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**A Table of Analytical  
Discrete Fourier Transforms**

William L. Briggs and Van Emden Henson



# A Table of Analytical Discrete Fourier Transforms

William L. Briggs\*      Van Emden Henson†

January 14, 1995

## Abstract

While most people rely on numerical methods (most notably the fast Fourier transform) for computing discrete Fourier transforms (DFTs), there it is still an occasional need to have analytical DFTs close at hand. Such a table of analytical DFTs is provided in this paper, along with comments and observations, in the belief that it will serve as a useful resource or teaching aid for Fourier practioners.

## 1 Introduction

The table of discrete Fourier transforms (DFTs) that comprises most of this paper was assembled using analytical methods. The table is not exhaustive, since a good symbolic package or additional patience could certainly produce a few more DFT pairs. However it does include many commonly encountered input sequences.

A few words of explanation are in order. We assume that a function  $f$  is sampled on the interval  $[-A/2, A/2]$  at  $N$  equally spaced points to produce the sequence  $f_n$ , where  $n = -N/2 + 1 : N/2$ . Of utmost importance is the fact that, when extended, the sequence  $f_n$  is  $N$ -periodic. This means that if the  $A$ -periodic extension of  $f$  is continuous at  $x = \pm A/2$ , then

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$f_{\pm N/2} = f(\pm A/2)$ . On the other hand, if the  $A$ -periodic extension of  $f$  is not continuous at  $x = \pm A/2$ , then  $f_{\pm N/2}$  must be defined as the average value of  $f(\pm A/2)$ . Similarly, at *any* point of discontinuity in  $[-A/2, A/2]$ , the sequence of samples  $f_n$  must be assigned the average of the function values across the discontinuity. This proviso is given the name *AVED: average values at endpoints and discontinuities*, and is noted in the table.

We can now define the DFT. The forward DFT is given by

$$F_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n \omega_N^{-nk}$$

for  $k = -N/2 + 1 : N/2$ , where  $i$  is the imaginary unit and  $\omega_N \equiv e^{i2\pi/N}$ , which defines another  $N$ -periodic sequence  $F_k$ . The choice of using a centered interval is somewhat arbitrary. Because of the periodicity of  $f_n$  and  $F_k$ , the DFT can be defined on any  $N$  consecutive points. For the record, the corresponding inverse DFT is given by

$$f_n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} F_k \omega_N^{nk}$$

for  $n = -N/2 + 1 : N/2$ . The table also lists the Fourier coefficients of each input function on the interval  $[-A/2, A/2]$ ,

$$c_k = \frac{1}{A} \int_{-A/2}^{A/2} f(x) e^{-i2\pi kx/A} dx \quad \text{for } k = 0, \pm 1, \pm 2, \dots,$$

in order to compare  $F_k$  with  $c_k$ .

Each entry of the table is arranged as follows.

Discrete input name	$f_n, n \in \mathcal{N}$	Graph of $F_k$
Graph of $f_n$	$F_k, k \in \mathcal{N}$	
	$ c_k - F_k , k \in \mathcal{N}$	
Continuum input name	$f(x), x \in I$	Graph of $ c_k - F_k $ $\max  c_k - F_k $
Graph of $f(x)$	$c_k, k \in \mathbf{Z}$	
	<b>Comments</b>	

The first column has two boxes. The upper box gives the name of the input, below which are graphs of the real and imaginary parts of the discrete input sequence. The lower box contains the name of the continuum input, and the corresponding continuum input graphs. The middle column has six boxes containing, in order from top to bottom, the formula of the input sequence  $f_n$ ; the analytic  $N$ -point DFT output  $F_k$ ; a measure of the error  $|c_k - F_k|$ ; the formula of the continuum input function  $f(x)$ ; the formula for the Fourier coefficients  $c_k$ ; an entry for comments, perhaps the most important of which is the AVED warning. This means that *average values at endpoints and discontinuities* must be used if the correct DFT is to be computed. The third column consists of two boxes. The upper box displays graphically the real and imaginary parts of the DFT. The lower box gives the maximum error  $\max |c_k - F_k|$ , and displays graphically the error  $|c_k - F_k|$  for a small (24-point) example.

A few comments and observations on the entries of the table might be useful.

- **Symmetry.** The well-known symmetries of the DFT and the Fourier coefficients are evident in the entries of the table: if  $f_n$  is a real, even sequence ( $f_{-n} = f_n$ ), then the resulting DFT sequence is also real and even; if  $f_n$  is a real, odd sequence ( $f_{-n} = -f_n$ ), then the resulting DFT sequence is pure imaginary and odd.
- **Exact DFTs.** The DFT is exact, meaning it reproduces the first  $N$  Fourier coefficients exactly, in several entries of the table. As shown in cases 1-4, if the input sequence is periodic and does not consist of frequencies greater than  $\omega_{max} = N/A$  cycles per unit length (or time), then the DFT is exact. This condition is just the Nyquist sampling condition.
- **Errors.** The table gives estimates of the difference  $|c_k - F_k|$  which should be interpreted in an asymptotic sense. If  $|c_k - F_k|$  is given as  $CN^{-p}$ , the meaning is that

$$\lim_{N \rightarrow \infty} \frac{|c_k - F_k|}{N^p} = C,$$

where  $0 < C < \infty$  is a constant. These estimates are obtained by a liberal use of Taylor series for large  $N$ . This asymptotic measure of error conforms with the pointwise errors in the DFT which can be obtained from the Poisson summation formula. Roughly speaking, if

the periodic extension of  $f$  has  $p - 1$  continuous derivatives, and  $f^{(p)}$  is bounded and piecewise monotone, then

$$|c_k - F_k| \leq \frac{C}{N^{p+1}} \quad \text{for } k = -\frac{N}{2} + 1 : \frac{N}{2},$$

where  $C$  is a constant for all  $N$  [1]. The case  $p = 0$  corresponds to functions  $f$  whose periodic extension is only piecewise continuous. A similar bound holds for DFT approximations of the Fourier transform of compactly supported functions.

For example, cases 5, 6, 6c, 7, 9, 10, 10a, 11, 13, 14, and 15b of the table are piecewise continuous functions ( $p = 0$ ), and all show asymptotic errors of the form  $CN^{-1}$  or  $CkN^{-2}$ . The latter term means that for the low frequency coefficients ( $|k| \sim O(1)$ ), the error can decrease as  $CN^{-2}$ ; however, for the high frequency coefficients ( $|k| \sim O(N/2)$ ), the error decreases as  $CN^{-1}$ . In cases 6b, 6d, 7, 8, 12, and 15a, the derivative of the input function is piecewise continuous ( $p = 1$ ), and the errors decrease as  $CN^{-2}$ . For input functions with higher degrees of smoothness, analytical DFTs are difficult to compute, however numerical experiments confirm the error estimates for larger values of  $p$ .

## 2 The Table of DFTs

The following notational conventions hold throughout the *Table of DFTs*:

$$\mathcal{N} = \left\{ -\frac{N}{2} + 1, \dots, \frac{N}{2} \right\}, \quad \mathcal{I} = \left[ -\frac{A}{2}, \frac{A}{2} \right], \quad \mathbf{Z} = \{0, \pm 1, \pm 2, \dots\};$$

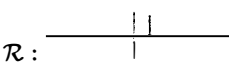
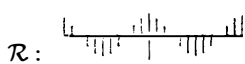
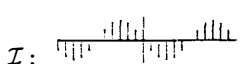
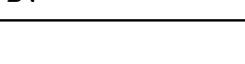
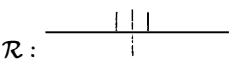
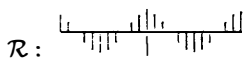


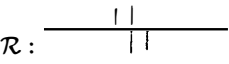

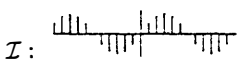
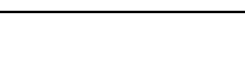
$$\omega_N = e^{i\frac{2\pi}{N}}, \quad \theta_k = \frac{2\pi k}{N}, \quad \delta(k) = \begin{cases} 1 & \text{if } k = 0, \\ 0 & \text{if } k \neq 0; \end{cases}$$

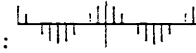
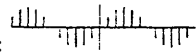
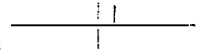




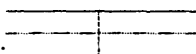
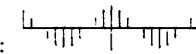


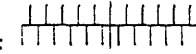


$$\hat{\delta}_N(k) = \begin{cases} 1 & \text{if } k = 0 \text{ or a multiple of } N, \\ 0 & \text{otherwise.} \end{cases}$$

AVED = average values at endpoints and discontinuities,

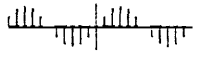
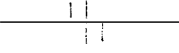
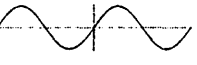
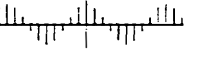

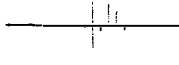


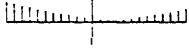
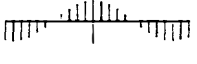
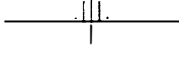

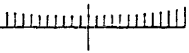

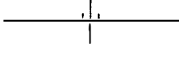

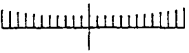
$C$  is a constant independent of  $k$  and  $N$ .

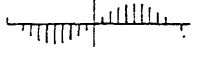

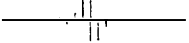
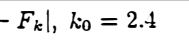
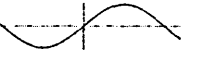

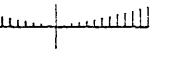
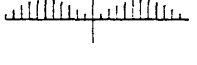
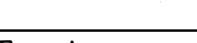
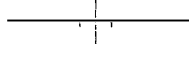
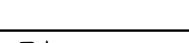
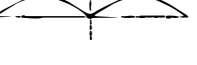
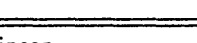
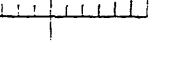
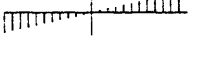


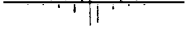
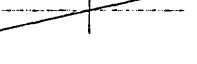
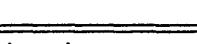
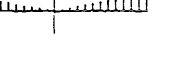

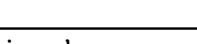

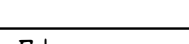
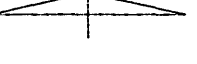
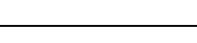
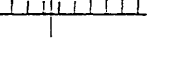
TABLE OF DFTS

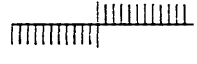
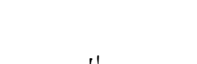
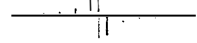
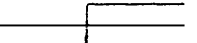
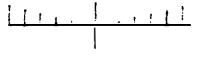

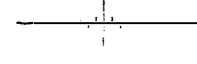
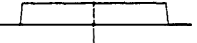
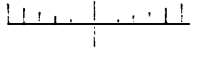

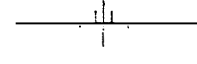
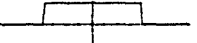
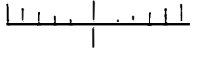
Discrete input name Graph of $f_n$	$f_n, n \in \mathcal{N}$	Graph of $F_k$
	$F_k, k \in \mathcal{N}$	
	$ c_k - F_k , k \in \mathcal{N}$	
Continuum input name Graph of $f(x)$	$f(x), x \in I$	Graph of $ c_k - F_k $ $\max c_k - F_k $
	$c_k, k \in \mathcal{Z}$	
	<b>Comments</b>	
1. Impulse $\mathcal{R}$ :  $\mathcal{I}$ :	$\delta(n - n_0)$	$F_k$ : 
	$\frac{1}{N} \omega_N^{-n_0 k}$	$\mathcal{R}$ : 
	Exact	$\mathcal{I}$ : 
1. None	-	
	-	
	$n_0 \in \mathcal{N}$	
2a. Paired impulses $\mathcal{R}$ :  $\mathcal{I}$ :	$\frac{1}{2}(\delta(n - n_0) + \delta(n + n_0))$	$F_k$ : 
	$\frac{1}{N} \cos(\frac{2\pi n_0 k}{N})$	$\mathcal{R}$ : 
	Exact	$\mathcal{I}$ : 
2a. None	-	
	-	
	$n_0 \in \mathcal{N}$	
2b. Paired impulses $\mathcal{R}$ :  $\mathcal{I}$ :	$\frac{1}{2}(\delta(n + n_0) - \delta(n - n_0))$	$F_k$ : 
	$\frac{i}{N} \sin(\frac{2\pi n_0 k}{N})$	$\mathcal{R}$ : 
	Exact	$\mathcal{I}$ : 
2b. None	-	
	-	
	$n_0 \in \mathcal{N}$	

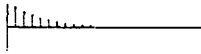
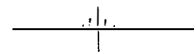

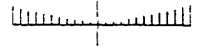
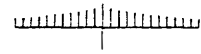
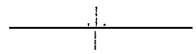
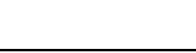
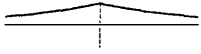
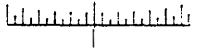


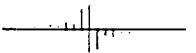
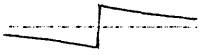
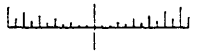
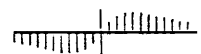



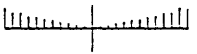
3. Complex harmonic $\mathcal{R}$ :  $\mathcal{I}$ : 	$\omega_N^{nk_0}$ $\hat{\delta}_N(k - k_0)$ (periodic) Exact	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ :
3. Complex harmonic $\mathcal{R}$ :  $\mathcal{I}$ : 	$e^{i\frac{2\pi k_0 x}{A}}$ $\delta(k - k_0)$ $k_0 \in \mathbf{Z}$	$\max  c_k - F_k  = 0$
3a. Constant $\mathcal{R}$ :  $\mathcal{I}$ :	1 $\hat{\delta}_N(k)$ (periodic) Exact	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ :
3a. Constant $\mathcal{R}$ :  $\mathcal{I}$ :	1 $\delta(k)$ Case 3: $k_0 = 0$	$\max  c_k - F_k  = 0$
4a. Cosine harmonic $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos(\frac{2\pi k_0 n}{N})$ $\frac{1}{2}(\hat{\delta}_N(k - k_0) + \hat{\delta}_N(k + k_0))$ Exact	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ :
4a. Cosine harmonic $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos(\frac{2\pi k_0 x}{A})$ $\frac{1}{2}(\delta(k - k_0) + \delta(k + k_0))$ $k_0 \in \mathbf{Z}$	$\max  c_k - F_k  = 0$
4b. Critical mode $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos(\pi n) = (-1)^n$ $\hat{\delta}_N(k - \frac{N}{2})$ Exact	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ :
4b. Critical mode $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos(\frac{\pi N x}{A})$ $\frac{1}{2}(\delta(k - \frac{N}{2}) + \delta(k + \frac{N}{2}))$ Case 4a: $k_0 = \frac{N}{2}$	$\max  c_k - F_k  = 0$

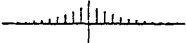
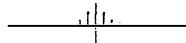

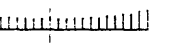
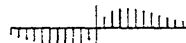
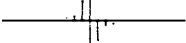
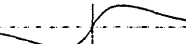
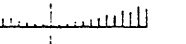


4c. Sine harmonic $\mathcal{R}$ :  $\mathcal{I}$ :	$\sin\left(\frac{2\pi k_0 n}{N}\right)$	$F_k$ and $c_k$ :
	$\frac{i}{2}(\hat{\delta}_N(k+k_0) - \hat{\delta}_N(k-k_0))$	$\mathcal{R}$ :
	Exact	$\mathcal{I}$ : 
4c. Sine harmonic $\mathcal{R}$ :  $\mathcal{I}$ :	$\sin\left(\frac{2\pi k_0 x}{A}\right)$	
	$\frac{i}{2}(\delta(k+k_0) - \delta(k-k_0))$	
	$k_0 \in \mathbf{Z}$	$\max  c_k - F_k  = 0$
5. Complex wave $\mathcal{R}$ :  $\mathcal{I}$ : 	$\omega_{N}^{nk_0}$	$F_k$ and $c_k$ :
	$\frac{\sin(\pi(k-k_0)) \sin(2\pi(k-k_0)/N)}{2N \sin^2(\pi(k-k_0)/N)}$	$\mathcal{R}$ : 
	$\sim C(k-k_0)N^{-2}$	$\mathcal{I}$ :
5. Complex wave $\mathcal{R}$ :  $\mathcal{I}$ : 	$e^{i\frac{2\pi k_0 x}{A}}$	$ c_k - F_k , k_0 = 2.4$
	$\frac{\sin(\pi(k-k_0))}{\pi(k-k_0)}$	
	AVED, $k_0 \notin \mathbf{Z}$	$\max \approx 10^{-2}, N = 24$
6a. Cosine $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos\left(\frac{\pi k n}{N}\right)$	$F_k$ and $c_k$ :
	$\frac{\cos(\pi k) \sin(\pi k n / 2)}{4N}$ $\times \left( \frac{\sin \theta_+}{\sin^2 \theta_+ / 2} - \frac{\sin \theta_-}{\sin^2 \theta_- / 2} \right)$	$\mathcal{R}$ : 
	$\sim C k_0 N^{-2}$	$\mathcal{I}$ :
6a. Cosine $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos\left(\frac{\pi k_0 x}{A}\right)$	$ c_k - F_k , k_0 = 2.4$
	$-\frac{2k_0 \cos(\pi k) \sin(\pi k_0 / 2)}{\pi(4k^2 - k_0^2)}$	
	$k_0 \notin \mathbf{Z}, \theta_{\pm} = \frac{\pi(2k \pm k_0)}{N}$	$\max \approx 10^{-3}, N = 24$
6b. Half cosine $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos\left(\frac{\pi n}{N}\right)$	$F_k$ and $c_k$ :
	$\frac{\cos(\pi k)}{4N} \left( \frac{\sin \theta_+}{\sin^2 \theta_+ / 2} - \frac{\sin \theta_-}{\sin^2 \theta_- / 2} \right)$	$\mathcal{R}$ : 
	$\sim CN^{-2}$	$\mathcal{I}$ :
6b. Half cosine $\mathcal{R}$ :  $\mathcal{I}$ :	$\cos(\pi x)$ on $(-\frac{1}{2}, \frac{1}{2}]$	$ c_k - F_k $
	$-\frac{2 \cos(\pi k)}{\pi(4k^2 - 1)}$	
	Case 6a: $\theta_{\pm} = \frac{\pi(2k \pm 1)}{N}, k_0 = 1, A = 1$	$\max \approx 10^{-3}, N = 24$

<p>6c. Sine</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$\sin\left(\frac{\pi k_0 n}{N}\right)$ $i \frac{\cos(\pi k) \sin(\pi k_0/2)}{4N}$ $\times \left( \frac{\sin \theta_+}{\sin^2 \theta_+/2} + \frac{\sin \theta_-}{\sin^2 \theta_-/2} \right)$ $\sim CkN^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>
<p>6c. Sine</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$\sin\left(\frac{\pi k x}{A}\right)$ $i \frac{4k \cos(\pi k) \sin(\pi k_0/2)}{\pi(4k^2 - k_0^2)}$ <p>AVED, <math>k_0 \notin \mathbb{Z}, \theta_{\pm} = \frac{\pi(2k \pm k_0)}{N}</math></p>	<p><math> c_k - F_k , k_0 = 2.4</math></p> <p><math>\mathcal{R}</math>: </p> <p>max <math>\approx 10^{-2}, N = 24</math></p>
<p>6d. Even sine</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$ \sin(2\pi n/N) $ $\frac{1+\cos(\pi k)}{4N} \left( \frac{\sin \theta_+}{\sin^2 \theta_+/2} - \frac{\sin \theta_-}{\sin^2 \theta_-/2} \right)$ $\sim C_1 N^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>
<p>6d. Even sine</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$ \sin(2\pi x/A) $ $\frac{1}{\pi} \frac{1-\cos(\pi k)}{1-k^2}$ <p><math>\theta_{\pm} = \frac{2\pi}{N}(k \pm 1), c_{\pm 1} = F_{\pm 1} = 0</math></p>	<p><math> c_k - F_k </math></p> <p><math>\mathcal{R}</math>: </p> <p>max <math>\approx 10^{-3}, N = 24</math></p>
<p>7. Linear</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$n/N$ $F_0 = 0, F_k = \frac{i \cos(\pi k) \sin \theta_k}{4N \sin^2 \theta_k/2}$ $\sim CkN^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>
<p>7. Linear</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$x/A$ $c_0 = 0, c_k = \frac{i \cos(\pi k)}{2\pi k}$ <p>AVED, <math>f_{-N/2} = 0</math></p>	<p><math> c_k - F_k </math></p> <p><math>\mathcal{R}</math>: </p> <p>max <math>\approx 10^{-2}, N = 24</math></p>
<p>8. Triangular wave</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$1 - 2 n /N$ $F_0 = \frac{1}{2}, F_k = \frac{1-\cos(\pi k)}{N^2 \sin^2 \theta_k/2}$ $\sim CN^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>
<p>8. Triangular wave</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>	$1 - 2 x /A$ $c_0 = \frac{1}{2}, c_k = \frac{1-\cos(\pi k)}{\pi^2 k^2}$	<p><math> c_k - F_k </math></p> <p><math>\mathcal{R}</math>: </p> <p>max <math>\approx 10^{-3}, N = 24</math></p>

<p>9. Rectangular wave</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$\begin{cases} -1, & -N/2 < n < 0 \\ 1, & 0 < n < N/2 \end{cases}$ $F_0 = 0, F_k = i \frac{(\cos(\pi k) - 1) \sin \theta_k}{2N \sin^2 \theta_k / 2}$ $\sim CkN^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>: </p>
<p>9. Rectangular wave</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$\begin{cases} -1, & -A/2 < x < 0 \\ 1, & 0 < x < A/2 \end{cases}$ $c_0 = 0, c_k = \frac{i(\cos(\pi k) - 1)}{\pi k}$ <p>AVED, <math>f_n = 0</math> for <math>n = 0, \pm N/2</math></p>	<p><math> c_k - F_k </math></p> <p></p> <p><math>\max \approx 10^{-2}, N = 24</math></p>
<p>10. Square pulse</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$\begin{cases} 1, &  n  < M/2 \\ 0, & M/2 <  n  < N/2 \end{cases}$ $F_0 = \frac{M}{N}, F_k = \frac{\sin(\pi k M/N) \sin \theta_k}{2N \sin^2 \theta_k / 2}$ $\sim CkN^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>
<p>10. Square pulse</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$\begin{cases} 1, &  x  < a/2 \\ 0, & a/2 <  x  < A/2 \end{cases}$ $c_0 = \frac{a}{A}, c_k = \frac{\sin(\pi k a/A)}{\pi k}$ <p>AVED, <math>0 &lt; \frac{a}{A} = \frac{M}{N} &lt; 1</math></p>	<p><math> c_k - F_k </math></p> <p></p> <p><math>\max \approx 10^{-2}, N = 24</math></p>
<p>10a. Square pulse</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$\begin{cases} 1, &  n  < N/4 \\ 0, & N/4 <  n  < N/2 \end{cases}$ $F_0 = \frac{1}{2}, F_k = \frac{\sin(\pi k/2) \sin \theta_k}{2N \sin^2 \theta_k / 2}$ $\sim CkN^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>
<p>10a. Square pulse</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$\begin{cases} 1, &  x  < 1/2 \\ 0, & 1/2 <  x  < 1 \end{cases}$ $c_0 = \frac{1}{2}, c_k = \frac{\sin(\pi k/2)}{\pi k}$ <p>AVED, Case 10: <math>A = 2a = 1</math></p>	<p><math> c_k - F_k </math></p> <p></p> <p><math>\max \approx 10^{-2}, N = 24</math></p>

11. Exponential  $\mathcal{R}$ : $\mathcal{I}$ :	$e^{-aAn/N}, 0 \leq n \leq N-1$ $\frac{\sigma(1-e_N^2) - i2\sigma e_N \sin \theta_k}{2N(1-2e_N \cos \theta_k + e_N^2)}$ $\sim CN^{-1}$	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ : 
11. Exponential $\mathcal{R}$ : $\mathcal{I}$ :	$e^{-ax}, 0 < x < A$ $\frac{\sigma(aA - i2\pi k)}{a^2 A^2 + 4\pi^2 k^2}$ AVED, $\sigma = 1 - e^{-aA}, e_N = e^{-aA/N}$	$ c_k - F_k , a = 2, A = 3$  $\max \approx 10^{-2}, N = 24$
12. Even exponential  $\mathcal{R}$ : $\mathcal{I}$ :	$e^{-aA n /N}$ $\frac{(1-e_N^2)(1-e_2 \cos(\pi k))}{N(1-2e_N \cos \theta_k + e_N^2)}$ $\sim CN^{-2}$	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ : 
12. Even exponential  $\mathcal{R}$ : $\mathcal{I}$ :	$e^{-a x },  x  < A/2$ $\frac{2aA(1-e_2 \cos(\pi k))}{a^2 A^2 + 4\pi^2 k^2}$ $e_2 = e^{-aA/2}, e_N = e^{-aA/N}$	$ c_k - F_k , a = 2, A = 1$  $\max \approx 10^{-3}, N = 24$
13. Odd exponential  $\mathcal{R}$ : $\mathcal{I}$ :	$(n/ n )e^{-aA n /N}$ $i \frac{2(e_N \sin \theta_k)(e_2 \cos(\pi k) - 1)}{N(1-2e_N \cos \theta_k + e_N^2)}$ $\sim CN^{-1}$	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ : 
13. Odd exponential  $\mathcal{R}$ : $\mathcal{I}$ :	$(x/ x )e^{-a x },  x  < A/2$ $i \frac{4\pi k(e_2 \cos(\pi k) - 1)}{a^2 A^2 + 4\pi^2 k^2}$ AVED, $e_2 = e^{-aA/2}, e_N = e^{-aA/N}$	$ c_k - F_k , a = 2, A = 1$  $\max \approx 10^{-2}, N = 24$
14. Linear/exponential  $\mathcal{R}$ : $\mathcal{I}$ :	$(nA/N)e^{-aA n /N}$ $i \frac{2A \sin \theta_k (\sigma_k (e_N^3 - e_N) + \dots)}{N^2 (e_N^4 - 4 \cos \theta_k (e_N^3 + e_N) + \dots)}$ $\frac{\dots + Ne_N e_N \cos(\pi k) (1 - e_N \cos \theta_k)}{\dots + 2e_N^2 (\cos(2\theta_k) + 2) + 1}$ $\sim CN^{-1}$	$F_k$ and $c_k$ : $\mathcal{R}$ :  $\mathcal{I}$ : 
14. Linear/exponential  $\mathcal{R}$ : $\mathcal{I}$ :	$x e^{-a x }$ $i \frac{2\pi k A}{d} (e_2 \cos(\pi k) + \frac{4aA}{d} (e_2 \cos(\pi k) - 1))$ $\sigma_k = 1 + (\frac{N}{2} - 1)e_2 \cos(\pi k)$ $e_2 = e^{-aA/2}, e_N = e^{-aA/N}$ $d = a^2 A^2 + 4\pi^2 k^2$ , AVED	$ c_k - F_k , a = 2, A = 3$  $\max \approx 10^{-3}, N = 24$

<p>15a. Cosine/exponential</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$e^{-a n /N} \cos(\pi n/N)$ $\frac{(1-e_N^2)+2e_2e_N \cos(\pi k) \sin \theta_+}{2N(1-2e_N \cos \theta_+ + e_N^2)}$ $+ \frac{(1-e_N^2)-2e_2e_N \cos(\pi k) \sin \theta_-}{2N(1-2e_N \cos \theta_- - e_N^2)}$ $\sim CN^{-2}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>
<p>15a. Cosine/exponential</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$e^{-a x } \cos(\pi x/A)$ $\frac{e_2 k_+ \cos(\pi k) + a}{A(k_+^2 + a^2)} - \frac{e_2 k_- \cos(\pi k) - a}{A(k_-^2 + a^2)}$ $e_2 = e^{-aA/2}, \quad e_N = e^{-aA/N}$ $\theta_{\pm} = \frac{\pi}{N}(2k \pm 1), \quad k_{\pm} = \frac{\pi}{A}(2k \pm 1)$ <p>AVED</p>	<p><math> c_k - F_k , a = 2, A = 3</math></p> <p></p> <p><math>\max \approx 10^{-3}, N = 24</math></p>
<p>15b. Sine/exponential</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$e^{-a n /N} \sin(\pi n/N)$ $i \left( \frac{1-e_N \cos \theta_+ + e_2 e_N \cos(\pi k) \sin \theta_+}{N(1-2e_N \cos \theta_+ + e_N^2)} - \frac{1-e_N \cos \theta_- - e_2 e_N \cos(\pi k) \sin \theta_-}{N(1-2e_N \cos \theta_- - e_N^2)} \right)$ $\sim CN^{-1}$	<p><math>F_k</math> and <math>c_k</math>:</p> <p><math>\mathcal{R}</math>:</p> <p><math>\mathcal{I}</math>: </p>
<p>15b. Sine/exponential</p> <p><math>\mathcal{R}</math>: </p> <p><math>\mathcal{I}</math>:</p>	$e^{-a x } \sin(\pi x/A)$ $\frac{ie_2 k_+ \sin((2k+1)\pi/2) + a}{A(k_+^2 + a^2)} - \frac{ie_2 k_- \sin((2k-1)\pi/2) + a}{A(k_-^2 + a^2)}$ $e_2 = e^{-aA/2}, \quad e_N = e^{-aA/N}$ $\theta_{\pm} = \frac{\pi}{N}(2k \pm 1), \quad k_{\pm} = \frac{\pi}{A}(2k \pm 1)$ <p>AVED</p>	<p><math> c_k - F_k , a = 2, A = 3</math></p> <p></p> <p><math>\max \approx 10^{-3}, N = 24</math></p>

## References

- [1] W.L. BRIGGS, V.E. HENSON, *The DFT: An Owner's Manual for the Discrete Fourier Transform*, SIAM Publications, 1995.

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22. H.Wang, R.E. Ewing and T.F. Russell, "Eulerian-Lagrangian Localized Adjoint Methods for Convection-Diffusion Equations and Their Convergence Analysis."
23. S. Ghosal, J. Mandel and R. Tezaur, "Fast Neural Networks for Domain Decomposition in Finite Element Analysis."
24. L.P. Franca and C. Farhat, "Bubble Functions Prompt Unusual Stabilized Finite Element Methods."
25. S. Ghosal and P. Č. Vaněk, "Fast Algebraic Multigrid for Discontinuous Optical Flow Estimation."
26. C. Liao, Z. Liu and C. Liu, "Implicit Multigrid Method for Modeling 3-D Laminar Diffusion Flames with NO<sub>x</sub> Prediction."
27. Z. Cai, R. Lazarov, T.A. Manteuffel and S.F. McCormick, "First-Order System Least-Squares for Second-Order Partial Differential Equations: Part I."
28. Z. Cai, T.A. Manteuffel and S.F. McCormick, "First-Order System Least-Squares for Second-Order Partial Differential Equations: Part II."
29. S.E. Payne, "The Fundamental Theorem of  $q$ -Clan Geometry."
30. S.E. Payne, "A Tensor Product Action on  $q$ -Clan Generalized Quadrangles with  $q = 2^e$ ."
31. J. Mandel, "Iterative Methods for  $p$ -Version Finite Elements: Preconditioning Thin Solids."
32. David C. Fisher, Kathryn Fraughnaugh and Larry Langley, " $P_3$ -Connected Graphs of Minimum Size."
33. J. Mandel and R. Tezaur, "Convergence of a Substructuring Method with Lagrange Multipliers."
34. P. Vaněk, J. Mandel and M. Brezina, "Algebraic Multigrid on Unstructured Meshes."
35. P. Vaněk and J. Křížková, "Two-Level Method on Unstructured Meshes with Convergence Rate Independent of the Coarse-Space Size."
36. P. Vaněk, J. Mandel and M. Brezina, "Algebraic Multigrid by Smoothed Aggregation for Second and Fourth Order Elliptic Problems."
37. H. J. Greenberg, "Mathematical Programming Models for Environmental Quality Control."