P_3-Connected Graphs of Minimum Size

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A graph is C_4*-free* if it does not contain any 4-cycles. A C_4-free graph is C_4*-saturated* if adding any edge creates a 4-cycle. Ollmann showed that an n-node C_4-saturated graph has at least $\frac{3}{2}(n-2)$ edges. A graph is P_3*-connected* if there is a path of length 3 between each pair of nonadjacent nodes. We show that an n-node P_3-connected graph has at least $\frac{3}{2}(n-2)$ edges. Since all C_4-saturated graphs are P_3-connected, this generalizes Ollmann's results. We also find all n-node P_3-connected graphs with $\left\lceil\frac{3}{2}(n-2)\right\rceil$ edges.

Let P_k be a path on k edges. Two nodes are P_k*-connected* if they are connected by a P_k. A graph is P_k*-connected* if all pairs of nonadjacent nodes are P_k-connected. The only P_1-connected graphs are complete graphs. A graph is P_2-connected if and only if its diameter is at most 2. In general, a P_k-connected graph has diameter at most k. However, a diameter k graph need not be P_k-connected. For example, P_k for $k \geq 3$ has diameter k and yet is not P_k-connected.

Let C_k be a cycle with k edges. A graph without C_k as a (not necessarily induced) subgraph is called C_k*-free.* A C_k*-saturated* graph is a maximal C_k-free graph in the sense that adding any edge creates a C_k. If adding an edge between two nodes creates a C_k, then a P_{k-1} must connect the nodes. So a C_k-saturated graph is P_{k-1}-connected. However, a P_{k-1}-connected graph need not be C_k-saturated (*e.g.*, Figure 3 (d) and (e)).

Assume throughout that n is the number of nodes and e is the number of edges in a graph. Ollmann [**1**] answered the question: *What is the minimum number of edges in a C_4-saturated graph?* He showed $e \geq \frac{3}{2}(n-2)$ for all C_4-saturated graph. He also described all C_4-saturated graphs with $e = \left\lceil\frac{3}{2}(n-2)\right\rceil$ showing that, given integrality, the bound is exact for $n > 4$. Later, Tuza [**2**] gave a much simpler proof of the bound and description.

Theorem 1 answers the question: *What is the minimum number of edges in a P_3-connected graph?* The answer is again $e \geq \frac{3}{2}(n-2)$. However since all C_4-saturated graphs are P_3-connected, this is a generalization of Ollmann's result. Theorem 2 describes the class of P_3-connected graphs with $e = \left\lceil\frac{3}{2}(n-2)\right\rceil$. In addition to the three found by Ollman (Figure 3 (a), (b), and (c)), two new families are given (Figure 3 (d) and (e)). Graphs in these two families have an odd number of nodes, and are P_3-connected but not C_4-saturated.

Lemma 1. *Let G be a connected graph. If each edge of G is in a C_3, then $e \geq \frac{3}{2}(n-1)$.*

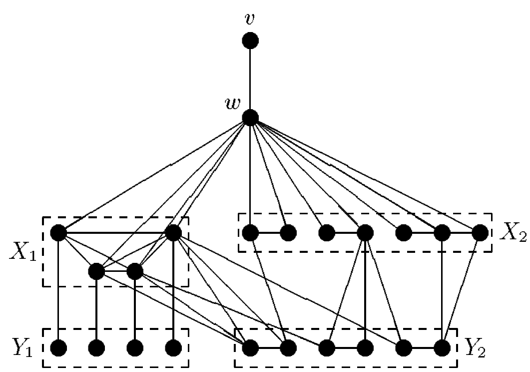

Figure 1. The construction in Case 1 of Theorem 1.

Proof. Since it is connected, G has a spanning tree T with $n-1$ edges. An edge of $G-T$ can form a C_3 with at most two edges of T. So $G-T$ has at least $\frac{1}{2}(n-1)$ edges. □

Theorem 1. *Let G be a P_3-connected graph. Then $e \geq \frac{3}{2}(n-2)$.*

Proof. This is trivially true for $n=1$. So assume $n \geq 2$. Let δ be the minimum degree of the nodes of G. Since G is connected, $\delta > 0$. If $\delta \geq 3$, then $e \geq \frac{3}{2}n$ giving the result. So assume δ is 1 or 2.

Case 1: $\delta = 1$**.** Let v be a degree 1 node with neighbor w. Let T be the breadth-first tree of G rooted at v. Let X and Y be the nodes that are distance 2 and 3, respectively, from v. Since all nodes are distance at most three from v, all nodes other than w and v are in X or Y. Thus $|X|+|Y| = n-2$. Because $x \in X$ is not adjacent to v, there is a path v, w, q, x. Since q is distance 2 from v, we have $q \in X$ and hence the nodes in X have degree at least 1 within X.

Let Y_1 be the degree 1 nodes in Y and $k = |Y_1|$. Let $Y_2 = Y - Y_1$. Let X_1 be the nodes of X adjacent to Y_1 and let $X_2 = X - X_1$. Then for all $y_1, y_2 \in Y_1$, there is a path y_1, x_1, x_2, y_2 where $x_1, x_2 \in X_1$. It is easy to see that each node in X_1 is adjacent to exactly one node in Y_1, and every pair of nodes in X_1 are adjacent. Thus $|X_1| = k$ and X_1 is a clique (see Figure 1). Since each node in X_2 and Y_2 is incident to at least one edge not in T, the number of edges not in T is

$$e(G-T) \geq \binom{|X_1|}{2} + \frac{|X_2|+|Y_2|}{2} = \binom{k}{2} + \frac{n-2-2k}{2} \geq \frac{n-4}{2}.$$

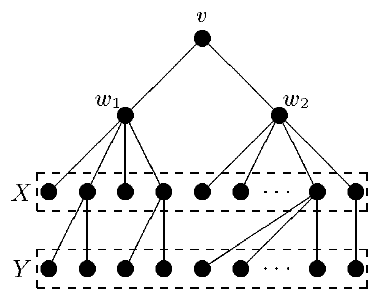

Figure 2. The tree formed in Case 2 of Theorem 1.

Thus $e \geq \frac{1}{2}(n-4) + n - 1 = \frac{3}{2}(n-2)$.

Case 2: $\delta = 2$ and a degree 2 node is not in a C_3. Let v have degree 2 with nonadjacent neighbors w_1 and w_2. Let T be the breadth-first tree of G rooted at v. Let X and Y be the nodes which are distance 2 and 3, respectively, from v (see Figure 2). Since a P_3 connects v to $x \in X$, there is an edge incident to x which is not in T. Further since $\delta = 2$, each $y \in Y$ is also incident to an edge not in T. So at least $\frac{1}{2}(|X| + |Y|) = \frac{1}{2}(n-3)$ edges are not in T. Therefore $e \geq n - 1 + \frac{1}{2}(n-3) = \frac{1}{2}(3n-5)$.

Case 3: $\delta = 2$ and each degree 2 node is in a C_3. Let S be the subgraph of all C_3's with one or more degree 2 nodes. Let $S_1, S_2, \ldots, S_m$ be the components of S. Lemma 1 shows that $e(S_i) \geq \frac{3}{2}(|S_i| - 1)$. Since the degree 2 nodes of two components are P_3-connected, there is an edge between every pair of components of S. So there are at least $\binom{m}{2}$ edges between components of S. Then the number of edges in the induced subgraph induced by the nodes of S is

$$e(\langle S \rangle) \geq \sum_{i=1}^{m} e(S_i) + \binom{m}{2} \geq \sum_{i=1}^{m} \frac{3}{2}(|S_i| - 1) + \binom{m}{2}$$
$$= \frac{3}{2}|S| + \frac{m^2}{2} - 2m \geq \frac{3}{2}|S| - 2.$$

Since all degree 2 nodes are in S, nodes not in S have degree 3 or more. Thus $e \geq \frac{3}{2}|S| - 2 + \frac{3}{2}(n - |S|) = \frac{3}{2}n - 2$. □

Theorem 2. *Let G be a P_3-connected graph with $e = \lceil \frac{3}{2}(n-2) \rceil$. Then G is in one of the families of graphs in Figure 3.*

Proof. The only cases in the proof of Theorem 1 where $e = \lceil \frac{3}{2}(n-2) \rceil$ are Cases 1 and 2.

Case 1: $\delta = 1$. Let v, w, X, Y, X_1, Y_1, X_2, Y_2, T, and k be as they were in Case 1 of Theorem 1. For the bound to hold $e(G-T) = \lceil \frac{1}{2}(n-4) \rceil$ and $k = 1$ or 2.

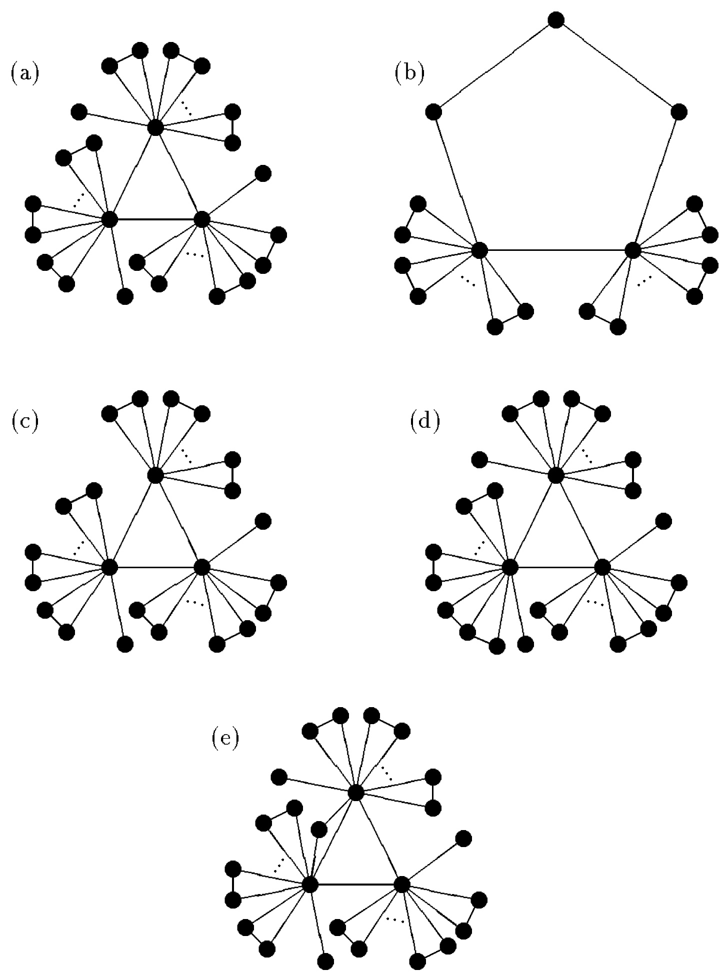

Figure 3 – Families of P_3-connected graphs with $\left\lceil \frac{3}{2}(n-2) \right\rceil$ edges. The dots indicate an arbitrary number (including zero) of pendant C_3's.

Case 1a: $k = 1$. Let $X_1 = \{x_1\}$ and $Y_1 = \{y_1\}$. Since every node in X and Y_2 has degree at least one in $G-T$, we have that $e(G-T) \geq \frac{1}{2}(|X|+|Y_2|) = \frac{1}{2}(n-3)$. So $e(G-T) = \frac{1}{2}(n-3)$ and hence n is odd and the edges of $G-T$ form a matching on $X \cup Y_2$. Since each node of X is adjacent to a node of X, the edges of $G-T$ form a matching on X and hence also form a matching on Y_2. Let $x_2 \in X_2$ so that x_1x_2 is an edge. Since x_1 is not adjacent to $X_2 - \{x_2\}$, if $y \in Y$ is not adjacent to x_1 or x_2, then y is not connected to y_1 by a P_3. So all nodes of Y_2 are adjacent to either x_1 or x_2.

Let $y_2, y_3 \in Y_2$ be such that y_2y_3 is an edge and suppose y_2x_1 and y_3x_2 are edges. Then y_2 is not adjacent to x_2 otherwise y_2 would be incident with two edges not in T. Similarly y_3 is not adjacent to x_1. Under these circumstances, y_2 is not P_3-connected to y_1 which is a contradiction. So the ends of any edge in Y are either both adjacent to x_1 or both adjacent to x_2. Thus G is in Family (c) where w, x_1, x_2 is the central C_3, and vw and x_1y_1 are pendant edges.

Case 1b: $k = 2$**.** Let $X_1 = \{x_1, x_2\}$ and $Y_1 = \{y_1, y_2\}$ where x_1y_1 and x_2y_2 are edges. Note that x_1x_2 must be an edge as well.

Assume n is even. Here $e(G-T) = \frac{1}{2}(n-4)$. For the inequalities to be equalities, each of the $n-4$ nodes in X and Y_2 has degree 1 in $G-T$, and there are no edges between X_1 and X_2. Then an argument similar to Case 1a shows that G is in Family (a) where w, x_1, x_2 is the central C_3 and vw, x_1y_1 and x_2y_2 are pendant edges.

Now assume n is odd. Here $e(G-T) = \frac{1}{2}(n-3)$. Since $|X|+|Y_2| = n-4$ and every node in $X \cup Y_2$ has degree at least 1 in $G-T$, the subgraph of $G-T$ restricted to $X \cup Y_2$ consists of a P_2 and isolated edges.

Since each node of X is adjacent to another node in X in $G-T$ and x_1x_2 is an edge in $G-T$, there are five possibilities for the P_2. If the P_2 is x_1, x_2, x_3 where $x_3 \in X_2$, then by arguments similar to Case 1a, we have that G is in Family (e) where w, x_1, x_2 is the central C_3, vw, x_1y_1 and x_2y_2 are pendant edges, and x_3 is adjacent to both w and x_2. If the P_2 is x_3, x_4, x_5 where $x_3, x_4, x_5 \in X_2$, then again by arguments similar to Case 1a, G is in Family (d) where w, x_1, x_2 is the central C_3, vw, x_1y_1 and x_2y_2 are pendant edges, and x_3, x_4, x_5 is the P_2 adjacent to w. If the P_2 is x_1, x_2, y_3 where $y_3 \in Y_2$, then x_1y_3 must be an edge so that y_3 is P_3-connected to y_2. So by arguments similar to Case 1a, G is in Family (e) where w, x_1, x_2 is the central C_3, vw, x_1y_1 and x_2y_2 are pendant edges, and y_3 is adjacent to both x_1 and x_2. If the P_2 is x_3, x_4, y_3 where $y_3 \in Y_2$, then y_3 cannot be P_3-connected to both y_1 and y_2. So this possibility is impossible. Finally, if the P_2 is y_3, y_4, y_5 where $y_3, y_4, y_5 \in Y_2$, then by arguments similar to Case 1a, G is in Family (d) where w, x_1, x_2 is the central C_3, vw, x_1y_1 and x_2y_2 are pendant edges, and y_3, y_4, y_5 form the P_2 which is adjacent to either x_1 or x_2.

Case 2: $\delta = 2$ **and a degree 2 node is not in a** C_3**.** Since $e \geq \frac{1}{2}(3n-5)$, we have that n is odd. Let v, w_1, w_2, X, Y, and T be as defined. All nodes save v, w_1 and w_2 have degree 1 in $G-T$. Since v is P_3-connected to each $x \in X$, every $x \in X$ has a neighbor in X. Let X_i be the neighbors of w_i in X. Since both w_1 and w_2 have degree at least 2, we have $X_i \neq \emptyset$. Since the nodes of X have degree 1 in X, the edges in X form a matching and $X_1 \cap X_2 = \emptyset$. Since each node in Y is incident to an edge in $G-T$, the edges in Y also form a matching, and each node in Y is adjacent to exactly one node in X.

Suppose there exists a pair of nodes $x_1 \in X_1$ and $x_2 \in X_2$ where x_1 has no neighbor in X_2 and x_2 has no neighbor in X_1. Since $x_1x_2 \notin E$, there is a path x_1, a, b, x_2. Our choice of x_1 and x_2 guarantees that $a \notin X_2$ and $b \notin X_1$. Since the edges within X form a matching, we also cannot have both $a \in X_1$ and $b \in X_2$. So at least one of a and b is not in X. Without loss of generality, assume $a \notin X$. If $a = w_1$, then $b \in X_1$, a contradiction. Thus $a \in Y$. Since no node in Y is adjacent to two nodes in X, we have $b \in Y$. But since a and b have degree 2, then b is not P_3-connected to x_1, a contradiction. Thus there is not a pair of nodes of this type.

Suppose $|X_1| > 1$ and $|X_2| > 1$. Then at least two edges are between X_1 and X_2, for otherwise a pair of the type precluded above would occur. Let x_1x_3 and x_2x_4 be these edges with $x_1, x_2 \in X_1$ and $x_3, x_4 \in X_2$. Since x_i can be adjacent to only one node in X, we have x_1 nonadjacent to x_2 and x_3 nonadjacent to x_4. Arguments similar to the ones above show that there are paths x_1, y_1, y_2, x_2 and x_3, y_3, y_4, x_4 where $y_i \in Y$. Since each node of Y has degree 2, the y_i are distinct, the neighbors of y_2 are y_1 and x_2, the neighbors of y_3 are y_4 and x_3, and there are no edges between $\{y_1, x_2\}$ and $\{y_4, x_3\}$. Thus y_2 is not P_3-connected to y_3, a contradiction. So either $|X_1| = 1$ or $|X_2| = 1$.

Without loss of generality, assume $X_1 = \{x_1\}$ and $X_2 = \{x_2, x_3, \ldots, x_k\}$ and where x_1x_2 is an edge. Let Y_1 be the nodes of Y adjacent to x_1. Recall that each node of Y is adjacent to exactly one node in X and the edges of Y form a matching.

Assume $|X_2| = 1$. Let $Y_2 = Y - Y_1$. Since w_1 is P_3-connected to nodes of Y_1, the edges in Y_1 form a matching. Similarly, the edges in Y_2 form a matching. So G is in Family (b) with v, w_1, x_1, x_2, w_2 forming the central C_5 with only x_1 and x_2 having pendant C_3's.

Otherwise $|X_2| > 1$. Let $y_1 \in Y_1$. Let y_2 be the neighbor of y_1 in Y. If y_2x_1 is an edge, then y_2 is not P_3-connected to x_3. If y_2x_2 is an edge, then y_2 is not P_3-connected to w_2. If y_2x_i is an edge for $i > 2$, then y_2 is not P_3-connected to x_1 because x_2 is not adjacent to x_i. So y_2 is not adjacent to X and hence y_1 cannot exist. Therefore, x_1 has degree 2 and is not adjacent to Y. Then replacing v by w_1 in the above argument we get that G is in Family (b). $\square$

References

1. L.T. Ollmann, $K_{2,2}$-saturated graphs with a minimal number of edges, Proceedings of the Third Southeast Conference on Combinatorics, Graph Theory, and Computing (1972) 367-392.

2. Z. Tuza, C_4-saturated graphs of minimum size, Acta Universitatis Carolinae–Mathematica et Physica (1989) 161-167.