_t(\alpha) + g_s(\alpha))$ (Put $\beta = (y_s + y_t)\alpha$)
$= (\gamma_{(y_s+y_t)^{-1}} \otimes \beta, (y_s + y_t)^{-2}(g_t(\beta) + g_s(\beta)))$.

To go further we need some kind of additional normalization on $\mathcal{C}$. **So from now on suppose $\mathcal{C}$ is τ^{-1}-normalized**, $\tau^{-1} \in \text{Aut}(F)$, i.e., $y_t = t^{\frac{1}{\tau}}$, $t \in F$. We want the image q-clan $\mathcal{C}^{i_s}$ to be τ^{-1}-normalized also. Then $(\gamma_{t^{\frac{-1}{\tau}}} \otimes \alpha, g_t(\alpha)) \in A(t)$ is mapped by i_s to $(\gamma_{(s+t)^{\frac{-1}{\tau}}} \otimes \beta, (s+t)^{\frac{-2}{\tau}}(g_t(\beta) + g_s(\beta)))$, for $\beta = (s+t)^{\frac{1}{\tau}}\alpha$. This image must be in $A^{i_s}((s+t)^{-1})$, since we want $\mathcal{C}^{i_s}$ also to be τ^{-1}-normalized. Hence $A^{i_s}_{(s+t)^{-1}} = (s+t)^{\frac{-2}{\tau}}(A_t + A_s)$. (Treat the case $s = t$ separately.) Put $x = (t+s)^{-1}$, so $t = s + x^{-1}$. Then

$$A^{i_s}_x = x^{\frac{2}{\tau}}(A_{s+x^{-1}} + A_s), \text{ for all } x,\ s \in F. \tag{37}$$

Here $i_s : 0 \mapsto \bar{0} = s^{-1}$, so $A^{i_s}_{\bar{0}} = A^{i_s}_{s^{-1}} = s^{\frac{-2}{\tau}} A_s$, for $0 \neq s \in F$. Then in $i_s = \theta(id, \begin{pmatrix} s^{\frac{1}{\tau}} & 1 \\ 1 & 0 \end{pmatrix} \otimes I)$, $a_4^2 A^{i_s}_{(a_2/a_4)^\tau} = s^{\frac{2}{\tau}} A^{i_s}_{s^{-1}} = A_s$, $a_2 a_3 = 1$, and $a_3^2 A^{i_s}_{(a_1/a_3)^\tau} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence i_s does have the form prescribed by Theorem 3.1.

By the F.T., the most general special automorphism of $G^\otimes$ mapping $\mathcal{J}(\mathcal{C})$ to $\mathcal{J}(\mathcal{C}^{i_s})$ and $A(\infty)$ to $A^{i_s}(\infty)$ is $\bar{\theta} = \theta(\sigma, \begin{pmatrix} 1 & \bar{0}^{\frac{1}{\tau}} \\ 0 & \mu \end{pmatrix} \otimes B)$, where $A^{i_s}_{\bar{t}} + A^{i_s}_{\bar{0}} \equiv \mu B^{-1} A^\sigma_t B^{-T}$, for $0 \neq \mu \in F$, $\sigma \in \text{Aut}(F)$, $B \in SL(2,q)$. Using Eq. 37 we may compute

$$\begin{gathered} \bar{\theta} = \theta(\sigma, \begin{pmatrix} 1 & \bar{0}^{\frac{1}{\tau}} \\ 0 & \mu \end{pmatrix} \otimes B) : \mathcal{J}(\mathcal{C}) \to \mathcal{J}(\mathcal{C}^{i_s}) : A(\infty) \mapsto A^{i_s}(\infty) \\ \text{iff (use } \bar{t} = \mu^\tau t^\sigma + \bar{0}) \ \ \mu B^{-1} A^\sigma_t B^{-T} \equiv A^{i_s}_{\bar{t}} + A^{i_s}_{\bar{0}} \\ = (\mu^2 t^{\frac{2\sigma}{\tau}} + \bar{0}^{\frac{2}{\tau}}) A_{\left(\frac{s\mu^\tau t^\sigma + 1 + \bar{0}s}{\mu^\tau t^\sigma + \bar{0}}\right)} + \mu^2 t^{\frac{2\sigma}{\tau}} A_s + \bar{0}^{\frac{2}{\tau}} A_{s+\bar{0}^{-1}}, \\ \text{for all } t \in F. \end{gathered} \tag{38}$$

(Here $\bar{0}$ is determined by condition (i) of Theorem 4.2.)

In applications it is usually easier to compute $A^{i_s}_t$ for all $t \in F$ before applying the F.T.

And then $\theta = \bar{\theta}\circ i_s^{-1} = \theta(\sigma, \begin{pmatrix} 1 & \bar{0}^{\frac{1}{\tau}} \\ 0 & \mu \end{pmatrix} \otimes B)\circ\theta(id, \begin{pmatrix} 0 & 1 \\ 1 & s^{\frac{1}{\tau}} \end{pmatrix} \otimes I) = \theta(\sigma, \begin{pmatrix} \bar{0}^{\frac{1}{\tau}} & (1+\bar{0}s)^{\frac{1}{\tau}} \\ \mu & \mu s^{\frac{1}{\tau}} \end{pmatrix} \otimes B)$.
We collect these results and slightly more in @ the following theorem. (Note: To achieve brevity in at least one aspect, we have more or less ignored the associated $GQ(\mathcal{C})$, the flock $\mathcal{F}(\mathcal{C})$ of a quadratic cone, etc. But the reader should keep in mind that the F.T. also asserts something about isomorphisms of these q-clan geometries.)

Theorem 5.1 *The most general special automorphism of $G^{\otimes}$ leaving $\mathcal{F}(\mathcal{C})$ invariant and mapping $A(\infty)$ to $A(s)$, for $s \in F$, is given by*

(i) $\theta(\sigma, \begin{pmatrix} \bar{0} & 1+\bar{0}s \\ \mu^\tau & \mu^\tau s \end{pmatrix}^{\frac{1}{\tau}} \otimes B) : A(t) \to A(t')$, *where*

(ii) $t' = \dfrac{s\mu^\tau t^\sigma + \bar{0}s + 1}{\mu^\tau t^\sigma + \bar{0}}$,

subject to the condition that there exist $\sigma \in Aut(F)$, $B \in SL(2,q)$, $0 \neq \mu \in F$, and a permutation $\pi : t \mapsto \bar{t} = \mu^\tau t^\sigma + \bar{0}$ for which

(iii) $\mu B^{-1} A_t^\sigma B^{-T} \equiv A_{\bar{t}}^{i_s} + A_{\bar{0}}^{i_s}$

(as in Eq. 38), for all $t \in F$.

In this case

(iv) $\theta(\sigma, \begin{pmatrix} \bar{0} & (1+\bar{0}s) \\ \mu^\tau & \mu^\tau s \end{pmatrix}^{\frac{1}{\tau}} \otimes B) : ((\alpha,\beta),c) \mapsto$

$((\alpha^\sigma,\beta^\sigma)[\begin{pmatrix} \bar{0} & 1+\bar{0}s \\ \mu^\tau & \mu^\tau s \end{pmatrix}^{\frac{1}{\tau}} \otimes B], \mu c^\sigma +$

$(\alpha^\sigma,\beta^\sigma)(I\otimes B)\begin{pmatrix} \bar{0}^{\frac{2}{\tau}} A_{(s+\bar{0}^{-1})} & \mu(1+\bar{0}s)^{\frac{1}{\tau}} P \\ 0 & \mu^2 A_s \end{pmatrix}(I\otimes B^T)(\alpha^\sigma,\beta^\sigma)^T)$.

Moreover, if $\bar{\theta} = \theta(\sigma, \begin{pmatrix} 1 & \bar{0} \\ 0 & \mu^\tau \end{pmatrix}^{\frac{1}{\tau}} \otimes B)$ is a typical special automorphism of $G^{\otimes}$ mapping $\mathcal{J}(\mathcal{C}^{i_s})$ to $\mathcal{J}(\mathcal{C}^{i_s})$ and leaving $A^{i_s}(\infty)$ invariant, then

(v) $\theta = i_s \cdot \bar{\theta} \cdot i_s^{-1} = \theta(\sigma, \begin{pmatrix} \mu^\tau + \bar{0}s^\sigma & \mu^\tau s + s^\sigma + \bar{0}s^{\sigma+1} \\ \bar{0} & 1+\bar{0}s \end{pmatrix}^{\frac{1}{\tau}} \otimes B)$ *is the typical special automorphism of $G^{\otimes}$ leaving $\mathcal{J}(\mathcal{C})$ invariant and fixing $A(s)$.*

We now know that every automorphism of the generalized quadrangle $GQ(\mathcal{C})$ fixing (∞) and $(\underline{0},0,\underline{0})$ has the form $\theta(\sigma, A\otimes B)$, where by Theorem 3.1 this notation completly identifies the map $\theta(\sigma, A\otimes B)$ as an automorphism of $G^{\otimes}$.

6 Ovals in $\bar{R}_\alpha$

We continue to assume that all q-clans are τ^{-1}-normalized.

For $\alpha \in PG(1,q)$, $A(t) \cap \mathcal{R}_\alpha = \{d(\gamma_{t^{\frac{1}{\tau}}} \otimes \alpha, g_t(\alpha)) = (\gamma_{t^{\frac{1}{\tau}}} \otimes d\alpha, g_t(d\alpha)) : d \in F\}$. So $A(t) \cap \mathcal{R}_\alpha$ is the point $\bar{p}_\alpha(t) = (\gamma_{t^{\frac{1}{\tau}}} \otimes \alpha, g_t(\alpha))$ in $\bar{\mathcal{R}}_\alpha$. And $\bar{\mathcal{O}}_\alpha = \{\bar{p}_\alpha(t) : t \in \tilde{F}\}$ is an oval in $\bar{\mathcal{R}}_\alpha$. The map (use $(a,b)^{(2)} = (a^2, b^2)$)

$$\pi_\alpha : \bar{\mathcal{R}}_\alpha \to PG(2,q) : (\gamma \otimes \alpha, c) \mapsto (\gamma^{(2)}, c)$$

is an isomorphism mapping $\bar{\theta}_\alpha$ to the oval

$$\mathcal{O}_\alpha = \{p_\alpha(t) = (\gamma_{t^{2/\tau}}, g_t(\alpha)) : t \in \tilde{F}\} \text{ in } PG(2,q). \tag{39}$$

Suppose that
$\theta = \theta(\sigma, A \otimes B) : ((\alpha,\beta),c) \mapsto ((\alpha^\sigma,\beta^\sigma)(A \otimes B), \mu c^\sigma +$
$(\alpha^\sigma,\beta^\sigma)(I \otimes B)\begin{pmatrix} a_4^2 A_{(a_2/a_4)^\tau} & a_2 a_3 P \\ 0 & a_3^2 A_{(a_1/a_3)^\tau} \end{pmatrix}(I \otimes B)^T(\alpha^\sigma,\beta^\sigma)^T)$ is an automorphism of $G^\otimes$ leaving $\mathcal{J}(\mathcal{C})$ invariant. Then θ induces an isomorphism from $\bar{\mathcal{R}}_\alpha$ to $\bar{\mathcal{R}}_{\alpha^\sigma B}$ mapping $\bar{\mathcal{O}}_\alpha$ to $\bar{\mathcal{O}}_{\alpha^\sigma B}$, and thus an automorphism of $PG(2,q)$ mapping $\mathcal{O}_\alpha$ to $\mathcal{O}_{\alpha^\sigma B}$. Put $\mu = \det(A)$, $A = \begin{pmatrix} a_4 & a_2 \\ a_3 & a_1 \end{pmatrix}$, $C = a_4^2 A_{(a_2/a_4)^\tau}$, $D = a_2 a_3 P$, $E = a_3^2 A_{(a_1/a_3)^\tau}$. Then Fig. 1 gives the maps involved:

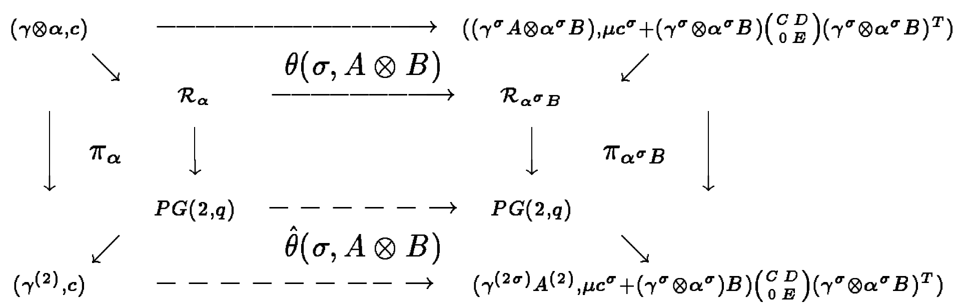

Fig. 1.

In this same context, suppose that $\alpha^\sigma B \equiv \alpha$, so $\theta(\sigma, A \otimes B)$ induces an automorphism of $\bar{\mathcal{R}}_\alpha$ (resp. $(PG(2,q))$ stabilizing $\bar{\mathcal{O}}_\alpha$ (resp., $\mathcal{O}_\alpha$). Making appropriate specializations in Fig. 1, we obtain Fig. 2.

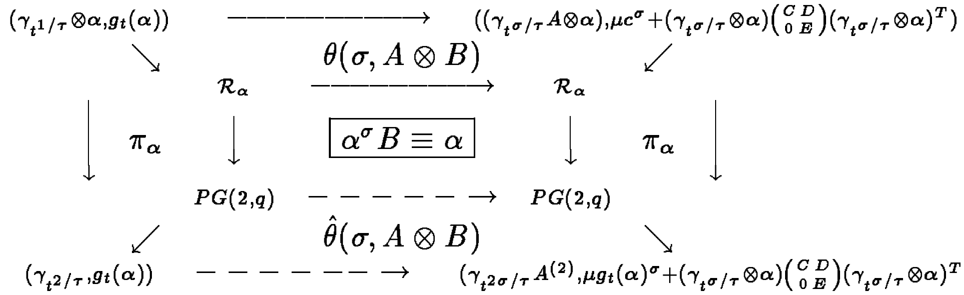

Fig. 2.

In the applications that follow for Subiaco q-clans, $\tau = 2$. In this case the bottom row of Fig. 2. appears as :

$$\begin{aligned}\hat{\theta}(\sigma, A\otimes B) : (\gamma_t, g_t(\alpha)) &\mapsto (\gamma_{t^\sigma}A^{(2)}, \mu g_t(\alpha)^\sigma + (\gamma_{t^{\sigma/2}}\otimes\alpha)\begin{pmatrix} C & D \\ O & E \end{pmatrix}(\gamma_{t^{\sigma/2}}\otimes\alpha)^T) \\ &= (\gamma_{t^\sigma}A^{(2)},\ \det(A)g_t(\alpha)^\sigma + (\gamma_{t^\sigma})(\alpha C\alpha^T, \alpha E\alpha^T)^T).\end{aligned} \tag{40}$$

From now on all q-clans are $\frac{1}{2}$-normalized. Given a q-clan $\mathcal{C} = \{A_t = \begin{pmatrix} x_t & t^{\frac{1}{2}} \\ 0 & z_t \end{pmatrix} : t \in F\}$, write $\mathcal{J}(\mathcal{C}) = \{A(t) : t \in \tilde{F}\}$ for the associated 4-gonal family in $G^\otimes$, and $\hat{\mathcal{J}}(\mathcal{C}) = \{\hat{A}(t) : t \in \tilde{F}\}$ for the 4-gonal family in G. Note that $\psi : G^\otimes \to G : ((\alpha,\beta), c) \mapsto ((\alpha,\beta), \sqrt{c}) : (\gamma_{t^{\frac{1}{2}}}\otimes\alpha, \alpha A_t\alpha^T) \mapsto (\gamma_{t^{\frac{1}{2}}}\otimes\alpha, \sqrt{\alpha A_t\alpha^T})$. So $\psi : A(t) \mapsto \hat{A}(t)$. With a little care we can use the same symbol for an automorphism $\theta(\sigma, A\otimes B)$ of $G^\otimes$ and the corresponding one $\psi^{-1}\cdot\theta(\sigma, A\otimes B)\cdot\psi$ of G.

$$\text{In } G^{\otimes}: \theta(\sigma, A\otimes B): ((\alpha,\beta),c) \mapsto ((\alpha^\sigma,\beta^\sigma)(A\otimes B), \mu c^\sigma + \\ (\alpha^\sigma,\beta^\sigma)(I\otimes B)\begin{pmatrix} C & D \\ O & E \end{pmatrix}(I\otimes B)^T(\alpha^\sigma,\beta^\sigma)^T). \tag{41}$$

$$\text{In } G: \theta(\sigma, A\otimes B): ((\alpha,\beta),c) \mapsto ((\alpha^\sigma,\beta^\sigma)(A\otimes B), \mu^{\frac{1}{2}} c^\sigma + \\ \sqrt{(\alpha^\sigma,\beta^\sigma)(I\otimes B)\begin{pmatrix} C & D \\ O & E \end{pmatrix}(I\otimes B)^T(\alpha^\sigma,\beta^\sigma)^T}). \tag{42}$$

7 Subiaco q-clans

For $q = 2^e$, $e \geq 4$, $F = GF(q)$, let $\delta \in F$ be chosen so that $x^2 + \delta x + 1$ is irreducible over F, i.e., $tr(\delta^{-2}) = tr(\delta^{-1}) = 1$. Then define the following functions on F.

$$\begin{array}{llcl} (i) & f(t) & = & \delta^2 t^4 + \delta^3 t^3 + (\delta^2 + \delta^4)t^2, \\ (ii) & g(t) & = & (\delta^2 + \delta^4)t^3 + \delta^3 t^2 + \delta^2 t, \\ (iii) & k(t) & = & (\nu(t))^2 = t^4 + \delta^2 t^2 + 1, \\ (iv) & F(t) & = & f(t)/k(t), \\ (v) & G(t) & = & g(t)/k(t). \end{array} \tag{43}$$

Then the **canonical Subiaco q -clan** (cf. [8]) is the set $\mathcal{C}(\delta) = \mathcal{C} = \{A_t = \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix} : t \in F\}$ where

$$\begin{array}{llcl} (i) & x_t & = & F(t) + (t/\delta)^{\frac{1}{2}}, \\ (ii) & y_t & = & t^{\frac{1}{2}}, \\ (iii) & z_t & = & G(t) + (t/\delta)^{\frac{1}{2}}. \end{array} \tag{44}$$

Using the group G, the authors of [8] determined the full automorphism group of G leaving $\hat{\mathcal{J}}(\mathcal{C})$ invariant. We want to recall several of the results from [8] but using the efficient tensor product notation of this paper. For each member $\hat{A}(s)$ of $\hat{\mathcal{J}}(\mathcal{C})$, $s \in \tilde{F}$, there is an involution I_s of G leaving $\hat{\mathcal{J}}(\mathcal{C})$ invariant and fixing $\hat{A}(s)$.

$$I_\infty = \theta(id, \begin{pmatrix} 1 & \delta^{\frac{1}{2}} \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1+\delta+\delta^2 & \delta^{\frac{3}{2}} \\ \delta^{\frac{1}{2}} + \delta^{\frac{5}{2}} & 1+\delta+\delta^2 \end{pmatrix}). \tag{45}$$

Here $(a_2/a_4)^2 = \delta$, and $A_\delta = \begin{pmatrix} 1+\delta^4+\delta^6 & \delta^{\frac{1}{2}} \\ 0 & 1+\delta^3+\delta^7 \end{pmatrix}$.

For $s \in F$, put $\bar{0} = \delta/\nu(s) = \delta/(s^2 + \delta s + 1)$. Then

$$
\begin{array}{llcl}
(i) & I_s & = & \theta\left(id, \begin{pmatrix} (1+\bar{0}s)^{\frac{1}{2}} & s\bar{0}^{\frac{1}{2}} \\ \bar{0}^{\frac{1}{2}} & (1+\bar{0}s)^{\frac{1}{2}} \end{pmatrix} \otimes B_s\right), \\
\text{where} & B_s & = & \begin{pmatrix} a(s) & b(s) \\ c(s) & a(s) \end{pmatrix} \text{ is defined by} \\
(ii) & a(s) & = & ((s^5+1)(1+\delta+\delta^2)+(s^4+s)(1+\delta))/\nu(s)^{\frac{5}{2}}, \\
(iii) & b(s) & = & (\delta^{\frac{3}{2}}s^5+\delta^{\frac{1}{2}}s^4+\delta^{\frac{3}{2}}s+\delta^{\frac{1}{2}}+\delta^{\frac{5}{2}})/\nu(s)^{\frac{5}{2}}, \\
(iv) & c(s) & = & ((\delta^{\frac{1}{2}}+\delta^{\frac{5}{2}})s^5+\delta^{\frac{3}{2}}s^4+\delta^{\frac{1}{2}}s+\delta^{\frac{3}{2}})/\nu(s)^{\frac{5}{2}}.
\end{array}
\tag{46}
$$

It is convenient to have a few specific involutions to compute with.

$$
\begin{array}{ll}
(i) & I_0 = \theta\left(id, \begin{pmatrix} 1 & 0 \\ \delta^{\frac{1}{2}} & 1 \end{pmatrix} \otimes \begin{pmatrix} 1+\delta+\delta^2 & \delta^{\frac{1}{2}}+\delta^{\frac{5}{2}} \\ \delta^{\frac{3}{2}} & 1+\delta+\delta^2 \end{pmatrix}\right). \\
(ii) & I_1 = \theta(id, P\otimes P) : ((\alpha,\beta),c) \mapsto ((\beta P, \alpha P), c+\alpha\circ\beta). \\
(iii) & I_\delta = \theta\left(id, \begin{pmatrix} 1+\delta & \delta^{\frac{3}{2}} \\ \delta^{\frac{1}{2}} & 1+\delta \end{pmatrix} \otimes \begin{pmatrix} 1+\delta^4+\delta^6+\delta^7 & \delta^{\frac{1}{2}}+\delta^{\frac{9}{2}}+\delta^{\frac{13}{2}} \\ \delta^{\frac{15}{2}} & 1+\delta^4+\delta^6+\delta^7 \end{pmatrix}\right).
\end{array}
\tag{47}
$$

In [8] it is shown that any two suitable δ give equivalent q-clans. So we lose no generality by choosing a particular one. Let ζ be a primitive element for $E = GF(q^2) \supseteq F$. Put $\lambda = \zeta^{q-1}$ and $\delta = \lambda + \lambda^{-1}$. Then $x^2 + \delta x + 1 = (x+\lambda)(x+\lambda^{-1})$ is irreducible over F. For each rational number a with denominator prime to $q+1$, put

$$
\begin{array}{ll}
(i) & [a] = \lambda^a + \lambda^{-a}. \text{ Then} \\
(ii) & [0] = 0, \text{ and } [1] = \delta. \\
(iii) & [a] = [b] \text{ iff } a \equiv \pm b (\bmod\ q+1). \text{ So } [a] = [-a]. \\
(iv) & [a][b] = [a+b] + [a-b]. \\
(v) & \text{For } \sigma = 2^i \in Aut(F),\ \ [a]^\sigma = [a\sigma].
\end{array}
\tag{48}
$$

In [8] it is observed that $I_0 \cdot I_\delta$ permutes the members of $\hat{\mathcal{J}}(\mathcal{C})$ in one cycle of length $q+1$. When $e \not\equiv 2 \pmod 4$, i.e., 5 does not divide $q+1$, it permutes the $\mathcal{R}_\alpha$ and hence the ovals $\mathcal{O}_\alpha$, in one cycle of length $q+1$. But when $e \equiv 2 \pmod 4$, so $5|(q+1)$, it permutes the $\mathcal{R}_\alpha$ in five cycles of length $(q+1)/5$. Let Ω_1 be the orbit containing $\mathcal{R}_\alpha$ with $\alpha = (1,1)$. The group $\mathcal{G}_0$ of all automorphisms of G leaving $\hat{\mathcal{J}}(\mathcal{C})$ invariant has two orbits on the $\mathcal{R}_\alpha$: the orbit Ω_1 and a second orbit Ω_2 of length $4(q+1)/5$.

$$
\begin{array}{ll}
(i) & I_0 \cdot I_\delta = \theta(id, M \otimes N), \text{ where } M = \begin{pmatrix} 1+\delta & \delta^{3/2} \\ \delta^{3/2} & 1+\delta+\delta^2 \end{pmatrix}, \quad N = B_0 B_\delta. \\
(ii) & M^j = \delta^{-\frac{1}{2}} \begin{pmatrix} [2j-\frac{1}{2}] & [2j] \\ [2j] & [2j+\frac{1}{2}] \end{pmatrix}. \\
(iii) & N^j = \delta^{-\frac{1}{2}} \begin{pmatrix} [10j+\frac{1}{2}] & [10j] \\ [10j] & [10j-\frac{1}{2}] \end{pmatrix}.
\end{array}
\tag{49}
$$

Proposition 7.1 *For all $e \geq 4$, $N^{-1} = M^5$.*

Proof: This is immediate from Eq. 49 parts (ii) and (iii), since $\det(N) = \det(M) = 1$. ∎

Theorem 7.2 *When $e \equiv 2$ (mod 4), $\mathcal{R}_\alpha \in \Omega_2$ for $\alpha = (1, \delta^{\frac{1}{2}})$.*

Proof: Since the orbit of $\mathcal{R}_{(1,1)}$ under $I_0 \cdot I_\delta$ is Ω_1, $\mathcal{R}_{(1,\delta^{\frac{1}{2}})} \in \Omega_1$ iff there is an integer j for which $(1,1)N^j \equiv (1, \delta^{\frac{1}{2}})$. This is iff
$([10j + \frac{1}{2}] + [10j], [10j] + [10j - \frac{1}{2}]) \equiv (1, [\frac{1}{2}])$ iff
$[10j] + [10j - \frac{1}{2}] = [\frac{1}{2}]([10j + \frac{1}{2}] + [10j])$ iff
$[10j] + [10j - \frac{1}{2}] = [10j] + [10j + 1] + [10j - \frac{1}{2}] + [10j + \frac{1}{2}]$ iff
$[10j + 1] = [10j + \frac{1}{2}]$ iff $\frac{1}{2} \equiv 0 \pmod{q+1}$ (impossible!) or $20j + \frac{3}{2} \equiv 0 \pmod{q+1}$.

When $5|(q+1)$, $20j + \frac{3}{2} \equiv 0 \pmod{q+1}$ implies $0 + \frac{3}{2} \equiv 0 \pmod 5$, impossible! ∎

Let $e \equiv 2 \pmod 4$, say $e = 2r$ with r odd, and put $\alpha = (1, \delta^{\frac{1}{2}})$. Then the oval $\mathcal{O}_\alpha$ has the following appearance.

$$\begin{array}{ll} (i) & \mathcal{O}_\alpha = \{(1, t, h_\alpha(t)) : t \in F\} \cup \{(0,1,0)\}, \text{ where} \\ (ii) & h_\alpha(t) = \frac{\delta^2 t^4 + \delta^5 t^3 + \delta^2 t^2 + \delta^3 t}{t^4 + \delta^2 t^2 + 1} + (t/\delta)^{\frac{1}{2}}. \end{array} \tag{50}$$

From [8] we know that the complete stabilizer of $\mathcal{O}_\alpha$ has order $5r$.

Clearly $(I_o \cdot I_\delta)^{\frac{q+1}{5}} = \theta(id, \delta^{-\frac{1}{2}} \begin{pmatrix} [\frac{4q-1}{10}] & [\frac{2(q+1)}{5}] \\ [\frac{2(q+1)}{5}] & [\frac{4q+9}{5}] \end{pmatrix} \otimes I)$ induces a linear collineation φ of $PG(2,q)$ stabilizing $\mathcal{O}_\alpha$. Put $x = \left[\frac{4(q+1)}{5}\right] / \left[\frac{4q-1}{5}\right]$ and $y = \left[\frac{4q+9}{5}\right] / \left[\frac{4(q+1)}{5}\right]$. Then from Eq. 40 we can work out the effect of φ to be

$$\begin{array}{rl} \varphi : (\gamma, c) \mapsto & (\delta^{-1}\gamma \begin{pmatrix} \left[\frac{4q-1}{5}\right] & \left[\frac{4(q+1)}{5}\right] \\ \left[\frac{4(q+1)}{5}\right] & \left[\frac{4q+9}{5}\right] \end{pmatrix}, \\ & c + (\gamma^{(\frac{1}{2})} \otimes (1, \delta^{\frac{1}{2}})) \begin{pmatrix} \left[\frac{4q-1}{5}\right] A_x & \left[\frac{4(q+1)}{5}\right] P \\ 0 & \left[\frac{4(q+1)}{5}\right] A_y \end{pmatrix} (\gamma^{(\frac{1}{2})} \otimes (1, \delta^{\frac{1}{2}}))^T). \end{array} \tag{51}$$

Put $\omega = \left[\frac{q+1}{5}\right]$, so $\omega^2 + \omega + 1 = 0$. Put $W_1 = \left[\frac{4q-1}{5}\right]$, $W_2 = \left[\frac{4q+9}{5}\right]$. Note that $W_1 + W_2 = \omega\delta$ and $W_1 W_2 = \omega^2 + \delta^2$. So W_1 and W_2 are the two roots in F of $x^2 + \omega\delta x + \omega^2 + \delta^2 = 0$. And then $x = \omega / W_1$, $y = \omega / W_2$. With the standard notation $A_t = \begin{pmatrix} x_t & t^{\frac{1}{2}} \\ 0 & z_t \end{pmatrix}$ for $t \in F$, the description of φ in Eq. 51 can be given a more conventional appearance.

$$\begin{array}{c} \varphi : (a, b, c) \mapsto (aW_1/\delta + b\omega/\delta, a\omega/\delta + bW_2/\delta, \\ c + aW_1(x_{\omega/W_1} + \left(\frac{W_1+W_2}{W_1}\right)^{\frac{1}{2}} + \delta z_{\omega/W_1}) + b\omega(x_{\omega/W_2} + (\frac{W_1+W_2}{W_2})^{\frac{1}{2}} + \delta z_{\omega/W_2})). \end{array} \tag{52}$$

This φ is a collineation of order 5 stabilizing $\mathcal{O}_\alpha$.

In principle we have an algorithm for computing that part of the stabilizer of $\mathcal{O}_\alpha$ contributed by $\mathrm{Gal}(F/GF(4))$. If we put $\delta_1 = \omega$ and $\delta_2 = \delta$, Theorem IV.4 of [8] gives an isomorphism of G of the form $\theta(id, A \otimes B)$ mapping $\hat{\mathcal{J}}(\mathcal{C}(\delta_1))$ to $\hat{\mathcal{J}}(\mathcal{C}(\delta_2))$ and mapping $[A_{\delta_1}(\infty)]$ to $[A_{\delta_2}(\infty)]$. Eq.(38) of [8] is $\bar{0}^2 + \delta\bar{0} + \omega\delta^2 + 1 = 0$ and has solutions $\bar{0} = \omega^2 W$, for $W \in \{W_1, W_2\}$. For $\sigma \in \mathrm{Aut}(F)$ with $\omega^\sigma = \omega$, $\theta_\sigma = \theta(\sigma, I \otimes I)$ is an automorphism of G stabilizing $\hat{\mathcal{J}}(\mathcal{C}(\delta_1))$ and mapping $\hat{A}(\infty)$ to itself. Therefore $\theta = \theta(id, A^{-1} \otimes B^{-1}) \cdot \theta(\sigma, I \otimes I) \cdot \theta(A \otimes B) = \theta(\sigma, A^{-\sigma}A \otimes B^{-\sigma}B)$ is the corresponding automorphism of G stabilizing $\hat{\mathcal{J}}(\mathcal{C}(\delta_2)) = \hat{\mathcal{J}}(\mathcal{C}(\delta)) = \hat{\mathcal{J}}(\mathcal{C})$ and mapping $\hat{A}(\infty)$ to itself. To compute θ is tedious but routine. But then one must compute the exponent i for which $\theta \cdot (I_0 \cdot I_\delta)^i$ leaves $\mathcal{R}_{(1,\delta^{\frac{1}{2}})}$ invariant. We leave this as an exercise for the tireless reader.

For the remainder of this section we suppose that $e \not\equiv 2 \pmod 4$, so that 5 does not divide $(q+1)$. Up to projective equivalence there is only one Subiaco oval $\mathcal{O}_\alpha$. The complete stabilizer of $\mathcal{O}_\alpha$ is known [4] to be induced by the group $\mathcal{G}_0$ of automorphisms of G leaving $\hat{\mathcal{J}}(\mathcal{C})$ invariant. And the order of this induced stabilizer is shown in [8] to be $2e$. Here we want to give these automorphisms explicitly.

Let $\delta = \lambda + \lambda^{-1}$ as above. Put $\sigma = 2$ and $\bar{0} = \delta^{-1}$ in Theorem V.I of [8] to obtain

$$\bar{\theta} = \theta(\sigma = 2, \begin{pmatrix} 1 & \delta^{-\frac{1}{2}} \\ 0 & \delta^{-\frac{1}{2}} \end{pmatrix} \otimes \begin{pmatrix} \delta^{-\frac{1}{4}} & \delta^{\frac{1}{4}} \\ \delta^{-\frac{1}{4}} + \delta^{3/4} & \delta^{5/4} \end{pmatrix}). \tag{53}$$

Then $\bar{\theta}$ is an automorphism of G stabilizing $\hat{\mathcal{J}}(\mathcal{C})$ and mapping $\hat{A}(\infty)$ to itself. According to [8] for each $\alpha \in PG(1,q)$ there is a unique j such that $\bar{\theta} \cdot (I_0 \cdot I_\delta)^j$ stabilizes $\mathcal{R}_\alpha$. The involution $I_1 = \theta(id, P \otimes P)$ fixes $\mathcal{R}_{(1,1)}$ and $\mathcal{O}_{(1,1)}$ has a pleasant form. But the j for which $\bar{\theta} \cdot (I_0 \cdot I_\delta)^j$ stabilizes $\mathcal{R}_{(1,1)}$ leads to rather messy formulae. However, using Eqs. 48 and 49 it is easy to prove the following for $\alpha = (0,1)$.

$$\begin{array}{ll} (i) & \bar{\theta} \cdot (I_0 \cdot I_\delta)^{3/20} \text{ stabilizes } \mathcal{R}_{(0,1)} \text{ and has companion automorphism } \sigma = 2. \\ (ii) & (I_0 \cdot I_\delta)^{-\frac{1}{40}} \text{ maps } \mathcal{R}_{(1,1)} \text{ to } \mathcal{R}_{(0,1)}, \text{ so that} \\ (iii) & \varphi = (I_0 \cdot I_\delta)^{\frac{1}{40}} \cdot I_1 \cdot (I_0 \cdot I_\delta)^{-\frac{1}{40}} \text{ is a (linear) involution fixing } \mathcal{R}_{(0,1)}. \end{array} \tag{54}$$

In the following computations put $T = [\frac{1}{5}]$, so $T^5 + T^3 + T = \delta$. Note that since M and N are symmetric with determinant 1 (see Eq. 49), $PMP = M^{-1}$, $PNP = N^{-1}$. And now it is amusing to use the tensor algebra to compute

$$\begin{aligned}\varphi &= \theta(id, M^{\frac{1}{40}} \otimes N^{\frac{1}{40}}) \cdot \theta(id, P \otimes P) \cdot \theta(id, M^{-\frac{1}{40}} \otimes N^{-\frac{1}{40}}) \\ &= \theta(id, M^{\frac{1}{40}}(PM^{-\frac{1}{40}}P)P \otimes N^{\frac{1}{40}}(PN^{-\frac{1}{40}}P)P) \\ &= \theta(id, M^{\frac{1}{20}}P \otimes N^{\frac{1}{20}}P) \\ &= \theta(id, \delta^{-\frac{1}{2}} \begin{pmatrix} [\frac{1}{10}] & [\frac{2}{5}] \\ [\frac{3}{5}] & [\frac{1}{10}] \end{pmatrix} \otimes \begin{pmatrix} 1 & \delta^{\frac{1}{2}} \\ 0 & 1 \end{pmatrix}) \\ &= \theta(id, \delta^{-\frac{1}{2}} \begin{pmatrix} T^{\frac{1}{2}} & T^2 \\ T^3 + T & T^{\frac{1}{2}} \end{pmatrix} \otimes \begin{pmatrix} 1 & \delta^{\frac{1}{2}} \\ 0 & 1 \end{pmatrix}).\end{aligned}$$

In the context of Figs. 1 and 2, $C = \delta^{-1}TA_{T^3}$, $D = xP$ for some constant x, $E = \delta^{-1}(T^6 + T^2)A_{(\delta+T^3)^{-1}}$, $A^{(2)} = \delta^{-1}\begin{pmatrix} T & T^4 \\ T^6 + T^2 & T \end{pmatrix}$. So applying Eq. 40 (and multiplying through by δ) we obtain

$$\hat{\varphi} : (x, y, z) \mapsto (Tx + (T^6 + T^2)y, T^4x + Ty, \delta z + Tz_{T^3}x + (T^6 + T^2)z_{(\delta+T^3)^{-1}}y). \tag{55}$$

Actually $\hat{\varphi} = I_s$ where $s = [\frac{2}{5}]/[\frac{3}{5}] = T/(T^2 + 1)$, because $I_s = \theta(id, \begin{pmatrix} (1 + \bar{0}s)^{\frac{1}{2}} & s\bar{0}^{\frac{1}{2}} \\ \bar{0}^{\frac{1}{2}} & (1 + \bar{0}s)^{\frac{1}{2}} \end{pmatrix} \otimes B_s)$ and s is easy to obtain as a ratio of off-diagonal elements of A.

$$\begin{array}{ll} (i) & \mathcal{O}_{(0,1)} = \{(1, t, h_\alpha(t)) : t \in F\} \cup \{(0, 1, 0)\}, \text{ where} \\ (ii) & h_{(0,1)}(t) = \frac{(\delta^4+\delta^2)t^3+\delta^3t^2+\delta^2t}{t^4+\delta^2t^2+1} + (t/\delta)^{\frac{1}{2}}. \end{array} \tag{56}$$

Also,

$$\bar{\theta} \cdot (I_0 \cdot I_\delta)^{3/20} = \theta(2, \frac{1}{\delta} \begin{pmatrix} [\frac{7}{10}] & [\frac{1}{5}] \\ [\frac{3}{10}] & [\frac{4}{5}] \end{pmatrix} \otimes \begin{pmatrix} \delta^{\frac{1}{4}} & \delta^{-\frac{1}{4}} \\ 0 & \delta^{-\frac{1}{4}} \end{pmatrix}) \tag{57}$$

yields a collineation $\hat{\theta}$ of $PG(2, q)$ stabilizing $\mathcal{O}_{(0,1)}$ and having companion automorphism $\sigma = 2$. In the context of Eq. 40,

$$A^{(2)} = \frac{1}{\delta^2} \begin{pmatrix} [\frac{7}{5}] & [\frac{2}{5}] \\ [\frac{3}{5}] & [\frac{8}{5}] \end{pmatrix} = \frac{1}{\delta^2} \begin{pmatrix} T^7 + T^3 + \delta & T^2 \\ T^3 + T & T^8 \end{pmatrix} = \begin{pmatrix} a_4^2 & a_2^2 \\ a_3^2 & a_1^2 \end{pmatrix}.$$

Then $C = a_4^2 A_{(a_2/a_4)^2}$, $E = a_3^2 A_{(a_1/a_3)^2}$, and $\mu = \det(A) = \delta^{-\frac{1}{2}}$, so that

$$\hat{\theta} : (x, y, z) \mapsto ((x^2, y^2)A^{(2)}, \ \ \delta^{-\frac{1}{2}}z^2 + x^2a_4^2z_{(a_2/a_4)^2} + y^2a_3^2z_{(a_1/a_3)^2}). \tag{58}$$

Then $\langle \hat{\theta}, \hat{\varphi} \rangle$ is the complete group of collineations of $PG(2, q)$ induced by $\mathcal{G}_0$ and stabilizing $\mathcal{O}_{(0,1)}$. In fact, $(\hat{\theta})^e = \hat{\varphi}$ and $< \hat{\theta}, \hat{\varphi} >=< \hat{\theta} >\cong C_{2e}$.

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