TWO–LEVEL METHOD ON UNSTRUCTURED MESHES WITH CONVERGENCE RATE INDEPENDENT OF THE COARSE–SPACE SIZE *

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January 19, 1995

Abstract. A two-level algorithm for unstructured meshes with a coarse-space size independent rate of convergence is developed based on polynomial smoothing of prolongators given by unknowns aggregation. A coarse level is generated algebraically using a system of possibly large subdomains. Numerical experiments confirm the theory and demonstrate that the method performs well also on a general class of problems with highly variable coefficients.

Key words. Unstructured meshes, two-level method, coarse-space size independent rate of convergence.

AMS(MOS) subject classifications. 65N55, 65F10

1. Introduction. When solving iteratively a large system of linear equations obtained by discretization of an elliptic boundary value problem, one needs to establish a mechanism of global distribution of information in order to supress the low energy components of the error. In the case of multilevel methods, this effect is achieved by creating a system of nested coarse spaces and combining the exact solution of the coarsest level problem with smoothing on other levels. In the case of the two-level method, it is desirable to create such a coarse space that the rate of convergence of the resulting iterative process depends as little as possible on the size of the coarse level problem and the computational cost involved in both setup and iteration is small. The most common way to achieve these properties is domain decomposition with a coarse-grid ([7, 3, 8, 2]), where instead of solving the original problem, one solves the reduced system associated with interfaces of the subdomains. The advantage of this approach is that the resulting reduced formulation typically couples interface nodes at the distance of the adjacent subdomains and, consequently, some global distribution is contained in the reduced problem itself. This is reflected in terms of functional analysis as relaxation of the energetic norm, which makes the method less sensitive to the choice of the coarse space, especially its size, and allows employing coarse spaces of rather small dimension without sacrificing the convergence rate.

However, there is a relatively large amount of the computational work involved, because we need to solve the local problem on each subdomain in every iteration. Moreover, most domain decomposition algorithms are very sensitive to errors in the

* This research was supported by NSF grants ASC-9121431, ASC-9217394 and partly by internal UWB grant GR135.

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solution of local problems which makes strigthforward application of local iterative solvers problematic.

The method presented in this paper is similar in spirit to the domain decomposition, but the solving of local problems is replaced by a substantially cheaper procedure, while the rate of convergence is proved to be coarse-space size independent. Our approach is based on the combination of a piecewice constant coarsening (in the discrete sense) given by a system of auxiliary subdomains, and smoothing of resulting prolongators using special recursively constructed polynomial smoothers. We only do such an amount of smoothing so that the coarse level matrix retains the structure of the subdomains adjacency.

Under regularity-free assumptions we provide the coarse-space size independent rate of convergence for uniformly V-elliptic problems discretized on shape regular meshes. The proposed method is a modification of the one suggested in [6] where certain power of Jacobi method has been used as prolongator smoothers. The main weakness of the algorithm from [6] is its poor convergence rate of $1-C(\frac{H}{h})^{1/2}$. This disadvantage is fully eliminated here by using more powerful recursively constructed prolongator smoothers of the same degree.

The theory presented is based mostly on algebraic arguments. The main observation is that the proposed two-level algorithm with prolongators smoothed by polynomials can be interpreted as the standard two-grid method for the solution of the system of linear algebraic equations with the polynomial $\mathcal{P}(A)$ instead of A, and with the piecewise constant coarse grid in discrete sense. We show that spectral properties of the smoothed matrix $\mathcal{P}(A)$ (which is never physically constructed) are similar to the spectral properties of the stiffness matrix obtained by discretization of the elliptic problem on a possibly very coarse grid of a gridsize H. It is then not surprising that the coarse space associated with a grid of a characteristic meshsize H is large enough to assure the $H-$independent rate of convergence of the method with the smoothed matrix. In this respect, the suggested method is close to a Schur type domain decomposition with the coarse-grid. In both cases, we essentially change the original problem to one with a weaker energetic norm. This results in a weaker dependence on the ratio $\frac{H}{h}$.

There are two advantages of our approach as compared to classical domain decomposition. First, we are getting a rate of convergence independent of the coarse space size, while the typical condition number of preconditioned domain decomposition is $(1 + C\log(\frac{H}{h}))^2$. Second, the computational cost we pay for large ratio $\frac{H}{h}$ is much smaller. For our method, in the case of a quasiuniform mesh, we need to use the polynomial of degree $O(\frac{H}{h})$ as a prolongator smoother, which requires $O(n\frac{H}{h})$ operations for the construction of the prolongator. The setup of local solvers involved in classical domain decomposition in 2D requires $O(\frac{1}{H^2})$ factorizations of the local problem, the cost of each factorization is $O((\frac{H}{h})^4)$.

All steps of the proposed method except for the solving of the coarse-grid problem can be implemented in parallel. In the case of the serial implementation and the optimal ratio $\frac{H}{h}$ the total cost of the solving of the 2D problem with n degrees of freedom

is $O(n^{1.125})$ operations. Numerical experiments confirm the theory and demonstrate that the method performs well also on a general class of problems with highly variable coefficients.

The paper is organized as follows: in Section 2 we review the basic two-level algorithm and formulate the model problem. The coarsenig is the subject of Section 3, followed by theory for quasiuniform meshes in Section 4. The complexity is estimated in Section 5. In Section 6 we generalize our method for shape regular meshes. Numerical results are presented in Section 7.

2. Problem formulation and the two-level algorithm. Let Ω be a polygonal domain in $\mathbb{R}^2$. The following variational problem will be considered: find $u \in H_0^1(\Omega)$ such that

$$a(u,v) \;=\; f(v) \;\text{ for all } v \in H_0^1(\Omega)\ , \tag{2.1}$$

where $a(.,.)$ is a symmetric, V-elliptic and continuous bilinear form on H_0^1 and f is a bounded linear functional in $L_2(\Omega)$. We assume that a constant is the local kernel of $a(.,.)$ away from essential boundary conditions (i.e. no zero-order term). Let $\tau_h = \{T_i\}$ denote a quasiuniform triangulation of Ω with a characteristic meshsize h. The corresponding P1-finite element space with constrained boundary degrees of freedom will be denoted by V_h . The basis $\{\phi_i\}_{i=1}^n$ of V_h is supposed to be scaled so that the standard finite element interpolator $\Pi_h \mathbf{x} = \sum_{i=1}^{n} x_i \phi_i$ preserves unity away from the essential boundary condition in the sense that for the unit vector $\mathbf{1}$

$$\Pi_h \mathbf{1} = 1$$

on all triangles T_i which are not adjacent to $\partial\Omega$. The conforming P1 finite element discretization of (2.1) leads to the solution of a system of linear algebraic equations

$$A\mathbf{x} \;=\; \mathbf{b} \tag{2.2}$$

with a positive definite stiffness matrix A.

The standard variational two-level algorithm is determined by an injective *prolongator* $p \;:\; \mathbb{R}^m \to \mathbb{R}^n$ ($m < n$) and *smoothers* $\mathcal{S}_{pre}$, $\mathcal{S}_{post}$ which are supposed to be linear iterative methods consistent with (2.2). The iteration step $\mathbf{x}_{i+1} = TGM(\mathbf{x}_i, \mathbf{b})$ is given by :

ALGORITHM 1.

1. $\mathbf{x} \leftarrow \mathcal{S}_{pre}(\mathbf{x}_i, \mathbf{b})$,
2. $\mathbf{d} \;=\; r(A\mathbf{x} - \mathbf{b})$,
3. *solve the coarse level problem* $A^c\mathbf{v} = \mathbf{d}$,
4. $\mathbf{x} \leftarrow \mathbf{x} - p\mathbf{v}$,
5. $\mathbf{x}_{i+1} \leftarrow \mathcal{S}_{post}(\mathbf{x}, \mathbf{b})$.

where $A^c \;=\; rAp$ *and* $r = p^T$.

The linear part of Algorithm 1 is of the form

$$S_{post}[I \;-\; p(rAp)^{-1}\, rA]S_{pre}\ ,$$

where S_{pre} and S_{post} are linear parts of smoothers $\mathcal{S}_{pre}$ and $\mathcal{S}_{post}$, respectively.

In what follows, $(.,.)$ denotes the standard inner product on $\mathbb{R}^n$ and $\|.\|$ the corresponding norm. For a (symmetric) positive definite matrix B we define $(.,.)_B = (B.,.)$ and $\|.\|_B = (B.,.)^{1/2}$. The induced matrix norms will be denoted by the same symbol as their vector counterparts. The restriction of an $n \times n$ matrix M (understood as the operator) to the subspace $T \subset \mathbb{R}^n$ will be denoted by M_T.

The following lemma summarizes several results well known from the theory of the two-level method:

LEMMA 2.1. *Let a matrix B be positive definite on $\mathbb{R}^n$, and $T = Ker(rB)$. Then*

1. *$I - p\,(rBp)^{-1}\,rB$ is a $B-$orthogonal projection onto T,*
2. *$\| S[I - p\,(rBp)^{-1}\,rB] \|_B \;\leq\; \|S_T\|_B$,*
3. *if $S = I - \varrho^{-1/2}\omega\, B$, $\omega \in (0,2)$ then for every $\mathbf{x} \in \mathbb{R}^n$*

$$\frac{\|S\mathbf{x}\|_B^2}{\|\mathbf{x}\|_B^2} \;\leq\; 1 - \frac{\|\,B\mathbf{x}\|^2}{\|\mathbf{x}\|_B^2} \cdot \frac{\omega}{\varrho(B)} \cdot (2-\omega)\ .$$

4. *Let us suppose that the following approximation condition is fulfilled : there exists a constant $C_{appr} > 0$ such that for every $\mathbf{u} \in \mathbb{R}^n$ there exists a $\mathbf{v} \in \mathbb{R}^m$ such that*

$$\|\ \mathbf{u}\ -\ p\mathbf{v}\ \|\ \leq\ C_{appr}\,\varrho(B)^{-\frac{1}{2}}\ \|\mathbf{u}\|_B\ . \tag{2.3}$$

Then

$$\frac{\|\,B\mathbf{u}\,\|}{\|\mathbf{u}\|_B}\ \geq\ C_{appr}^{-1}\,\varrho(B)^{1/2} \quad \textit{for every}\ \ \mathbf{u} \in T$$

Proof. Statements 1.–3. are trivial. The proof of 4. can be found in [4], Lemma 6.4.8. □

3. Coarsening strategy in the quasiuniform case. The input data of the method are the system of linear algebraic equations and the system of subdomains covering the domain Ω. For simplicity, we assume that the given subdomains are almost squares sharing edges and verices, although both method and theory can easily be extended to the case of more general decompositions.

We will cover Ω by an auxiliary square grid of a gridsize H so that no node of τ_h lies on the lines of the grid. (If appropriate, the grid can be deformed by a Lipschitz maping.) The grid divides Ω into a system of subdomains Ω_i. Let us define $\mathcal{D} = \{\ \Omega_i : \partial\Omega_i \cap \partial\Omega = \emptyset\ \}$, $m = \mathrm{card}(\mathcal{D})$ and for the time being we assume that each $\Omega_i \in \mathcal{D}$ contains at least one node.

Our prolongators are constructed in two steps. In the first step, we generate a tentative prolongator $p : \mathbb{R}^m \to \mathbb{R}^n$ using the unknowns aggregation technique based on $\mathcal{D}$. Each subdomain $\Omega_i \in \mathcal{D}$ (understood as the aggregate of degrees of freedom) gives rise to one degree of freedom on the coarse level.

$$p_{ij} = \begin{cases} 1 & \text{if the } i\text{-th node of } \tau_h \text{ is in } \Omega_j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.1}$$

In the second step, we apply Chebyschev-like polynomials in order to suppress high energy components in Range(p).

Let $\bar{\lambda}_0$ be a given bound of $\varrho(A)$ satisfying

$$\varrho(A) \leq \bar{\lambda}_0 \leq C\varrho(A)\,. \tag{3.2}$$

For construction of the *smoothed prolongator* we set $A_0 = A$ and define for $i = 0, 1, \ldots$ polynomial smoothers S_i, P_i recursively by

$$S_i = I - \frac{\omega}{\bar{\lambda}_i} A_i\ , \quad P_i = S_i S_{i-1} \ldots S_0\ , \tag{3.3}$$

where

$$A_i = S_{i-1}^2 A_{i-1},\ \bar{\lambda}_i = \frac{1}{9}\bar{\lambda}_{i-1}$$

and $\omega = \frac{4}{3}$. The choice of ω will be justified in Lemma 4.1.

For a given $L \geq 0$, we define the smoothed prolongator by

$$p_L \ = \ P_{L-1} p_0\ ,$$

where $p_0 = p$.

Here, the crucial question is the one of the choice of the parameter L. Intuitively, large L improves the convergence but causes the increase of the fill-in of the coarse-level matrix $A_L^c = r_L A p_L$, $r_L = p_L^T$. It is obvious that the nonzero structure of A_0^c (no smoothing) is one of the 9-point scheme corresponding to the adjacency of subdomains Ω_j in $\mathcal{D}$.

If the ratio $\frac{H}{h}$ is sufficiently large (which is true in the practically interesting cases) the structure of matrices A_L^c and A_0^c is the same even for some nonzero L. In our coarsening, we choose maximal L such that the structure of A_L^c is still a 9-point scheme.

Now we are ready to write down the implementation of Algorithm 1.

Setup phase:

Construct p using the unknowns aggregation (3.1), set $p_0 = p$.

For i = 1,2, ...

$p_i \ = \ S_{i-1} p_{i-1}$, where S_{i-1} is defined recursively by (3.3),

stop when the structure of A_{i+1}^c fails to follow the nine point scheme.

Set $L = i$, $A_L^c = p_L^T A p_L$.

Let us stress out that it is not necessary to assemble the polynomials S_{i-1}, as the operation $p_i \ = \ S_{i-1} p_{i-1}$ can be performed using Horner's scheme. Further, there is no need to assemble the matrix A_{i+1}^c in order to check its structure. Let $\mathbf{e}_j$ be the j-th canonical basis vector of $\mathbb{R}^m$. Considering possible numerical zeroes as nonzeroes, the structure of A_i^c corresponds to overlaps of supports of coarse space basis functions $\Pi p_i \mathbf{e}_j$. Supports of nonsmoothed basis functions $\Pi p \mathbf{e}_j$ are of the form

$$\text{supp}\ \Pi p \mathbf{e}_j = \cup T_k,$$

where the union expands over all k such that at least one vertex of triangle T_k lies in Ω_j. Applying the polynomial smoother of degree l causes adding l strips of elements to the support of each basis function. Thus, checking of the nonzero structure can be done using the test of intersections of lists of elements. Using the same argument, we can easilly conclude that if the ratio $\frac{H}{h}$ is large enough then for L obtained by the setup we have

$$c\frac{H}{h} \leq \deg(P_{L-1}) \leq C\frac{H}{h} \tag{3.4}$$

where deg denotes the degree of the polynomial.

Iteration phase:

In Algorithm 1 we use $p = p_L$ and smoothers $\mathcal{S}_{pre}$, $\mathcal{S}_{post}$ given by Chebyschev type iterative methods with linear parts $S_{pre} = P_{L-1}$, $S_{post} = S_L$ consistent with (2.2).

The advantage of this particular choice of multigrid components is that it allows our method to be viewed as the standard TGM for solution of the system of linear algebraic equations with the matrix A_L instead of A. As $A_L = P_{L-1}^2 A$, $p_L = P_{L-1}p_0$ and $P_L = S_L P_{L-1}$ the linear part of our method will be

$$S_L[I - p_L(r_L A p_L)^{-1} r_L A]P_{L-1} = P_L[I - p(r_0 A_L p_0)^{-1} r_0 A_L] \ . \tag{3.5}$$

The injectivity of p_L (and the invertibility of A_L^c) follows from [6], Lemma 5.2.

4. Theory. In view of (3.5), the convergence of the proposed method, depends on the operator norm $\|P_L[I - p_0(r_0 A_L p_0)^{-1} r_0 A_L]\|_{A_L}$. To this end, let estimate the spectral radius $\varrho(A_L)$.

LEMMA 4.1. *There exist constants C_1, C_2 independent of the ratio $\frac{H}{h}$ and the number of elements in τ_h such that*

$$\varrho(A_L) \leq C_1 \left(\frac{1}{9}\right)^L \varrho(A) \leq C_2 \left(\frac{h}{H}\right)^2 .$$

Proof. First we will prove

$$\varrho(A_i) \ \leq \ \bar{\lambda}_i \ , \quad \text{for } i = 0, 1, \ldots . \tag{4.1}$$

For $i = 0$ the statement follows from (3.2). Assume that (4.1) is valid for some $i > 0$. From (3.3) and $\omega = \frac{4}{3}$ we have

$$\begin{aligned} \varrho(A_{i+1}) \ &= \ \varrho(\,(I - \frac{\omega}{\bar{\lambda}_i} A_i)^2 \, A_i) \ = \ \max_{t \in \sigma(A_i)} (1 - \frac{\omega}{\bar{\lambda}_i} t)^2 \, t \ \leq \\ &\leq \ \max_{t \in (0, \bar{\lambda}_i]} (1 - \frac{\omega}{\bar{\lambda}_i} t)^2 \, t \ = \ \frac{1}{9}\bar{\lambda}_i \ = \ \bar{\lambda}_{i+1}, \end{aligned}$$

concluding the proof of (4.1). Now the first inequality of the statement follows from $\bar{\lambda}_{i+1} = \frac{1}{9}\bar{\lambda}_i$ and (3.2).

To prove the second inequality we need the lower estimate of L. ¿From (3.4) we have (for $\frac{H}{h}$ large enough) $\deg(P_L) \geq C\frac{H}{h}$. The definition (3.3) gives

$$\deg(P_{L-1}) = \sum_{i=0}^{L-1} \deg(S_i) = \sum_{i=0}^{L-1} 3^i < \frac{3}{2}3^{L-1}$$

and therefore $3^L \geq C\frac{H}{h}$. Using this estimate the second inequality of the statement follows from the first one. □

LEMMA 4.2. *There exists a constant C independent of the ratio $\frac{H}{h}$ and the number of elements in τ_h such that for every $\mathbf{x} \in Ker(r_0 A)$*

$$\|A\mathbf{x}\| \geq C\frac{h}{H}\|\mathbf{x}\|_A,$$

where $r_0 = p_0^T$ is the unsmoothed restriction.

Proof. As $\varrho(A) \leq C$, to prove the statement it is sufficient to verify (2.3) for $B = A$ and $C_{appr} \leq C\frac{H}{h}$. This has been proved in [6], Lemma 6.4. □

The smoother P_L in (3.5) can be written as the product $P_L = S_L P_{L-1}$. Since $\varrho(S_L) < 1$, $\varrho(P_{L-1}) < 1$ and A_L, S_L, P_{L-1} are commuting, we have

$$\| S_L P_{L-1}\mathbf{x}\|_{A_L} \leq \min\{ \| S_L\mathbf{x} \|_{A_L}, \| P_{L-1}\mathbf{x} \|_{A_L} \}. \tag{4.2}$$

In the next lemma we will show that for an error $\mathbf{e} \in \mathrm{Range}(I - p_0(r_0 A_L p_0)^{-1} r A_L)$ (i.e. $\mathbf{e} \in Ker(r_0 A_L)$) either S_L or P_{L-1} is an efficient smoother.

LEMMA 4.3. *There exist positive constants C_1, C_2 independent of the ratio $\frac{H}{h}$ and the number of elements in τ_h such that for every $\mathbf{x} \in Ker(r_0 A_L)$ at least one from the following estimates is valid:*

1. $\| S_L\mathbf{x} \|_{A_L} \leq (1 - C_1)\|\mathbf{x}\|_{A_L}$,
2. $\| P_{L-1}\mathbf{x} \|_{A_L} \leq (1 - C_2)\|\mathbf{x}\|_{A_L}$.

Proof. Let us start with the estimate of $\frac{\|A_L\mathbf{x}\|^2}{\|\mathbf{x}\|^2_{A_L}}$. For every $\mathbf{x} \in \mathbb{R}^n$ (3.3) yields

$$\frac{\|A_L\mathbf{x}\|^2}{\|\mathbf{x}\|^2_{A_L}} = \frac{(AP^2_{L-1}\mathbf{x}, AP^2_{L-1}\mathbf{x})}{(AP^2_{L-1}\mathbf{x}, \mathbf{x})} =$$

$$= \frac{(AP^2_{L-1}\mathbf{x}, AP^2_{L-1}\mathbf{x})}{(AP^2_{L-1}\mathbf{x}, P^2_{L-1}\mathbf{x})} \cdot \frac{(AP^2_{L-1}\mathbf{x}, P^2_{L-1}\mathbf{x})}{(AP^2_{L-1}\mathbf{x}, \mathbf{x})} \quad . \tag{4.3}$$

Now, consider $\mathbf{x} \in Ker(r_0 A_L)$. As $A_L = AP^2_{L-1}$ we have $P^2_{L-1}\mathbf{x} \in Ker(r_0 A)$. Therefore Lemma 4.2 gives

$$\frac{(AP^2_{L-1}\mathbf{x}, AP^2_{L-1}\mathbf{x})}{(AP^2_{L-1}\mathbf{x}, P^2_{L-1}\mathbf{x})} \geq C(\frac{h}{H})^2.$$

For the second term in (4.3) we have

$$\frac{(AP_{L-1}^2\mathbf{x}\,,P_{L-1}^2\mathbf{x})}{(AP_{L-1}^2\mathbf{x}\,,\,\mathbf{x})}\;=\;\frac{(A_L\mathbf{x}\,,P_{L-1}^2\mathbf{x})}{(A_L\mathbf{x}\,,\,\mathbf{x})}\;=\;\frac{\|\,P_{L-1}\mathbf{x}\,\|_{A_L}^2}{\|\mathbf{x}\|_{A_L}^2}$$

and consequently

$$\frac{\|A_L\mathbf{x}\|^2}{\|\mathbf{x}\|_{A_L}^2}\geq C(\frac{h}{H})^2\cdot\frac{\|\,P_{L-1}\mathbf{x}\,\|_{A_L}^2}{\|\mathbf{x}\|_{A_L}^2}. \tag{4.4}$$

Let us consider some $\alpha\in(0,1)$. Two cases can occur:

(i) $\frac{\|\,P_{L-1}\mathbf{x}\,\|_{A_L}}{\|\mathbf{x}\|_{A_L}}\;\leq\;\alpha$. Then the statement 2. is trivially valid with $C_2=1-\alpha$.

(ii) $\frac{\|\,P_{L-1}\mathbf{x}\,\|_{A_L}}{\|\mathbf{x}\|_{A_L}}\;>\;\alpha$. In this case (4.4) and Lemma 4.1 imply

$$\frac{\|A_L\mathbf{x}\|^2}{\|\mathbf{x}\|_{A_L}^2}\;\geq\;C\alpha(\frac{h}{H})^2\;\geq\;C\alpha\varrho(A_L).$$

The proof of 1. follows from the statement 3. of the Lemma 2.1. □

THEOREM 4.4. *Let* $\mathbf{e_i}$ *denotes the error after i-th iteration. Then for the problem (2.1) discretized on a quasiuniform mesh it holds that*

$$\|\;\mathbf{e_{i+1}}\|_{A_L}\;\leq\;(1-C)\|\;\mathbf{e_i}\|_{A_L}\,,$$

where the constant C *does not depend on the ratio* $\frac{H}{h}$ *and the number of elements in* τ_h.

Proof. The proof follows from (3.5), statement 2 of the Lemma 2.1, (4.2) and Lemma 4.3. □

5. Computational complexity and choice of H. The purpose of this section is to analyze the complexity of the proposed method and suggest an optimal choice of the coarse grid parameter H. We will consider two kinds of direct methods used to solve the coarse-level problem: *Choleski decomposition* and the more sophisticated *nested dissection*, taking advantage of the fact that the coarse problem belongs to a square grid, (see [5]).

The setup phase consists of generation of p_L, A_L^c and factorization of A_L^c. The construction of p_0 takes $O(n)$ operations. ¿From (3.4), the degree of the prolongator smoother $P_{L-1}=S_{L-1}\ldots S_0$ is $O(\frac{H}{h})$. As all matrices involved are sparse the total expense of generating the smoothed prolongator $p_L=P_{L-1}p_0$ and the coarse matrix $A_L^c=r_LAp_L$ is $O(n\frac{H}{h})$ operations. The order of the coarse-level matrix A_L^c is $O(\frac{1}{H^2})$, its bandwidth is $O(\frac{1}{H})$. Choleski factorization of A^c requires $O(\frac{1}{H^4})$ operations. If the nested dissection is used the expenditure of the factorization step is $O(\frac{1}{H^3})$.

Summing up, the setup phase requires

$$O(n\frac{H}{h})+O(\frac{1}{H^4})$$

operations if Choleski decomposition is used and

$$O(n\frac{H}{h}) + O(\frac{1}{H^3})$$

in the case of the nested dissection.

The iteration consists of the smoothing steps using the iterative method with linear parts $S_{pre} = P_{L-1}$ and $S_{post} = S_L$, computation of the residual, prolongating, restricting and solving the coarse-level problem with the matrix A_L^c. Prolongating, restricting and evaluating of the residual takes $O(n)$ operations. From (3.4) the cost of smoothing is $O(n\frac{H}{h})$. The back substitution requires $O(\frac{1}{H^3})$ operations in the case of Choleski decomposition and $O(\frac{1}{H^2})$ operations if the nested dissection is used. Therefore the total cost of the iteration phase is

$$O(n\frac{H}{h}) + (\frac{1}{H^3})$$

if the Choleski decomposition is used and

$$O(n\frac{H}{h}) + (\frac{1}{H^2})$$

in the case of the nested dissection.

Taking into account the fact that $n < C\frac{1}{h^2}$, the requirement that the cost of the setup phase be the same as the cost of the iteration phase yields the following result:

THEOREM 5.1. *If the Choleski decomposition is used for the solving of the coarse-level problem, choosing $H \approx h^{3/5}$, the system (2.2) can be solved to the level of truncation error in $O(n^{1.2})$ operations. For the case of the nested dissection and the choice $H \approx h^{3/4}$ the cost of the solving of (2.2) is $O(n^{1.125})$.*

6. Shape regular meshes. The proposed method can easily be generalized for the case of the shape regular (i.e. locally quasiuniform) meshes. Except for two arguments, the theory is algebraic and independent of properties of the mesh and the bilinear form $a(.,.)$.

The first, geometrical argument is (3.4). Roughly speaking, (3.4) means that it is possible to apply the prolongator smoother of degree greater than $C\frac{H}{h}$ to the tentative prolongator p_0 without undesirable increase of the fill-in of the coarse-level matrix. In other words, every line $\mathcal{L}$ connecting two subdomains Ω_i, $\Omega_j \in \mathcal{D}$ such that

- $\partial\Omega_i \cap \partial\Omega_j = \emptyset$,
- $\partial\Omega_i \cap \partial\Omega_k \neq \emptyset$ and $\partial\Omega_j \cap \partial\Omega_k \neq \emptyset$ for some subdomain $\Omega_k \in \mathcal{D}$

intesects approximately $\frac{H}{h}$ elements in Ω_k (if $\mathcal{L} \cap \Omega_k \neq \emptyset$).

The second argument is the approximation property of the coarse-space given by p_0 needed in the proof of Lemma 4.2:
For every $\mathbf{u} \in \mathbb{R}^n$ there exists $\mathbf{v} \in \mathbb{R}^m$ such that

$$\|\mathbf{u} - p_0\mathbf{v}\| \leq C\frac{H}{h}\|\mathbf{u}\|_A. \tag{6.1}$$

The proof of (6.1) (see [6]) is a consequence of the fact that the coarse-space Range($\Pi_h p_0$) is able to approximate the constant in local sense, scaled Poincaré-Friedrichs inequality on Ω_i and the equivalence of discrete and continuous L_2 norms on V_h with a scaling factor h.

Further, there is a formal trouble with the ratio $\frac{H}{h}$ which is meaningless in the case of a nonquasiuniform mesh and it must be replaced by a more general term.

In order to generalize the proposed method for shape-regular meshes it is sufficient for the subdomains in $\mathcal{D}$ to be ruled by the local properties of the mesh. For $\mathcal{D}$ we require:

ASSUMPTION 1.

(i) Each subdomain Ω_i can be mapped onto a unit square $\widehat{\Omega}$ by a mapping F_i so that for its Jacobian ∂F_i it holds that for $x \in T_j$

$$\mathrm{cond}(\partial F_i(x), \frac{h_j}{\widehat{h}} I) < C, \tag{6.2}$$

where h_j is a diameter of the element T_j. In other words Ω_i can be mapped onto $\widehat{\Omega}$ in such a way that the mapped mesh is a quasiuniform mesh of a characteristic meshsize $\widehat{h}$. (mapped elements can be curved)

(ii) Let us set $\Omega_{int} = \bigcup_{i=1}^{m} \Omega_i$. Each subdomain Ω_i lies in the interior of Ω and $\Omega \setminus \Omega_{int}$ consists of at most $C\widehat{h}^{-1}$ strips of elements closest to $\partial\Omega$.

(iii) (conformity) For any subdomain Ω_i, its edge e is either an edge of another subdomain or a part of $\partial\Omega_{int}$.

The graph algorithm generating the system of subdomains $\mathcal{D}$ which tends to satisfy ASSUMPTION1 is described in [11].

The number $\widehat{h}^{-1}$ is a natural generalization of the ratio $\frac{H}{h}$. The inequality (3.4) is trivially valid with $\widehat{h}^{-1}$ instead of $\frac{H}{h}$.

Due to (6.2) the H^1-seminorm is scaled uniformly, i.e. for $\widehat{u}(\widehat{x}) = u(F_i^{-1}(\widehat{x}))$, $\widehat{x} \in \widehat{\Omega}$

$$c|u|_{H^1(\Omega_i)} \leq |\widehat{u}|_{H^1(\widehat{\Omega})} \leq C|u|_{H^1(\Omega_i)}$$

with constants c, C independent of $\widehat{h}$ and i. Let us define the discrete l_2 norm by $\|u\|_{l_2(\Omega_i)} = (\sum_{v_j \in \Omega_i} u^2(v_j))^{1/2}$, where v_j are nodes of the triangulation. Standard manipulations give the uniform equivalence

$$c\|\widehat{u}\|_{L_2(\widehat{\Omega})} \leq \widehat{h}\|u\|_{l_2(\Omega_i)} \leq C\|\widehat{u}\|_{L_2(\widehat{\Omega})}.$$

The Poincaré-Friedrichs inequality on $\widehat{\Omega}$ and two equivalences above imply the scaled Poincaré-Friedrichs inequality on Ω_i :
For every $u \in H^1(\Omega_i)$ such that $\int_{\Omega_i} u dx = 0$

$$\|u\|_{l_2(\Omega_i)} \leq C\widehat{h}^{-1}|u|_{H^1(\Omega_i)}. \tag{6.3}$$

Following the proof of Lemma 6.4 in [6] and using (6.3) we get the generalization of (6.1):
For every $\mathbf{u} \in \mathbb{R}^n$ there exists $\mathbf{v} \in \mathbb{R}^m$ such that

$$\|\mathbf{u} - p_0\mathbf{v}\| \leq C\widehat{h}^{-1}\|\mathbf{u}\|_A. \tag{6.4}$$

Replacing $\frac{H}{h}$ by $\widehat{h}$ in Section 4 and using (6.4) in the proof of Lemma 4.2 we get the following result:

THEOREM 6.1. *Let the* ASSUMPTION *1 be satisfied and let* $\mathbf{e_i} \in \mathbb{R}^n$ *denote the error after the i-th iteration. Then for the problem (2.1) discretized on the shape regular mesh it holds that*

$$\| \mathbf{e_{i+1}}\|_{A_L} \leq (1 - C)\| \mathbf{e_i}\|_{A_L},$$

where the constant C *does not depend on* $\widehat{h}$ *and the number of elements.*

7. Numerical experiments. The efficiency of the algorithm is demonstrated by four experiments below. In Experiment 2 (planar elasticity on highly unstructured mesh) the standard block approach (see e.g [11], section 5.1) has been used in order to handle the vector problem with 2 degrees of freedom per node. The system of subdomains has been generated by graph Algorithm 2 in [11] (with the patemeter $\varepsilon = 0$) used 2 times to get sufficiently large subdomains. In experiments 3 and 4 smoothers S_L and P_{L-1} were polynomials in the matrix $D^{-1}A$ instead of A.

All experiments were performed on an IBM RS-6000/360 with 128 MBytes of memory. The residual was measured in the l^2 norm. The rate of convergence is computed as an average reduction of l^2-norm of the residual per iteration.

EXPERIMENT 1:

Problem	Laplace equation
Boundary conditions	Dirichlet
Mesh	regular square mesh
Number of nodes	160 000
Number of equations	160 000
Size of coarse-grid	2 916
Stopping condition	relative residual smaller than 10^{-5}
Rate of convergence	0.091
Real time of computation	87 s
CPU time of computation	92 s

EXPERIMENT 2:

Problem	planar elasticity
Poisson ratio	0.3
Boundary conditions	Dirichlet and Neumann
Mesh	see *Figure 7.1*
Number of nodes	10 679
Number of equations	21 358
Stopping condition	relative residual smaller than 10^{-5}
Size of coarse-grid	118 nodes, 236 degrees of freedom
Average number of nonzeroes per column	8.6
Rate of convergence	0.079
Real time of computation	9 s
CPU time of computation	11 s

EXPERIMENT 3:

Problem	Second order problem (7.1)
Boundary conditions	Dirichlet
Mesh	regular square mesh
Number of nodes	160 000
Number of equations	160 000
Size of coarse-grid	2 916
Stopping condition	relative residual smaller than 10^{-5}
Rate of convergence	0.103
Real time of computation	106 s
CPU time of computation	99 s

$$-\frac{\partial}{\partial x}\, a(x,y)\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\, a(x,y)\frac{\partial u}{\partial y} = f(x,y) \tag{7.1}$$

where

$$a = \begin{cases} 10^{-2} & \text{on } (0,\ 0.5) \times (0,\ 0.5) \\ 10^{+2} & \text{on } (0,\ 0.5) \times [0.5,\ 1) \\ 1 & \text{on } (0,\ 1) \quad \times [0.5,\ 1). \end{cases} \tag{7.2}$$

EXPERIMENT 4:

Problem	3D problem (7.3)
Boundary conditions	Dirichlet
Mesh	regular square mesh
Number of nodes	21 952
Number of equations	21 952
Size of coarse-grid	27 (27-point scheme)
Stopping condition	relative residual smaller than 10^{-5}
Rate of convergence	0.199
Real time of computation	76 s
CPU time of computation	52 s

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}(w_{ij}(x,y)\frac{\partial u}{x_j}) = f(x,y), \tag{7.3}$$

with random coefficients $w_{11} = \exp(rn_1)$, $w_{22} = \exp(rn_2)$, $w_{33} = \exp(rn_3)$, $w_{ij} = 0$ for $i \neq j$, rn_i is a random number uniformly distributed in the in terval $[\ln(10^{-2}), \ln(10^2)]$.

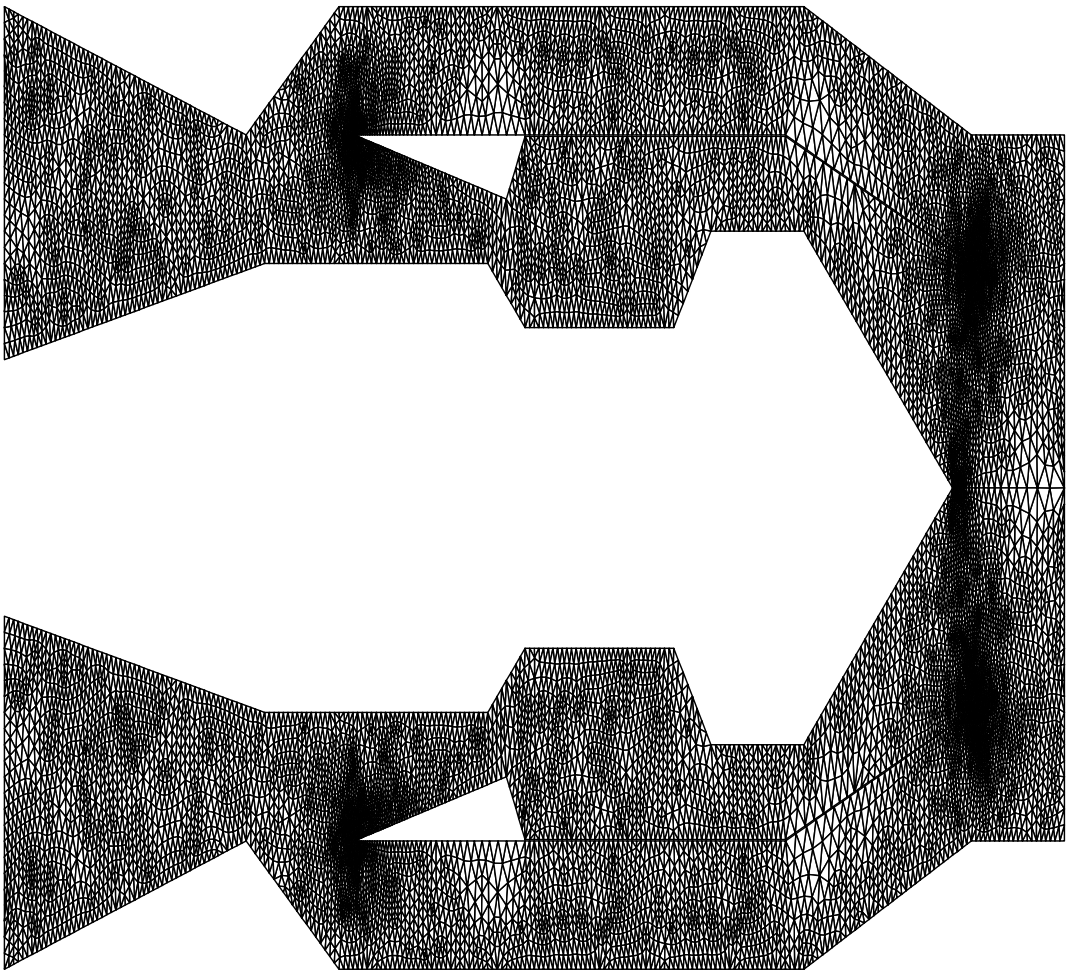

FIG. 7.1. *Mesh 1 (Courtesy of Charbel Farhat, Center for Aerospace Engineering, Unive rsity of Colorado, Boulder)*

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