ALGEBRAIC MULTIGRID BY SMOOTHED AGGREGATION FOR SECOND AND FOURTH ORDER ELLIPTIC PROBLEMS*

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Abstract. An algebraic multigrid algorithm is developed based on prolongations by smoothed aggregation. Coarse levels are generated automatically. Guidelines for the selection of method components are presented based on energy considerations. Efficiency of the resulting algorithm is demonstrated by computational results.

Key words. Algebraic multigrid, unstructured meshes, automatic coarsening, biharmonic equation

AMS(MOS) subject classifications. 65N55, 65F10

1. Introduction. Multigrid methods are very efficient iterative solvers for systems of algebraic equations arising from finite element and finite difference discretizations of elliptic boundary value problems. The main principle of multigrid methods is to complement the local exchange of information in point-wise iterative methods by a global one utilizing several related systems, called *coarse levels,* with a smaller number of variables. The coarse levels are often obtained as a hierarchy of discretizations with different characteristic meshsizes, but this requires that the discretization is controlled by the iterative method. To solve linear systems produced by existing finite element software, one needs to create an artificial hierarchy of coarse problems. The principal issue is then to obtain computational complexity and approximation properties similar to those for nested meshes, using only information in the matrix of the system and as little extra information as possible.

Such algebraic multigrid method that uses the system matrix only was developed by Ruge, et al. [10, 4, 11]. The prolongations were based on the matrix of the system by partial solution from given values at selected coarse points [1]. The coarse grid points were selected so that each point would be interpolated to via so-called strong connections.

Our approach is based on *smoothed aggregation* introduced recently by Vaněk [14, 13]. First the set of nodes is decomposed into small mutually disjoint subsets. A tentative piecewise constant interpolation (in the discrete sense) is then defined on those subsets as piecewise constant for second order problems, and piecewise linear for fourth order problems. The prolongation operator is then obtained by smoothing the output of the tentative prolongation and coarse level operators are defined variationally. Multigrid method based on such prolongations converges very fast for a wide range of problems

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including those with strongly anisotropic and discontinuous coefficients and, in addition, it has a remarkably low computational complexity since the typical coarsening ratio is about three in each dimension.

Almost optimal theoretical bounds for our method were given by the authors in [15] for second order problems and under natural assumptions on the coarse level hierarchy that tend to be satisfied by our coarsening algorithm, namely that the coarsening is by about the factor of three, and that the aggregates of the nodes are based on aggregated elements that form a reasonable mesh of macroelements. A bound on the energy of the coarse level basis functions was proved and used to verify the assumptions of the multilevel regularity-free approach of Bramble, Pasciak, Wang, and Xu [3]. The theory can be extended to fourth order problems once similar energy bounds are available for that case.

The part of this paper dealing with second order problems is based on [15]. The algorithm for fourth order problems is new. For more details and theory for the second order case, see [15].

For other multigrid approaches to the biharmonic equation, see [5, 9, 16, 8]. For a multigrid theory for the biharmonic equation with non-nested finite element spaces, see [2].

1.1. Basic Multigrid Algorithm. For reference, we state the basic multigrid algorithm for the solution of the system of linear algebraic equations $Ax = b$. First, a preprocessing stage creates full rank prolongation matrices P_l of size $n_l \times n_{l+1}$, $l = 1, \ldots, L-1$. by an automatic coarsening process described below. The coarse level matrices are defined by

$$A_1 = A, \qquad A_{l+1} = P_l^T A_l P_l, l = 1, \ldots, L-1.$$

The iterations then proceed as follows.

ALGORITHM 1 (BASIC MULTIGRID). *To solve the system $A_l x^l = b^l$, do:*

Pre-smoothing: *do ν_1 times $x^l \leftarrow \mathcal{S}^l(x^l, b^l)$*

Coarse grid correction:

- *let $b^{l+1} \leftarrow P_l^T(b^l - A_l x^l)$*
- *If $l+1 = L$, solve $A_{l+1}x^{l+1} = b^{l+1}$ by a direct method, otherwise apply γ iterations of this algorithm on level $l+1$, starting with initial guess $x^{l+1} = 0$*
- *correct the solution on level l by $x^l \leftarrow x^l + P_l x^{l+1}$*

Post-smoothing: *do ν_2 times $x^l \leftarrow \mathcal{S}^l(x^l, b^l)$.*

We use $\nu_1 = \nu_2 = \gamma = 1$ with the pre-smoothing iteration consisting of one forward iteration of the Gauss-Seidel followed by one iteration of backward SOR. The post-smoothing iteration consists of one forward SOR iteration followed by an iteration of backward Gauss-Seidel. The over-relaxation parameter used is 1.85 in both pre and post-smoothing.

Each level is associated with basis functions $\{\varphi_i^l\}_{i=1}^{n_l}$. The basis functions on the finest level are given as finite element shape functions, while the coarse level basis

functions are determined from the prolongations by

$$\begin{bmatrix} \varphi_1^{k+1} \\ \vdots \\ \varphi_{n_{k+1}}^{k+1} \end{bmatrix} = P_k^T \begin{bmatrix} \varphi_1^k \\ \vdots \\ \varphi_{n_k}^k \end{bmatrix} \qquad k = 1, \ldots, L-1.$$

2. Algebraic Multigrid for Second Order Problems. Consider discretization by standard conforming linear finite elements of a second order elliptic variational problem

$$u \in V: \quad a(u,v) = f(v) \qquad \forall v \in V \tag{2.1}$$

where $V = H^1_{\Gamma_D}(\Omega)$ denotes the Sobolev space of H^1 functions vanishing on $\Gamma_D \subset \partial\Omega$, $\mu(\Gamma_D) > c\mu(\partial\Omega)$, Ω a domain in $\mathbb{R}^2$. The bilinear form

$$a(u,v) = \int_\Omega \sum_{i,j} a_{ij} \partial_i u \partial_j v \tag{2.2}$$

is assumed to be symmetric, V-elliptic, and bounded,

$$c_1 \|u\|^2_{H^1(\Omega)} \leq a(u,u) \leq c_2 \|u\|^2_{H^1(\Omega)}, \qquad \forall u \in V. \tag{2.3}$$

Moreover we assume, that the finite element basis forms a decomposition of unity

$$\sum_{i=1}^{n_1} \varphi_i^1 = 1 \tag{2.4}$$

away from essential boundary conditions.

2.1. Construction of Prolongations for second order elliptic problems. The prolongation operators are chosen to achieve low energy of coarse basis functions, leading to good theoretical estimates of the convergence of the iterations, as well as by sparsity considerations to achieve low computational complexity of the iterations. We are looking for prolongations that satisfy the following properties. First we specify the desired properties of the support of the coarse shape functions (or, equivalently, the allowed nonzeros of the prolongation matrices), and then the numerical values of the nonzero entries.

(AMG1) Coarse supports should follow strong couplings. We require that every two nodes in the support of a coarse basis function can be connected by a path of strong couplings. Two nodes i and j on level l are strongly coupled if $|a^l_{ij}|$ is relatively large compared with $\sqrt{|a^l_{ii} a^l_{jj}|}$. Essentially, we want to assure that the algorithm will provide the semi-coarsening in the case of solving of the anisotropic problem ([6], [12]). Algebraically, the anisotropy is reflected in the coefficients of the stiffness matrix in the sense that the neighboring nodes are strongly coupled in the direction of anisotropy.

(AMG2) Bounded intersection. Support of each basis function intersects a bounded number of supports of other basis functions on the same level only. The number of intersections does not depend on the level. This property guarantees sparsity of the resulting coarse-level matrices.

(AMG3) Decomposition of unity. Every coarse space V_l should represent the constant function exactly, aside from essential boundary condition. This requirement is motivated by the need to bound locally the error of a coarse grid approximation $P_l v^{l+1}$ of a fine grid function u^l in terms of the energy $(u^l)^T A_l u^l$ and by the fact that the constant function has zero energy because of (2.2). Because of (2.4), this is equivalent to the requirement that the columns of each prolongation matrix form a decomposition of unity

$$\sum_{j=1}^{n_{l+1}} P_{ij} = 1, \qquad l = 1, \ldots, L-1,$$

for all rows i that do not correspond to degrees of freedom adjacent to an essential boundary condition. For generalizations, see Sections 3.1 and 3.3.

(AMG4) Small energy of coarse basis functions. We require that the energy of the coarse space basis functions be almost minimal in the sense that

$$\frac{a(\varphi_i^l, \varphi_i^l)}{||\varphi_i^l||^2_{L^2(\Omega)}} \leq C \inf_{u \in H_0^1(\mathrm{supp}\varphi_i^l)} \frac{a(u,u)}{||u||^2_{L^2(\Omega)}}.$$

Note that in the case of uniformly V-elliptic problems the requirement above together with bounded intersections of supports of basis functions (AMG2) assures the standard inverse inequality on each coarse space.

(AMG5) Uniform l^2 equivalence. Discrete l_2 norms on all spaces V_l should be uniformly equivalent up to diagonal scaling. The scaling may depend on the measure of the support of basis function and type of degree of freedom. For the algorithm described in this section, such uniform equivalence has been proved in [15].

We now construct prolongations P_l based on the matrix A_l. First we create a tentative piecewise constant prolongator satisfying all of the above properties except for the energy bound in (AMG4). This prolongator will then be smoothed to satisfy (AMG4), while preserving the other properties.

We start by specifying a disjoint decomposition of the set of nodes on level l. Every component of the decomposition on level l (so-called *aggregate*) gives rise to one degree of freedom on level $l+1$.

Motivated by the requirement (AMG1) above, for a given ε we define the *strongly-coupled neighborhood of node* i as

$$N_i^l(\varepsilon) = \{j : \ |a_{ij}| \geq \varepsilon \sqrt{a_{ii} a_{jj}}\} \cup \{i\} \tag{2.5}$$

ALGORITHM 2 (AGGREGATION). *Let the matrix A_l of order n_l and $\varepsilon \in [0,1)$ be given. Generate a disjoint covering $\{C_i^l\}_{i=1}^{n_{l+1}}$ of the set $\{1,\ldots,n_l\}$ as follows.*

Initialization *Set $R = \{1,\ldots,n_l\}$ and $j = 0$.*

Step 1 *Select disjoint strongly coupled neighborhoods as the initial attempted covering: If there exists a strongly coupled neighborhood $N_i^l(\varepsilon) \subset R$, set $j \leftarrow j+1$, $C_j^l \leftarrow N_i^l(\varepsilon)$, $R \leftarrow R \setminus C_j^l$. Repeat until R does not contain any strongly coupled neighborhood.*

Step 2 *Add each remaining $i \in R$ to one of the sets already selected to which it is strongly connected, if possible:*

Copy $\tilde{C}_k^l = C_k^l$, $k = 1,\ldots,j$

If there exists $i \in R$ and k such that $N_i^l(\varepsilon) \cap \tilde{C}_k^l \neq \emptyset$ then set $C_k^l \leftarrow C_k^l \cup \{i\}$. Repeat until no such i exists.

Step 3 *Make the remaining $i \in R$ into aggregates that consist of subsets of strongly coupled neighborhoods: If there exists $i \in R$, set $j \leftarrow j+1$ and $C_j^l = R \cap N_i^l(\varepsilon)$. Repeat until $R = \emptyset$.*

Define the tentative prolongation $\tilde{P}_l$ by the aggregates C_i^l:

$$(\tilde{P}_l)_{ij} = \begin{cases} 1 \text{ if } i \in C_j^l \\ 0 \text{ otherwise} \end{cases} \tag{2.6}$$

The piecewise constant prolongation $\tilde{P}_l$ will now be improved by a smoothing to get the final prolongation matrix P_l. We choose a simple Jacobi smoother, giving the prolongation matrix

$$P_l = (I - \omega D^{-1} A_l^F)\tilde{P}_l \tag{2.7}$$

where $A_l^F = (a_{ij}^F)$ is the *filtered matrix* given by

$$a_{ij}^F = \left\{ \begin{array}{c} a_{ij} \text{ if } \ j \in N_i^l(\varepsilon) \\ 0 \quad \text{otherwise} \end{array} \right\} \text{ if } i \neq j, \qquad a_{ii}^F = a_{ii} - \sum_{j=1, j\neq i}^{n_l} (a_{ij} - a_{ij}^F), \tag{2.8}$$

and D denotes the diagonal of A_l^F.

When applying Algorithm 2 to uniformly elliptic problems, one usually obtains the coarsening by about a factor of 3 in each dimension and the resulting coarse level matrix A_{l+1} tends to follow the nonzero pattern of the 9-point stencil. The filtration (2.8) has little or no effect in this case.

In the case of anisotropic problems, however, the application of the smoother with the unfiltered matrix would make the supports of basis functions overlap extensively in the direction of week connections. Here the filtration prevents the undesired overlaps of the coarse space basis functions. By construction, A_l^F typically makes the nonzero pattern of A_{l+1} follow the 9-point stencil as in the uniformly V-elliptic case. It also assures that a constant remains the local kernel of A_l^F at every point where constant is the local kernel of A_l. Consequently, for problems without zero-order term the final prolongator P_l satisfies the decomposition of unity away from the essential boundary conditions.

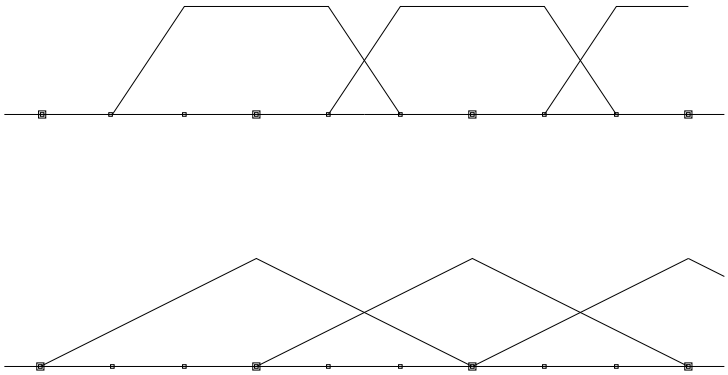

FIG. 2.1. *The basis functions given by aggregation and the corresponding smoothed basis for 1D Laplacian, using the smoother* $I - 2/3D^{-1}A$

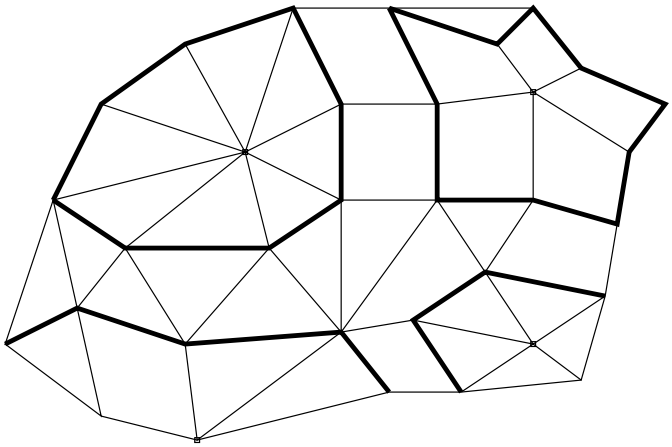

FIG. 2.2. *Typical 2D aggregates*

Fig. 2.1 shows the 1D coarse basis functions resulting from prolongation by aggregation and the smoothed aggregation. Note that for the 1D Laplace operator and the choice of $\omega = 2/3$ in (2.7), the smoothed coarse space basis is exactly the one of $P1$-finite elements. Fig. 2.2 shows the typical aggregates obtained on an unstructured grid. The corresponding supports are formed by adding one belt of elements to the aggregates. The smoothing adds at most one more belt of adjacent elements.

We choose

$$\varepsilon = 0.08\ (\frac{1}{2})^{l-1}, \qquad \omega = \frac{2}{3}.$$

The theory for the above method can be found in [15].

3. Generalizations.

3.1. High order elements and unscaled problems. The decomposition of unity (2.4) may be violated in practice. In such a case, in order to construct coarse spaces representing the constant function exactly we need the representation of unity

with respect to finite element basis of finest space V_1 as user input data. More specifically, we need the vector $\alpha \in \mathbb{R}^{n_1}$ satisfying

$$\sum_{i=1}^{n_1} \alpha_i \varphi_i^l = 1$$

away from essential boundary conditions.

The definition (2.6) of the auxiliary prolongators remains in place for all levels but level 1; we define $\tilde{P}_1$ as

$$\tilde{P}_{1_{ij}} = \begin{cases} \alpha_i \text{ if } i \in C_j^1 \\ 0 \text{ otherwise} \end{cases} \tag{3.1}$$

Thus, the unit constant function is represented by the vector $\alpha = (\alpha_i)_{i=1}^{n_1}$ on the finest level, while on levels 2 to L, the unit constant fucntion is represented by vectors of all ones. The process can be easily generalized to the nonscalar case using the block approach described in Section 3.2. It was applied to the problem from Example No. 1 of Section 5 modified by scaling the basis functions randomly in the interval $[0.01, 1]$. The results are summed up in Example No. 5.

3.2. Vector problems. In the case of nonscalar problems, the coarsening algorithm as described in Section 2 is likely to produce aggregates of physically incompatible degrees of freedom causing deterioration of convergence. This phenomenon can, however, be overcome by using so-called block approach, which consists in replacing the scalar operations on the level of degree of freedom by their block counterparts on the level of node. Let n_d denote the number of degrees of freedom per node (assumed to be constant) and $df(i)$ be the list of degrees of freedom associated with the node i. The communication between the neighboring nodes k, l can now be expressed in the form of a matrix selection A_{kl} of order n_d

$$A_{kl} = A(df(k), df(l)). \tag{3.2}$$

The definition of strongly coupled neighborhood of node i (2.5) is now replaced by

$$N_i^l(\epsilon) = \{\, j : \ \|A_{ij}\| \geq \epsilon \sqrt{\|A_{ii}\| \|A_{jj}\|} \} \cup \{i\}, \tag{3.3}$$

where $\|.\|$ is a matrix norm. Further, in the definition of auxiliary prolongations (2.6), we replace the numbers 1 and 0 by identity and zero matrices of order n_d, respectively. The efficiency of this generalization is demonstrated by Experiments No. 1 and No. 5 in Section 5.

3.3. Absolute term. Consider now (2.1) modified by adding a positive absolute term

$$a(u, v) = \int_\Omega \sum_{i,j} a_{ij} \partial_i u \partial_j v + quv, \qquad q > 0.$$

In this case, the prolongation smoothers lose its constant-preserving property because the constant is no longer locally in the kernel of A_l. Fortunately, the presence of the absolute term improves the condition number of A_l, thus compensating for the loss of the preservation of a constant.

For large q, the absolute term also has the effect of boosting the diagonal dominance in certain (block) columns. The nodes corresponding to these columns are then treated by Algorithm 2 as isolated nodes, and the coarsening process may stall. Note that the same phenomenon may also result from certain treatments of the essential boundary conditions. This difficulty can easily be defeated by a simple modification. Removing these nodes from the set R in Algorithm 2 prevents the stalling. At the same time, it does not harm the convergence of the overall method, because the smoothers $\mathcal{S}^l$ are very efficient at approximating values in numerically isolated nodes.

4. Method for High order problems. For the elliptic problems of order $2K$, $K > 1$ requirements on prolongators have to be slightly stronger. Instead of decomposition of unity **(AMG3)** we now need the more general requirement

(AMG3') Every coarse space V_l must represent polynomials of degrees up to $K-1$ exactly, away from the essential boundary conditions. As in the case of second order problems, this requirement is motivated by the need to control the coarse-grid approximation of $P_l v^{l+1}$ of u^l by energy $(u^l)^T A_l u_l$ and by the fact that norm and seminorm are equivalent on the factor space H^K modulo polynomials of degree of up to $K-1$.

Second, the small energy of coarse basis functions **(AMG4)** must be replaced by its straightforward generalization

(AMG4') We require that the energy of coarse space basis functions be almost minimal in the sense that

$$\frac{a(\varphi_i^l, \varphi_i^l)}{||\varphi_i^l||^2_{L^2(\Omega)}} \leq C \inf_{u \in H_0^K(\mathrm{supp}\varphi_i^l)} \frac{a(u,u)}{||u||^2_{L^2(\Omega)}}.$$

Unfortunately, the construction of prolongators resulting in the coarse spaces satisfying **(AMG3')** for $K > 1$ is not possible without additional user input. In order to be able to approximate the polynomials with degrees of up to $K-1$ by coarse space functions exactly, we need their representation with respect to the finest level basis $\{\varphi_i^1\}_{i=1}^{n_1}$. Finally, assumption **(AMG5)** may be satisfied with different scaling for each type of degree of freedom.

For the elliptic problem of order $2K$ on the domain $\Omega \subset \mathbb{R}^d$, we need vectors $p^{(0)}, p^{(ij)} \in \mathbb{R}^{n_1}$, $i = 1 \ldots, K-1$, $j = 1, \ldots d$ satisfying

$$\sum_{k=1}^{n_1} p_k^{(0)} \varphi_k^1 = 1, \quad \sum_{k=1}^{n_1} p_k^{(ij)} \varphi_k^1 = x_j^i \tag{4.1}$$

away from the essential boundary conditions. For example, to solve the biharmonic equation in 2D, we need $p^{(0)}$, $p^{(11)}$, and $p^{(12)}$, the representations with respect to the fine-level basis of the the planes $z = 1$, $z = x$, $z = y$, respectively.

The coarsening technique we are using is a natural generalization of the concept of smoothed aggregation described in Section 2.1. The aggregation step (2.6) can

be viewed as a restriction of the unit vector to aggregates C_i^l, which gives rise to one degree of freedom on the level $l+1$ for each C_i^l. Here, tentative prolongators will be generated by restricting all the vectors p^0, p^{jk} to the aggregates C_i^1. Each aggregate will be represented by a set of degrees of freedom, where every degree of freedom corresponds to one of the vectors $p^{(0)}$, $p^{(jk)}$ (see Fig. 4.1). The shape of the basis functions derived from the nonconstant polynomials depends on the position of the aggregate. More specifically, being far away from the origin, basis functions derived from polynomials of higher degree contain a large low degree polynomial component which results in the violation of the uniform equivalence of discrete and continuous L_2-norms. This undesirable effect is suppressed by a local l_2 Gram-Schmidt orthogonalization process performed on each aggregate C_i^1. (see Fig. 4.2). Again, the resulting prolongator will be smoothed by the Jacobi smoother (see Fig. 4.3).

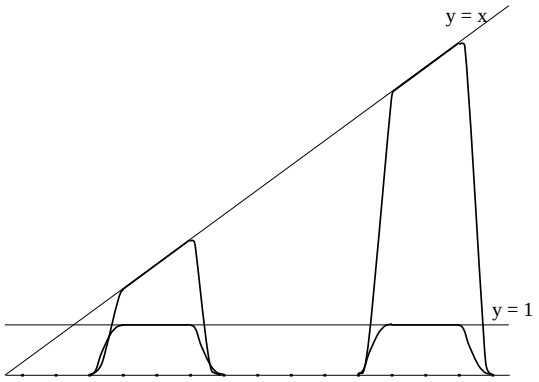

FIG. 4.1. *The coarse-space basis given by the restriction of p^0 and p^{11} onto aggregates of nodes*

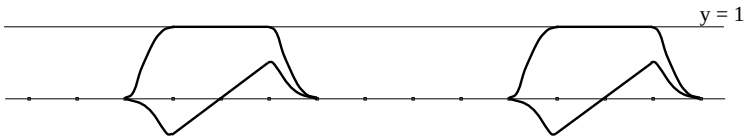

FIG. 4.2. *The coarse-space basis after l_2 Gram-Smidt modification*

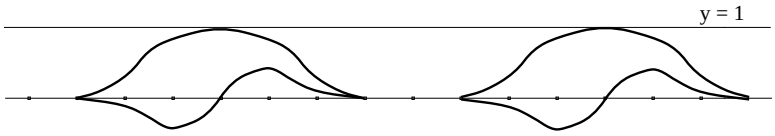

FIG. 4.3. *The final smoothed basis*

The following is a generalization of the algorithm of Section 2.1 to the case of problems of order $2K$, $K \geq 1$.

ALGORITHM 3 (COARSENING OF HIGH-ORDER PROBLEMS). *We assume the number of degrees of freedom per node on the finest level to be constant. Let n_1 be the number of nodes on the finest level, and $df^1(i)$ denotes the list of degrees of freedom associated*

with the node i. We set $p^{1,(0)} = p^{(0)}, p^{1,(ij)} = p^{(ij)}$, $i = 1, \ldots, K-1$, $j = 1, \ldots, d$ (see (4.1)).

Step 1 - Decomposition. *Generate the disjoint covering $\{C_i^l\}_{i=1}^{n_{l+1}}$ of the set of nodes $\{1, \ldots, n_l\}$ using the Algorithm 2, where the strongly coupled neighborhood of i is defined by (3.3) and A_{ij} is the selection $A^l(df^l(i), df^l(j))$.*

Step 2 - Restriction. *For each aggregate C_i^l define the index set D_i^l of all degrees of freedom associated with nodes in C_i^l, i.e.*

$$D_i^l = \bigcup_{j \in C_i^l} df^l(j).$$

For every D_i^l generate auxiliary sparse vectors $v^{l,i,1}, \ldots, v^{l,i,n_p}$ by

$$v^{l,i,1} = p^{l,(0)}|_{D_i^l},\ v^{l,i,2} = p^{l,(1,1)}|_{D_i^l},\quad v^{l,i,3} = p^{l,(1,2)}|_{D_i^l},\quad \ldots, v^{l,i,n_p} = p^{l,(K-1,d)}|_{D_i^l},$$

where $2K$ is the order of equation, d is the number of space variables ($\Omega \subset \mathbb{R}^d$), and $n_p = (K-1)d + 1$ is the number of the user supplied polynomials. $v|_I$ denotes the restriction of the vector to the index set in the sense that $(v|_I)_i = v_i$ iff $i \in I$, zero otherwise.

Step 3 - Gram-Schmidt modification. *For each aggregate C_i^l update the set of associated sparse vectors generated in Step 2 by l_2 Gram-Schmidt orthogonalization process in the ordering $v^{l,i,1}$, $v^{l,i,2}, \ldots, v^{l,i,n_p}$ (i.e., vectors derived from low-degree polynomials are processed first). Note that the representation of the unity $v^{l,i,1}$ remains unchanged by the process.*

Step 4 - Building of auxiliary prolongators. *Generate the auxiliary prolongator $\tilde{P}_l$ whose $n_p(i-1)+j$-th column consists of the vector $v^{l,i,j}$ and create the corresponding coarse-level list of degrees of freedom associated with node i*

$$df^{l+1}(i) = \{n_p(i-1)+1,\ n_p(i-1)+2, \ldots, n_p i\}, \quad i = 1, \ldots, n_{l+1}.$$

Step 5 - Representation of polynomials on the coarse-level. *Generate vectors $p^{l+1,(0)}, p^{l+1,(11)}, \ldots, p^{l+1,(K-1,d)}$ satisfying*

$$p^{l,(0)} = \tilde{P}_l p^{l+1,(0)},\quad p^{l,(11)} = \tilde{P}_l p^{l+1,(11)}, \ldots,\ p^{l,(K-1,d)} = \tilde{P}_l p^{l+1,(K-1,d)}.$$

As $\{C_i^l\}$ is a disjoint covering, the columns corresponding to different aggregates are l_2-orthogonal and consequently, the global Gram matrix given by columns of $\tilde{P}_l$ is a block matrix. Therefore, $p^{l+1,(0)}$, $p^{l+1,(11)}, \ldots, p^{l+1,(K-1,d)}$ can be computed by solving the local problems with Gram matrices generated by the columns of prolongator $\tilde{P}_l$ associated with C_i^l

$$G_i^l = \{(v^{l,i,j}, v^{l,i,k})_{l_2}\}_{j,k=1}^{n_p}.$$

Step 6 - Final smoothing. *Improve the prolongator $\tilde{P}_l$ by smoothing step (2.7), (2.8), where scalar entries a_{ij} are replaced by blocks $A_{ij} = A^l(df^l(i), df^l(j))$.*

REMARK 4.1. Note that the final smoothed coarse basis functions resemble the standard shape functions for the Hermitean element with one degree of freedom for the value at the node and one degree of freedom for each derivative. This is true regardless of the choice of basis functions in the original problem (finest level), and makes an algebraic coarsening possible.

For the results of application of Algorithm 3, see Experiments 6 and 7 in Section 5.

REMARK 4.2. Efficient solution in the case of nonscalar problems of second order may also need the use of the coarsenig technique described in this section. For example, in the case of 3D elasticity, the energy norm is not equivalent to $(H^1)^3$-seminorm on the factorspace modulo constant in each field in the local sense, and consequently, the approximation property of the coarse space depends on the global constant of V-ellipticity, which can be very small if, for example, displacements are prescribed only on a rather small part of the boundary.

In order to eliminate the dependence of the convergence on boundary conditions, we need the prolongator to support the local kernels of the form, which will typically assure the desired local equivalence on the factorspace modulo kernel (i.e., local Korn's inequality on macroelements).

Thus, it is reasonable to build prolongators supporting the entire local kernels of the bilinear form instead of just a constant in each field. This can be achieved by supplying the representation of the basis vectors of the the kernel in place of the vectors $p^{(0)}, p^{(11)} \ldots$. A similar technique that builds the coarse space from local generators of the nullspace is used in the so-called Balancing Domain Decomposition [7].

5. Numerical Experiments . The experiments in this section demonstrate the favorable behavior of the method. The code is available through anonymous ftp to `tiger.denver.colorado.edu` , directory /pub/faculty/pvanek. The experiments were performed on an IBM RS-6000/360 with 128 MBytes of memory.

The residual was measured in the l^2 norm. The iteration process was stopped once the relative residual became smaller than 10^{-5}. In all the experiments $V(1,1)$ cycle has been used. By algebraic complexity we mean the number of nonzero entries in the matrices on all the levels divided by the number of nonzeros in the matrix on finest level.

The rate of convergence is computed as an average reduction of l^2-norm of residual per iteration.

Results of experiments are summed up in Tab. 5. The description of testing problems follows.

EXPERIMENT NO. 1: Planar elasticity on unstructured mesh (Fig. 5.2). Poisson ratio 0.3, number of nodes 10610, number of degrees of freedom 21358. Boundary conditions : Dirichlet and Neumann.

EXPERIMENT NO. 2: Large anisotropic problem (5.1) with jumps in coefficients as in Fig. 5.1 and $q(x,y) = 0$. Number of nodes 10^6. The problem has been discretized on the regular square grid.

experiment No.	rate of convergence	algebraic complexity	CPU time	real time
1	0.08	1.23	5s	5s
2	0.10	1.56	768s	7892s
3	0.21	1.14	134s	233s
4 a/b/c	0.11/0.10/0.10	1.65/1.65/1.65	85/85/85s	95/96/91s
5	0.09	1.24	13s	13s
6	0.26	1.37	64s	77s
7	0.31	1.48	114s	121s

TABLE 5.1
Results of numerical experiments.

EXPERIMENT NO. 3: 3D problem (5.2) with random coefficients

$$w_{11} = \exp(rn_1),\ w_{22} = \exp(rn_2),\ w_{33} = \exp(rn_3),$$

where rn_i is a random number uniformly distributed in the interval $[\ln(10^{-2}), \ln(10^2)]$. Number of nodes 68921. The problem was discretized on the regular square grid.

$$-\frac{\partial}{\partial x}\, a(x,y)\frac{\partial u}{\partial x} - \frac{\partial}{\partial y}\, b(x,y)\frac{\partial u}{\partial y} + q(x,y)\, u = f(x,y) \ \text{ on } (0,1)\times(0,1), \qquad u = 0 \ \text{ on } \ \partial\Omega. \tag{5.1}$$

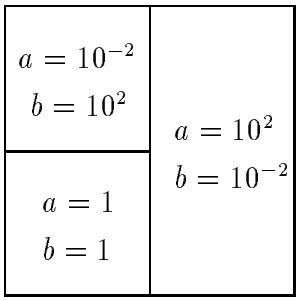

FIG. 5.1. *The coefficients $a(x,y)$, $b(x,y)$*

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}(w_{ij}(x,y)\frac{\partial u}{x_j}) = f(x,y), \ \text{ on } (0,1)\times(0,1), \qquad u = 0 \ \text{ on } \ \partial\Omega. \tag{5.2}$$

EXPERIMENT NO. 4: 2D anisotropic problem (5.1) with jumps in coefficients as in Fig. 5.1 and a) $q(x,y) = 0.1$, b) $q(x,y) = 1$, c) $q(x,y) = 10$. Number of nodes 160000. The problem was discretized on the regular square grid.

EXPERIMENT NO. 5: Planar elasticity on an unstructured mesh (Fig. 5.2) discretized by finite elements with randomly scaled basis. Poisson ratio 0.3, number of nodes 10610, number of degrees of freedom 21358. Boundary conditions : Dirichlet and Neumann.

EXPERIMENT NO. 6: Biharmonic problem discretized on the rectangular square grid. Number of degrees of freedom 48400. Boundary conditions: essential.

EXPERIMENT NO. 7: Fourth order problem (5.3) with coefficients given by (5.4) discretized on regular square grid. Number of degrees of freedom 48400. Boundary conditions: essential.

$$\frac{\partial^2}{\partial x^2}\left(a(x,y)\frac{\partial^2 u}{\partial x^2}\right)+\frac{\partial^2}{\partial x^2}\left(b(x,y)\frac{\partial^2 u}{\partial x^2}\right)=f(x,y) \quad \text{on } (0,1)\times(0,1) \tag{5.3}$$

$$a(x,y)=1, \quad b(x,y)=e^{16xy} \tag{5.4}$$

The second order problems are discretized by $P1$ finite elements. The fourth order problems are discretized by a 27-point difference formula with Lagrangean degrees of freedom.

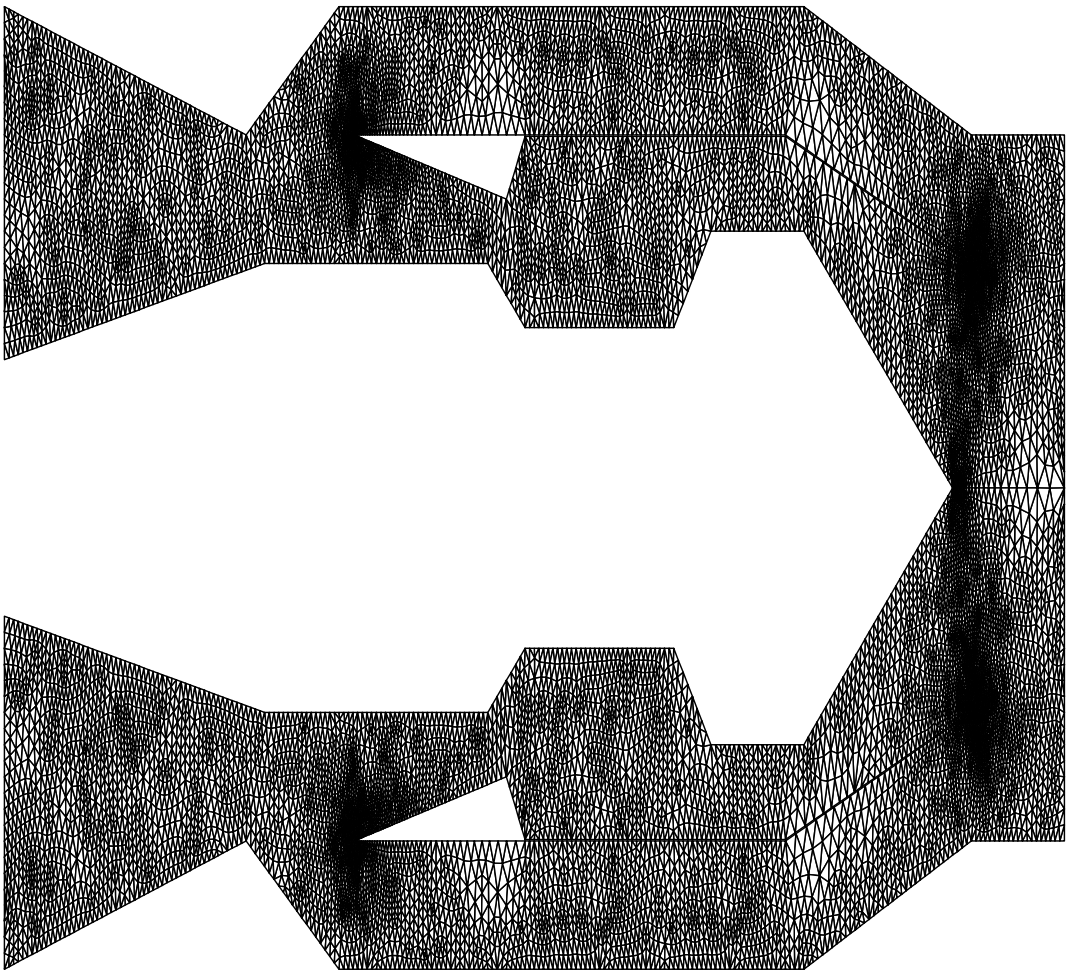

FIG. 5.2. *Mesh 1 (Courtesy of Charbel Farhat, Center for Aerospace Engineering, University of Colorado, Boulder)*

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