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Elements in the Analysis of Domain
Decomposition Methods**

Jin-sheng Gu and Xian-cheng Hu

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SOME ESTIMATES WITH NONCONFORMING ELEMENTS IN THE ANALYSIS OF DOMAIN DECOMPOSITION METHODS[‡]

Jin-sheng Gu and Xian-cheng Hu

(Department of Applied Mathematics, Tsinghua University, Beijing 100084, China)

Abstract

The goal of this paper is to establish some essential estimates, especially the so-called extension theorem, for the nonconforming finite elements with their continuity only at the vertices or the edge midpoints of the elements of the quasi-uniforming mesh^[4,5,6,13,14,21]. As in the conforming discrete cases^[2,3,20], these estimates play key roles in the analysis of nonoverlap domain decomposition methods with nonconforming elements^[8,9,11,12]. It is well-known that the regularity of the solution of an elliptic problem depends on the smoothness of the domain and the boundary condition^[7,15,16]. In Sect.1, we present advanced error estimates of the nonconforming approximate solution under weaker assumption on the regularity, based on the second Strang lemma^[5]. In Sect.2, we construct and analyse the conforming interpolation operator I_h , which acts as a bridge between the nonconforming element spaces and the conforming element spaces. Based on the results of Sect.1 and Sect.2, plus the known estimates for the conforming element spaces^[2,5,18], we obtain, in Sect.3, a series of fundamental inequalities for the nonconforming element spaces, some of which are the extension theorem^[8,10], the Poincaré inequalities and the maximum norm estimate. We must point out that their applications are omitted here for the length of the present paper, which can be referred to [8,9,11,12].

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§1 Advanced Error Estimations

For simplicity of the exposition, we consider the elliptic boundary value problem on a bounded open polygonal domain $\Omega \subset \mathbb{R}^2$

$$u \in H_0^1(\Omega) : a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega) \quad (1.1)$$

where

$$a(u, v) = \int_{\Omega} \nabla u \nabla v, \quad (f, v) = \int_{\Omega} f v, \quad f \in H^{-1}(\Omega).$$

It is well-known that (1.1) has a unique solution $u \in H_0^1(\Omega)$ (cf.[7,15,16]).

Suppose that $\Omega_h = \{e\}$ is a quasi-uniform mesh of Ω , i.e., Ω_h satisfies

$$\sup_{e \in \Omega_h} \inf_{B_r \supset e} r \leq ch, \quad \inf_{e \in \Omega_h} \sup_{B_r \subset e} r \geq Ch, \quad (1.2)$$

where, B_r is a region bounded by the circle of radius r , $h = \max_{e \in \Omega_h} h_e$ is the mesh parameter, and $h_e = \inf_{B_r \supset e} r$. Here and later, c and C , with or without subscript, denote generic positive constants independent of h . Let V_h be the finite element space on Ω_h , and π_h be the corresponding interpolation operator. V_h can be the space of Wilson elements^[21], Carey membrane elements^[4], Wilson-like elements^[14], which are only continuous at the vertices of each $e \in \Omega_h$. Also, V_h can be the space of Crouzeix-Raviart elements^[6], quartic rectangular elements^[13], which are only continuous at the edge midpoints of each $e \in \Omega_h$. For briefness, the former are called the nonconforming elements of the first kind and the latter are called the nonconforming elements of the second kind. V_h can be written in the following general form

$$V_h = \left\{ v : v|_e \text{ is a polynomial of finite order, } v \text{ is continuous at the} \right. \\ \left. \text{vertices (edge midpoints) of } e, \quad \forall e \in \Omega_h \right\},$$

$$V_h^0 = \left\{ v \in V_h : v(x) = 0, \quad \forall \text{ interpolation point } x \in \partial\Omega \right\}.$$

Denote $A(w, v) = \sum_{e \in \Omega_h} \int_e \nabla w \nabla v$, $|v|_{1,\Omega,h} = \sqrt{A(v, v)}$. Obviously, $A(\cdot, \cdot), |\cdot|_{1,\Omega,h}$ are respectively, the inner product of V_h^0 and its induced norm. The nonconforming finite element discrete problem of (1.1) is

$$u_h \in V_h^0 : A(u_h, v) = (f, v), \quad \forall v \in V_h^0 \quad (1.3)$$

Theorem 1.1.^[5] (*The second Strang lemma.*) Let u, u_h be the solutions of (1.1) and (1.3), respectively. Then

$$|u - u_h|_{1,\Omega,h} \leq c \left\{ \inf_{v \in V_h^0} |u - v|_{1,\Omega,h} + \sup_{w \in V_h^0 \setminus \{0\}} \frac{E(u, w)}{|w|_{1,\Omega,h}} \right\}, \quad (1.4)$$

where

$$E(u, w) = A(u, w) - (f, w) = \sum_{e \in \Omega_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} w, \quad (1.5)$$

$\mathbf{n} = (\nu_1, \nu_2)^T$ represents the unit outward normal vector of e .

Let L_e be the linear (bilinear) interpolation operator on e with the vertices of e as its interpolation points. For any measurable set z , define the mean value operator $M_z : L^2(z) \rightarrow \mathfrak{R}$ by

$$\forall v \in L^2(z), \quad M_z v \in \mathfrak{R}, \quad M_z v = \frac{1}{\text{meas}(z)} \int_z v.$$

Let \hat{e} be the reference element, which is a square or an isosceles right triangle with $O(1)$ as its area. If there exists an invertible affine mapping

$$x = F_{\hat{e}}(\hat{x}) = B\hat{x} + b : \quad \hat{e} \rightarrow e \quad (1.6)$$

such that $e = F_{\hat{e}}(\hat{e})$, then we call that e is affine equivalent to \hat{e} . Here, $B \in \mathfrak{R}^{2 \times 2}$ is nonsingular. For any function v defined on e , let \hat{v} be the corresponding function defined on \hat{e} such that $\hat{v}(\hat{x}) = (v \circ F_{\hat{e}})(\hat{x})$, $\forall \hat{x} \in \hat{e}$.

Lemma 1.2. Let $u \in H^{1+\varepsilon}(\Omega)$, ($\varepsilon > 0$), be the solution of (1.1). Then

1) For the first kind nonconforming element space V_h , $\forall w \in V_h^0$, we have

$$E(u, w) \leq c \left\{ \sum_{e \in \Omega_h} \sum_{i=1}^2 \int_{\partial e} \left| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right|^2 \right\}^{\frac{1}{2}} \left(\sum_{e \in \Omega_h} \int_{\partial e} |w - L_e w|^2 \right)^{\frac{1}{2}}; \quad (1.7)$$

2) For the second kind nonconforming element space V_h , $\forall w \in V_h^0$, we have

$$E(u, w) \leq c \left\{ \sum_{e \in \Omega_h} \sum_{i=1}^2 \int_{\partial e} \left| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right|^2 \right\}^{\frac{1}{2}} \left(\sum_{e \in \Omega_h} \int_{\partial e} |w - M_e w|^2 \right)^{\frac{1}{2}}. \quad (1.8)$$

Proof. Let's prove (1.7) first. $\forall w \in V_h^0$, $L_e w$ is a linear (bilinear) function on e , and for each $e \in \Omega_h$, we have $L_e w$, which result in a piecewise linear (bilinear) function on Ω_h , denoted $L_e w$ still. Obviously, $L_e w \in H^1(\Omega)$, thus $E(u, L_e w) = 0$, and $E(u, w) = E(u, w - L_e w)$.

On the other hand, one-by-one analysis shows that

$$\int_{\partial e} \nu_i (w - L_e w) = 0, \quad i = 1, 2. \quad (1.9)$$

Furthermore, we get (1.7) by the Schwarz inequality.

Now, lets take the Carey membrane elements as an example to prove (1.9). Let (x_i, y_i) , $i = 1, 2, 3$ be the vertices of the triangle element e . F_i denotes the opposite edge of (x_i, y_i) , and its length is l_i . Let λ_i be the corresponding area coordinate. The unit outward normal vector $\mathbf{n} = (\nu_1, \nu_2)^T$ of e is

$$\nu_1 = \frac{y_{i+2} - y_{i+1}}{l_i}, \quad \nu_2 = -\frac{x_{i+2} - x_{i+1}}{l_i}, \quad \text{on } F_i, \quad i = 1, 2, 3, \quad (1.10)$$

where $x_4 = x_1$, $y_4 = y_1$, $x_5 = x_2$, $y_5 = y_2$ for notational convenience. Obviously, in order to prove (1.9) is right for the Carey elements, it suffices to show

$$\int_{\partial e} \nu_i (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) = 0, \quad i = 1, 2. \quad (1.11)$$

Note that, from (1.10), it is easy to obtain

$$\begin{aligned} \int_{\partial e} \nu_1 (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) &= \sum_{i=1}^3 \int_{F_i} \nu_1 (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) \\ &= \sum_{i=1}^3 \int_{F_i} \nu_1 \lambda_{i+1} \lambda_{i+2} = \sum_{i=1}^3 \frac{y_{i+2} - y_{i+1}}{l_i} \int_0^{l_i} \frac{s}{l_i} \frac{l_i - s}{l_i} ds = \sum_{i=1}^3 \frac{1}{6} (y_{i+2} - y_{i+1}) = 0. \end{aligned}$$

Thus, (1.11) is true for $i = 1$. Similarly, (1.11) is true for $i = 2$.

By now, we get (1.7).

Next, we prove (1.8). For the Crouzeix–Raviart elements, let F be the edge of the triangle element $e \in \Omega_h$. Obviously, $\int_F \nu_i w = 0$, $i = 1, 2, \forall F \subset \partial\Omega$; if F is the common edge of e_1, e_2 , then it follows from the linearity of w on F and the continuity of w at the midpoint of F that $\int_F \nu_i w_F = 0$, $i = 1, 2$. Here, w_F is the jump of w on F . Therefore, we have

$$\begin{aligned} E(u, w) &= \sum_{e \in \Omega_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e \in \Omega_h} \sum_{F \subset \partial e} \int_F \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e \in \Omega_h} \sum_{F \subset \partial e} \int_F \left[\frac{\partial u}{\partial \mathbf{n}} - M_F \left(\frac{\partial u}{\partial \mathbf{n}} \right) \right] w \\ &= \sum_{e \in \Omega_h} \sum_{F \subset \partial e} \int_F \left[\frac{\partial u}{\partial \mathbf{n}} - M_F \left(\frac{\partial u}{\partial \mathbf{n}} \right) \right] (w - M_F w) \\ &\leq \sum_{e \in \Omega_h} \left\{ \sum_{F \subset \partial e} \int_F \left| \frac{\partial u}{\partial \mathbf{n}} - M_F \left(\frac{\partial u}{\partial \mathbf{n}} \right) \right|^2 \right\}^{\frac{1}{2}} \left(\sum_{F \subset \partial e} \int_F |w - M_F w|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{e \in \Omega_h} \left\{ \sum_{F \subset \partial e} \sum_{i=1}^2 \int_F \left| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right|^2 \right\}^{\frac{1}{2}} \left(\sum_{F \subset \partial e} \int_F |w - M_e w|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which results in (1.8).

For the quartic rectangular elements, let F_i , $i = 1, 2, 3, 4$ be the four edges of $e \in \Omega_h$, and a_i be their corresponding midpoints. l_i is the length of F_i . a_5 denote the centroid of e . There exists an invertible affine mapping $x = F_{\hat{e}}(\hat{x}) : \hat{e} \rightarrow e$, \hat{e} is the reference element as shown in Fig. 1.1.

It is easy to see that

$$E(u, w) = \sum_{e \in \Omega_h} \int_{\partial e} \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e \in \Omega_h} \sum_{i=1}^4 \int_{F_i} \frac{\partial u}{\partial \mathbf{n}} w = \sum_{e \in \Omega_h} \sum_{i=1}^4 \int_{F_i} \frac{\partial u}{\partial \mathbf{n}} (w - w(a_i)). \quad (1.12)$$

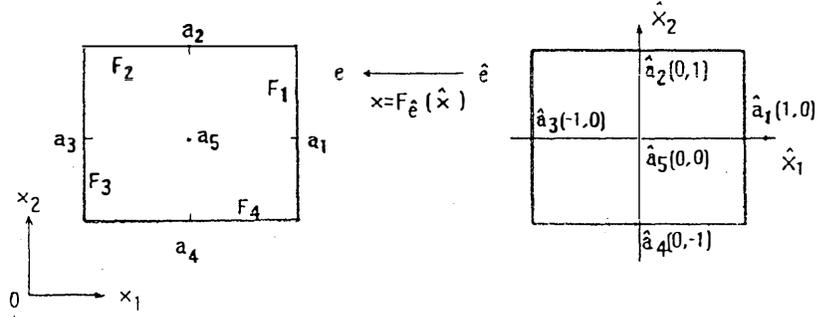


Fig. 1.1

By the affine mapping $x = F_{\hat{e}}(\hat{x}) : \hat{e} \rightarrow e$, we obtain that

$$\int_{F_i} \nu_j (w - w(a_i)) = \frac{l_i}{2} \int_{\hat{F}_i} \nu_j (\hat{w} - \hat{w}(\hat{a}_i)), \quad j = 1, 2. \quad (1.13)$$

It follows from Fig. 1.1 and [13] that

$$\int_{\hat{F}_2} \nu_1 (\hat{w} - \hat{w}(\hat{a}_2)) = 0, \quad \int_{\hat{F}_4} \nu_1 (\hat{w} - \hat{w}(\hat{a}_4)) = 0,$$

$$\hat{w} - \hat{w}(\hat{a}_1) = \frac{\hat{w}(\hat{a}_2) - \hat{w}(\hat{a}_4)}{2} \hat{x}_2 + \frac{\hat{w}(\hat{a}_2) + \hat{w}(\hat{a}_4) - 2\hat{w}(\hat{a}_5)}{2} \phi(\hat{x}_2), \quad \text{on } \hat{F}_1,$$

where $\phi(t) = \frac{1}{2}(5t^4 - 3t^2)$. Since $\int_{-1}^1 \hat{x}_2 d\hat{x}_2 = 0$, $\int_{-1}^1 \phi(\hat{x}_2) d\hat{x}_2 = 0$, we have $\int_{\hat{F}_1} \nu_1 (\hat{w} - \hat{w}(\hat{a}_1)) = 0$. In the same manner, it is easy to get $\int_{\hat{F}_3} \nu_1 (\hat{w} - \hat{w}(\hat{a}_3)) = 0$.

Therefore, by (1.13), we obtain that $\sum_{i=1}^4 \int_{F_i} \nu_1 (w - w(a_i)) = 0$.

Similarly, $\sum_{i=1}^4 \int_{F_i} \nu_2 (w - w(a_i)) = 0$ can be established.

Furthermore, it follows from (1.12) and the Schwarz inequality that

$$\begin{aligned} E(u, w) &= \sum_{e \in \Omega_h} \sum_{i=1}^4 \int_{F_i} \left[\sum_{j=1}^2 \nu_j \left(\frac{\partial u}{\partial x_j} - M_e \left(\frac{\partial u}{\partial x_j} \right) \right) \right] (w - w(a_i)) \\ &\leq \sum_{e \in \Omega_h} \left(\sum_{i=1}^4 \sum_{j=1}^2 \int_{F_i} \left| \frac{\partial u}{\partial x_j} - M_e \left(\frac{\partial u}{\partial x_j} \right) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^4 \int_{F_i} |w - w(a_i)|^2 \right)^{\frac{1}{2}} \\ &\leq \sum_{e \in \Omega_h} \left(\sum_{i=1}^4 \sum_{j=1}^2 \int_{F_i} \left| \frac{\partial u}{\partial x_j} - M_e \left(\frac{\partial u}{\partial x_j} \right) \right|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^4 \int_{F_i} |w - M_e w|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

which leads to (1.8). Hence the proof of the lemma is complete. \square

Lemma 1.3. For $\varepsilon \in (0, 1)$, we have

$$\|M_e w\|_{0,e} \leq \|w\|_{0,e}, \quad \forall w \in L^2(e) \quad (1.14)$$

$$\|w - M_e w\|_{0,e} \leq ch_e^\varepsilon \|w\|_{\varepsilon,e}, \quad \forall w \in H^\varepsilon(e). \quad (1.15)$$

Proof. By the Schwarz inequality, it is easy to get (1.14). We next prove (1.15). Note that $M_e c = c$, $\forall c \in \mathfrak{R}$, thus the error estimate of the finite element interpolation^[5] yields

$$\|w - M_e w\|_{0,e} \leq ch_e |w|_{1,e} \leq ch_e \|w\|_{1,e}, \quad \forall w \in H^1(e)$$

It follows from (1.14) that $\|w - M_e w\|_{0,e} \leq c \|w\|_{0,e}$, $\forall w \in L^2(e)$. Therefore, $I - M_e : L^2(e) \rightarrow L^2(e)$, $I - M_e : H^1(e) \rightarrow L^2(e)$ are bounded linear operators. Note that $H^\varepsilon(e)$ is the interpolation space between $L^2(e)$ and $H^1(e)$. Then (1.15) can be established by the interpolation theorem of Sobolev spaces^[1,15]. \square

Lemma 1.4. If e is affine equivalent to the reference element \hat{e} , $1 > \varepsilon \geq \frac{1}{2}$, then

$$\int_{\partial e} w^2 \leq c \{ h_e^{-1} \|w\|_{0,e}^2 + h_e^{2\varepsilon-1} |w|_{\varepsilon,e}^2 \}, \quad \forall w \in H^\varepsilon(e)$$

where $|w|_{\varepsilon,e}^2 \triangleq \int_e \int_e \frac{|w(x) - w(y)|^2}{|x - y|^{2+2\varepsilon}} dx dy$.

Proof. In the reference element \hat{e} , the trace theorem yields

$$\int_{\partial \hat{e}} (\hat{w})^2 \leq c \|\hat{w}\|_{\varepsilon,\hat{e}}^2 \leq c \{ \|\hat{w}\|_{0,\hat{e}}^2 + |\hat{w}|_{\varepsilon,\hat{e}}^2 \}.$$

It follows from the affine equivalence of e to \hat{e} that

$$\|\hat{w}\|_{0,\hat{e}}^2 \leq ch_e^{-2} \|w\|_{0,e}^2, \quad |\hat{w}|_{\epsilon,\hat{e}}^2 \leq ch_e^{2\epsilon-2} |w|_{\epsilon,e}^2, \quad \int_{\partial\hat{e}} (\hat{w})^2 \geq ch_e^{-1} \int_{\partial e} w^2.$$

With the above inequalities, Lemma 1.4 holds. \square

Lemma 1.5. *Let $u \in H^{1+\epsilon}(\Omega)$ be the solution of (1.1). $1 > \epsilon \geq \frac{1}{2}$. We have*

$$E(u, w) \leq ch^\epsilon \|u\|_{H^{1+\epsilon}(\Omega)} |w|_{1,\Omega,h}, \quad \forall w \in V_h^0$$

Proof. (1.15) gives

$$\begin{aligned} \left\| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right\|_{0,e} &\leq ch_e^\epsilon \left\| \frac{\partial u}{\partial x_i} \right\|_{\epsilon,e} \leq ch_e^\epsilon \|u\|_{1+\epsilon,e}, \\ \left| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right|_{\epsilon,e} &= \left| \frac{\partial u}{\partial x_i} \right|_{\epsilon,e} \leq \|u\|_{1+\epsilon,e}, \end{aligned}$$

which combining Lemma 1.4 yield

$$\begin{aligned} &\sum_{e \in \Omega_h} \sum_{i=1}^2 \int_{\partial e} \left| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right|^2 \\ &\leq c \sum_{e \in \Omega_h} \sum_{i=1}^2 \left\{ h_e^{-1} \left\| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right\|_{0,e}^2 + h_e^{2\epsilon-1} \left| \frac{\partial u}{\partial x_i} - M_e \left(\frac{\partial u}{\partial x_i} \right) \right|_{\epsilon,e}^2 \right\} \quad (1.16) \\ &\leq c \sum_{e \in \Omega_h} \sum_{i=1}^2 \left\{ h_e^{-1} h_e^{2\epsilon} \|u\|_{1+\epsilon,e}^2 + h_e^{2\epsilon-1} \|u\|_{1+\epsilon,e}^2 \right\} \leq ch^{2\epsilon-1} \|u\|_{H^{1+\epsilon}(\Omega)}^2. \end{aligned}$$

On the other hand, It follows from the interpolation error estimates and the inverse inequalities^[5] that

$$\begin{aligned} \|w - L_e w\|_{0,e} &\leq ch_e^2 |w|_{2,e} \leq ch_e |w|_{1,e}, & \|w - M_e w\|_{0,e} &\leq ch_e |w|_{1,e}, \\ |w - L_e w|_{1,e} &\leq ch_e |w|_{2,e} \leq c |w|_{1,e}, & |w - M_e w|_{1,e} &\leq c |w|_{1,e}. \end{aligned}$$

Furthermore, Lemma 1.4 indicates

$$\begin{aligned} \sum_{e \in \Omega_h} \int_{\partial e} |w - L_e w|^2 &\leq c \sum_{e \in \Omega_h} \left\{ h_e^{-1} \|w - L_e w\|_{0,e}^2 + h_e |w - L_e w|_{1,e}^2 \right\} \\ &\leq c \sum_{e \in \Omega_h} h_e |w|_{1,e}^2 \leq ch |w|_{1,\Omega,h}^2. \end{aligned} \quad (1.17)$$

Similarly, we have

$$\sum_{e \in \Omega_h} \int_{\partial e} |w - M_e w|^2 \leq ch |w|_{1,\Omega,h}^2. \quad (1.18)$$

It follows from the substitution of (1.16), (1.17) and (1.18) into (1.7) and (1.8) that Lemma 1.5 is right. \square

Lemma 1.6. *Suppose that $\hat{\pi} : H^1(\hat{e}) \rightarrow L^2(\hat{e})$ is the linear operator on the reference element \hat{e} , which satisfies that there exists a positive constant $c_1 = c_1(\hat{\pi})$, such that*

$$\|\hat{\pi}\hat{v}\|_{0,\hat{e}} \leq c_1\|\hat{v}\|_{1,\hat{e}}, \quad \forall \hat{v} \in H^1(\hat{e}) \quad (1.19)$$

$$\hat{\pi}\hat{c} = \hat{c}, \quad \forall \hat{c} \in \mathfrak{R} \quad (1.20)$$

Suppose that the element e is affine equivalent to \hat{e} and π_e is a linear mapping on e defined by

$$\widehat{\pi_e v} = \hat{\pi}\hat{v}, \quad \forall v \in H^1(e) \quad (1.21)$$

Then there exists a positive constant $c_2 = c_2(\hat{\pi}, \hat{e})$, such that

$$\|v - \pi_e v\|_{0,e} \leq c_2 h_e |v|_{1,e}, \quad \forall v \in H^1(e)$$

Proof. $\forall \hat{c} \in \mathfrak{R}$, (1.19) and (1.20) yield

$$\|\hat{v} - \hat{\pi}\hat{v}\|_{0,\hat{e}} = \|(\hat{v} + \hat{c}) - \hat{\pi}(\hat{v} + \hat{c})\|_{0,\hat{e}} \leq (1 + c_1)\|\hat{v} + \hat{c}\|_{1,\hat{e}}, \quad \forall \hat{v} \in H^1(\hat{e})$$

Furthermore, it follows from Theorem 3.1.1^[5] that

$$\|\hat{v} - \hat{\pi}\hat{v}\|_{0,\hat{e}} \leq (1 + c_1) \inf_{\hat{c} \in \mathfrak{R}} \|\hat{v} + \hat{c}\|_{1,\hat{e}} \leq c_3 |\hat{v}|_{1,\hat{e}}, \quad \forall \hat{v} \in H^1(\hat{e})$$

On the other hand, (1.21) implies $\hat{v} - \hat{\pi}\hat{v} = \hat{v} - \widehat{\pi_e v}$, hence Theorem 3.1.2^[5] gives

$$\|v - \pi_e v\|_{0,e} \leq c |\det B|^{\frac{1}{2}} \|\hat{v} - \hat{\pi}\hat{v}\|_{0,\hat{e}}, \quad |\hat{v}|_{1,\hat{e}} \leq c \|B\| |\det B|^{-\frac{1}{2}} |v|_{1,e},$$

where $\det B$ is the determinant of matrix B in the affine equivalence of e and \hat{e} , $\|B\|$ represents the Euclidean norm of B in \mathfrak{R}^2 . It follows from Theorem 3.1.3^[5] that $\|B\| \leq h_e/\hat{\rho}$, where $\hat{\rho} \triangleq \sup_{B, \hat{c}\hat{e}} r$. Obviously, $\hat{\rho} = O(1)$.

With the above facts, we end the proof of Lemma 1.6. \square

Remark 1.1. By the imbedding theorem^[1], the Schwarz inequality and the trace theorem, it is easy to see that (1.19) is true for finite element interpolation operators.

Let $P(e)$ be the polynomial space on e of finite order. Define the L^2 projection operator $Q_e : L^2(e) \rightarrow P(e)$ as follows

$$(Q_e v, w)_{L^2(e)} = (v, w)_{L^2(e)}, \quad \forall w \in P(e) \quad (1.22)$$

Lemma 1.7. *Suppose that the element e is affine equivalent to the reference element \hat{e} , we have*

$$|Q_e v|_{1,e} \leq c|v|_{1,e}, \quad \forall v \in H^1(e) \quad (1.23)$$

$$\|v - Q_e v\|_{0,e} \leq ch_e|v|_{1,e}, \quad \forall v \in H^1(e) \quad (1.24)$$

Proof. Analogously to (1.22), we can define the L^2 projection operator on \hat{e} , which is denoted $Q_{\hat{e}}$. It is easy to see that

$$\widehat{Q_e v} = Q_{\hat{e}} \hat{v},$$

which implies that (1.23) is equivalent to

$$|Q_{\hat{e}} \hat{v}|_{1,\hat{e}} \leq c|\hat{v}|_{1,\hat{e}}, \quad \forall \hat{v} \in H^1(\hat{e}) \quad (1.25)$$

It follows from the definition of $Q_{\hat{e}}$ and the Schwarz inequality that

$$\|Q_{\hat{e}} \hat{v}\|_{0,\hat{e}} \leq \|\hat{v}\|_{0,\hat{e}}. \quad (1.26)$$

Note that the norms of the finite dimensional space $P(\hat{e})$ are equivalent, thus we obtain

$$\begin{aligned} |Q_{\hat{e}} \hat{v}|_{1,\hat{e}} &\leq c\|Q_{\hat{e}} \hat{v}\|_{0,\hat{e}} \leq c\|\hat{v}\|_{0,\hat{e}} \leq c\|\hat{v}\|_{1,\hat{e}}, \\ |Q_{\hat{e}} \hat{v}|_{1,\hat{e}} &= |Q_{\hat{e}}(\hat{v} + \hat{c})|_{1,\hat{e}} \leq c\|\hat{v} + \hat{c}\|_{1,\hat{e}}, \quad \forall \hat{c} \in \mathfrak{R} \end{aligned}$$

Furthermore, Theorem 3.1.1^[5] yields

$$|Q_{\hat{e}} \hat{v}|_{1,\hat{e}} \leq c \inf_{\hat{c} \in \mathfrak{R}} \|\hat{v} + \hat{c}\|_{1,\hat{e}} \leq c|\hat{v}|_{1,\hat{e}},$$

which is (1.25). Therefore, (1.23) holds.

(1.26) implies that

$$\|\hat{v} - Q_{\hat{e}} \hat{v}\|_{0,\hat{e}} \leq 2\|\hat{v}\|_{0,\hat{e}} \leq 2\|\hat{v}\|_{1,\hat{e}},$$

$$\|\hat{v} - Q_{\hat{e}} \hat{v}\|_{0,\hat{e}} = \|(\hat{v} + \hat{c}) - Q_{\hat{e}}(\hat{v} + \hat{c})\|_{0,\hat{e}} \leq 2\|(\hat{v} + \hat{c})\|_{1,\hat{e}}, \quad \forall \hat{c} \in \mathfrak{R}$$

which combining Theorem 3.1.2^[5], Theorem 3.1.3^[5] and Theorem 3.1.1^[5] leads to

$$\|v - Q_e v\|_{0,e} \leq ch_e \|\hat{v} - Q_{\hat{e}} \hat{v}\|_{0,\hat{e}} \leq ch_e \inf_{\hat{c} \in \mathfrak{R}} \|\hat{v} + \hat{c}\|_{1,\hat{e}} \leq ch_e |\hat{v}|_{1,\hat{e}} \leq ch_e |v|_{1,e}.$$

Hence, (1.24) is established. \square

Lemma 1.8. *Let $\hat{e}, \hat{\pi}, e, \pi_e$ be those in Lemma 1.6. $0 \leq \varepsilon < 1$. If π_e satisfies*

$$|v - \pi_e v|_{1,e} \leq ch_e^k |v|_{k+1,e}, \quad \forall v \in H^{k+1}(e) \quad (1.27)$$

for some integer $k \geq 1$ then

$$|v - \pi_e v|_{1,e} \leq ch_e^\varepsilon |v|_{\varepsilon+1,e}, \quad \forall v \in H^{\varepsilon+1}(e) \quad (1.28)$$

Proof. It follows from the inverse inequality, Lemma 1.6 and Lemma 1.7 that

$$\begin{aligned} |v - \pi_e v|_{1,e} &\leq |v - Q_e v|_{1,e} + |Q_e v - \pi_e v|_{1,e} \\ &\leq c \left\{ |v|_{1,e} + h_e^{-1} \|Q_e v - \pi_e v\|_{0,e} \right\} \\ &\leq c \left\{ |v|_{1,e} + h_e^{-1} \|Q_e v - v\|_{0,e} + h_e^{-1} \|v - \pi_e v\|_{0,e} \right\} \\ &\leq c |v|_{1,e}. \end{aligned}$$

Thus (1.28) is true for $\varepsilon = 0$. Furthermore, with (1.27) and the interpolation theorem of Sobolev spaces^[1,15], we know that (1.28) is true for $\varepsilon > 0$. \square

Note that (1.27) is the finite element error estimate. It follows from Theorem 1.1, Lemma 1.5 and Lemma 1.8 that

Theorem 1.9. *Let $u \in H^{1+\varepsilon}(\Omega)$ be the solution of (1.1). $\varepsilon \geq \frac{1}{2}$. If u_h is the solution of (1.3), then*

$$|u - u_h|_{1,\Omega,h} \leq ch^\varepsilon \|u\|_{H^{1+\varepsilon}(\Omega)}.$$

§2 The Conforming Interpolation Operator I_h

First of all, we construct another mesh $\tilde{\Omega}_h$ of Ω based on Ω_h as follows: for the nonconforming elements of the first kind, if e is a quadrilateral, then e is divided into two triangles by connecting the opposite vertices of e as shown in Fig. 2.1; for the nonconforming elements of the second kind, e is divided into several triangles by connecting the interpolation points on e as shown in Fig. 2.2 and Fig. 2.3. Denote $\tilde{\Omega}_h = \{\tilde{e}\}$. \tilde{e} is a triangle. Let $S^h(\Omega)$ be the piecewise linear continuous finite element space on $\tilde{\Omega}_h$.

The conforming interpolation operator $I_h : V_h \rightarrow S^h(\Omega)$ is defined as follows

$$\forall v \in V_h, I_h v \in S^h(\Omega), \text{ such that}$$

1) for the nonconforming elements of the first kind,

$$(I_h v)(b) = v(b), \quad \forall \text{ vertice } b \text{ of } e, \forall e \in \Omega_h$$

2) for the nonconforming elements of the second kind,

$$(I_h v)(x) = \begin{cases} v(a), & \forall \text{ interpolation point } x = a \in \bar{\Omega} \\ \frac{1}{2}(v(a_1) + v(a_2)), & \forall \text{ vertice } x \text{ of } e \in \Omega_h, \\ & x \text{ is not the corner point of } \Omega \\ 0, & \forall \text{ corner point } x \text{ of } \Omega, \\ & a_1, a_2 \in \partial\Omega, v(a_1) \cdot v(a_2) = 0 \\ v(a_1), & \forall \text{ corner point } x \text{ of } \Omega, \\ & a_1, a_2 \in \partial\Omega, v(a_1) \cdot v(a_2) \neq 0 \end{cases}$$

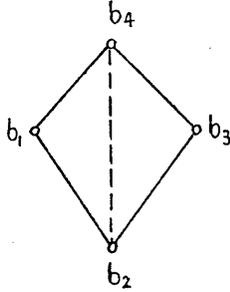


Fig. 2.1

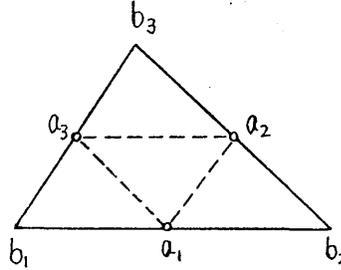


Fig. 2.2

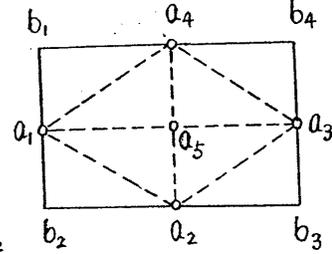


Fig. 2.3

Here, a_1, a_2 represent the midpoints of any two edges of the elements with x as their common endpoint. Generally, if possible, we select a_1, a_2 such that a_1, x, a_2 are in a line. But, when $x \in \partial\Omega$ is the vertice of some element e , we select $a_1, a_2 \in \partial\Omega$. We point out that although there might be different way to selected a_1, a_2 , we have

Theorem 2.1. *If the conforming interpolation operator I_h on V_h is defined as above, then*

$$\|v - I_h v\|_{L^2(\Omega)} \leq ch|v|_{1,\Omega,h}, \quad \forall v \in V_h \quad (2.1)$$

$$|v - I_h v|_{1,\Omega,h} \leq c|v|_{1,\Omega,h}, \quad \forall v \in V_h \quad (2.2)$$

$$\max_{e \in \Omega_h} \|v - I_h v\|_{L^\infty(e)} \leq c|v|_{1,\Omega,h}, \quad \forall v \in V_h \quad (2.3)$$

Proof. $\forall v \in V_h$, if (2.1) is true, then the inverse inequalities yield

$$\begin{aligned} |v - I_h v|_{1,\Omega,h}^2 &= \sum_{e \in \Omega_h} |v - I_h v|_{1,e}^2 \\ &\leq \sum_{e \in \Omega_h} ch^{-2} |v - I_h v|_{0,e}^2 \leq ch^{-2} \|v - I_h v\|_{L^2(\Omega)}^2 \leq c |v|_{1,\Omega,h}^2, \end{aligned}$$

$$\max_{e \in \Omega_h} \|v - I_h v\|_{L^\infty(e)} \leq \|v - I_h v\|_{L^\infty(e_0)} \leq ch^{-1} \|v - I_h v\|_{0,e_0} \leq c |v|_{1,\Omega,h}.$$

Here, $e_0 \in \Omega_h$. Thus, (2.1) implies (2.2) and (2.3). Therefore, it suffices to show (2.1).

For the nonconforming elements of the first kind, I_h is the (piecewise) linear interpolation operator on e . It follows from the interpolation error estimate and the inverse inequality that

$$\|v - I_h v\|_{L^2(\Omega)}^2 = \sum_{e \in \Omega_h} \|v - I_h v\|_{0,e}^2 \leq \sum_{e \in \Omega_h} ch^4 |v|_{2,e}^2 \leq \sum_{e \in \Omega_h} ch^2 |v|_{1,e}^2 \leq ch^2 |v|_{1,\Omega,h}^2,$$

hence, (2.1) is established in this case.

For the nonconforming elements of the second kind, construct a function \tilde{v} on Ω , such that $\tilde{v}|_e$ is the unique piecewise linear continuous function determined by $v|_{\tilde{e}}$, and $\tilde{v}|_e$ is linear on $\tilde{e} \subset e$, $\forall e \in \Omega_h$. Refer to Fig. 2.2 and Fig. 2.3. Obviously, $I_h v = I_h \tilde{v}$, on Ω . It follows from the interpolation error estimates and the inverse inequalities that

$$\|v - \tilde{v}\|_{0,e}^2 = \sum_{\tilde{e} \subset e} \|v - \tilde{v}\|_{0,\tilde{e}}^2 \leq \sum_{\tilde{e} \subset e} ch^4 |v|_{2,\tilde{e}}^2 = ch^4 |v|_{2,e}^2 \leq ch^2 |v|_{1,e}^2,$$

$$|v - \tilde{v}|_{1,e}^2 = \sum_{\tilde{e} \subset e} |v - \tilde{v}|_{1,\tilde{e}}^2 \leq \sum_{\tilde{e} \subset e} ch^2 |v|_{2,\tilde{e}}^2 \leq c |v|_{1,e}^2,$$

$$\|v - \tilde{v}\|_{L^2(\Omega)} \leq ch |v|_{1,\Omega,h}, \quad (2.4)$$

$$|v - \tilde{v}|_{1,\Omega,h} \leq c |v|_{1,\Omega,h},$$

$$|\tilde{v}|_{1,\Omega,h} \leq c |v|_{1,\Omega,h}. \quad (2.5)$$

If we can show that

$$\|\tilde{v} - I_h \tilde{v}\|_{L^2(\Omega)} \leq ch |\tilde{v}|_{1,\Omega,h}, \quad (2.6)$$

then it follows from $I_h v = I_h \tilde{v}$, on Ω , (2.4) and (2.5) that

$$\begin{aligned} \|v - I_h v\|_{L^2(\Omega)} &\leq \|v - \tilde{v}\|_{L^2(\Omega)} + \|\tilde{v} - I_h \tilde{v}\|_{L^2(\Omega)} \\ &= \|v - \tilde{v}\|_{L^2(\Omega)} + \|\tilde{v} - I_h \tilde{v}\|_{L^2(\Omega)} \\ &\leq ch |v|_{1,\Omega,h} + ch |\tilde{v}|_{1,\Omega,h} \leq ch |v|_{1,\Omega,h}, \end{aligned}$$

which is (2.1). Hence, it left to prove (2.6).

For the quartic rectangular elements, $I_h \tilde{v} = \tilde{v}$, on \tilde{e}_i , $i = 5, 6, 7, 8$. Refer to Fig. 2.4. Therefore, $\|\tilde{v} - I_h \tilde{v}\|_{0,e}^2 = \sum_{i=1}^4 \|\tilde{v} - I_h \tilde{v}\|_{0,\tilde{e}_i}^2$.

On the other hand, on \bar{e} , it is easy to see that

$$\tilde{v}(b_1) = v(b_1) = v(a_1) + v(a_4) - v(a_5) = \tilde{v}(a_1) + \tilde{v}(a_4) - \tilde{v}(a_5).$$

If b_1 is not the corner point of Ω , without loss of generality, we assume that $(I_h \tilde{v})(b_1) = \frac{1}{2}(\tilde{v}(a_1) + \tilde{v}(a_6))$. Then

$$\begin{aligned} \|\tilde{v} - I_h \tilde{v}\|_{0,\bar{e}_1}^2 &\leq ch|\tilde{v}(b_1) - (I_h \tilde{v})(b_1)| \\ &\leq ch|\tilde{v}(a_1) + \tilde{v}(a_4) - \tilde{v}(a_5) - \frac{1}{2}(\tilde{v}(a_1) + \tilde{v}(a_6))| \\ &\leq ch\{|\tilde{v}(a_1) - \tilde{v}(a_5)| + |\tilde{v}(a_4) - \tilde{v}(a_1)| + |\tilde{v}(a_4) - \tilde{v}(a_6)|\} \\ &\leq ch\{|\tilde{v}|_{1,\bar{e}_5} + |\tilde{v}|_{1,\bar{e}_9}\}. \end{aligned}$$

If b_1 is the corner point of Ω , without loss of generality, assume that $(I_h \tilde{v})(b_1) = \tilde{v}(a_1)$. Then

$$\|\tilde{v} - I_h \tilde{v}\|_{0,\bar{e}_1} \leq ch|\tilde{v}(b_1) - (I_h \tilde{v})(b_1)| \leq ch|\tilde{v}(a_4) - \tilde{v}(a_5)| \leq ch|\tilde{v}|_{1,\bar{e}_5}.$$

By now, we obtain that

$$\|\tilde{v} - I_h \tilde{v}\|_{0,e} \leq ch\{|\tilde{v}|_{1,e} + |\tilde{v}|_{1,\bar{e}}\},$$

where \bar{e} is the union of the elements adjacent to e . Furthermore, summing up with $e \in \Omega_h$, we get (2.6).

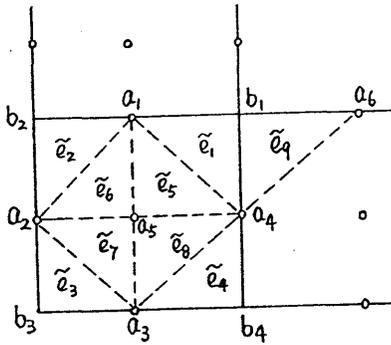


Fig. 2.4

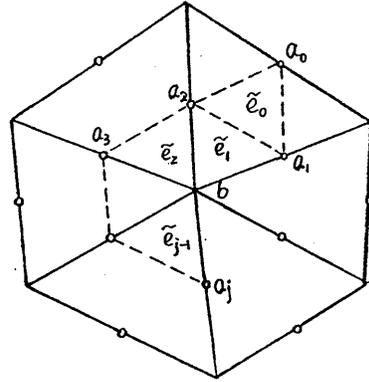


Fig. 2.5

For the Crouzeix–Raviart elements, $\forall e \in \Omega_h$, $\tilde{v} - I_h \tilde{v} = 0$, on \tilde{e}_0 . Refer to Fig. 2.5. It suffices to consider $\|\tilde{v} - I_h \tilde{v}\|_{0, \tilde{e}_1}$. Here, $\tilde{e}_0, \tilde{e}_1 \subset e$. In \bar{e} , it is easy to see that

$$\begin{aligned}\tilde{v}(a_1) &= (I_h \tilde{v})(a_1), \quad \tilde{v}(a_2) = (I_h \tilde{v})(a_2), \\ \tilde{v}(b_1) &= v(b_1) = v(a_1) + v(a_2) - v(a_0) = \tilde{v}(a_1) + \tilde{v}(a_2) - \tilde{v}(a_0).\end{aligned}$$

Without loss of generality, assume that $(I_h \tilde{v})(b) = \frac{1}{2}(\tilde{v}(a_i) + \tilde{v}(a_j))$, where a_i, a_j ($i \leq j$) are the midpoints of any two edges of the elements with b as their endpoint. It follows from the quasi-uniformness of Ω_h that there exists a positive integer J , independent of h , such that $j \leq J$. An elementary calculation yields

$$\begin{aligned}\|\tilde{v} - I_h \tilde{v}\|_{0, \tilde{e}_1} &\leq ch |\tilde{v}(b) - (I_h \tilde{v})(b)| \\ &\leq ch |\tilde{v}(a_1) + \tilde{v}(a_2) - \tilde{v}(a_0) - \frac{1}{2}(\tilde{v}(a_i) + \tilde{v}(a_j))| \\ &\leq ch \{ |\tilde{v}(a_2) - \tilde{v}(a_0)| + |\tilde{v}(a_1) - \tilde{v}(a_i)| + |\tilde{v}(a_1) - \tilde{v}(a_j)| \}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}h |\tilde{v}(a_2) - \tilde{v}(a_0)| &\leq c |\tilde{v}|_{1, \tilde{e}_0}, \\ h |\tilde{v}(a_1) - \tilde{v}(a_i)| &\leq h \sum_{k=1}^{i-1} |\tilde{v}(a_k) - \tilde{v}(a_{k+1})| \leq c \sum_{k=1}^{i-1} |\tilde{v}|_{1, \tilde{e}_k}, \\ h |\tilde{v}(a_1) - \tilde{v}(a_j)| &\leq c \sum_{k=1}^{j-1} |\tilde{v}|_{1, \tilde{e}_k}.\end{aligned}$$

Hence

$$\|\tilde{v} - I_h \tilde{v}\|_{0, \tilde{e}_1} \leq c \left\{ |\tilde{v}|_{1, \tilde{e}_0} + \sum_{k=1}^{j-1} |\tilde{v}|_{1, \tilde{e}_k} \right\}.$$

Summing up with $\tilde{e}_1 \subset e$, $e \in \Omega_h$ leads to (2.6). By now, the proof of Theorem 2.1 is finished. \square

§3 The Fundamental Inequalities

In this section, $d = \text{diam}\Omega$ represents the diameter of Ω .

Theorem 3.1. (*Poincaré inequalities in the nonconforming space V_h*)

1)

$$\|v\|_{L^2(\Omega)}^2 \leq c \left(d^2 |v|_{1, \Omega, h}^2 + d^{-2} \left| \int_{\Omega} v \right|^2 \right), \quad \forall v \in V_h$$

2)

$$d^{-2}\|v\|_{L^2(\Omega)}^2 \leq c|v|_{1,\Omega,h}^2, \quad \forall v \in V_h^0$$

3) If $v \in V_h$, $v(x) = 0$, \forall interpolation point $x \in \Gamma$, where $\Gamma \subset \partial\Omega$ has at least two interpolation points, then

$$d^{-2}\|v\|_{L^2(\Omega)}^2 \leq c|v|_{1,\Omega,h}^2.$$

Proof. $\forall v \in V_h$, $I_h v \in S^h(\Omega) \subset H^1(\Omega)$. It follows from the Poincaré inequality in $H^1(\Omega)$ that

$$\begin{aligned} \|I_h v\|_{L^2(\Omega)}^2 &\leq c\left(d^2|I_h v|_{H^1(\Omega)}^2 + d^{-2}\left|\int_{\Omega} I_h v\right|^2\right) \\ &\leq c\left\{d^2|I_h v|_{H^1(\Omega)}^2 + d^{-2}\int_{\Omega} v^2 + d^{-2}\left|\int_{\Omega} (v - I_h v)\right|^2\right\} \\ &\leq c\left\{d^2|v|_{1,\Omega,h}^2 + d^2|v - I_h v|_{1,\Omega,h}^2 + d^{-2}\int_{\Omega} v^2 + d^2\left(\int_{\Omega} 1^2\right)\|v - I_h v\|_{L^2(\Omega)}^2\right\}. \end{aligned}$$

Furthermore, by the triangle inequality and Theorem 2.1, we obtain

$$\begin{aligned} \|v\|_{L^2(\Omega)}^2 &\leq 2\left\{\|v - I_h v\|_{L^2(\Omega)}^2 + \|I_h v\|_{L^2(\Omega)}^2\right\} \\ &\leq c\left\{d^2|v|_{1,\Omega,h}^2 + d^{-2}\int_{\Omega} v^2 + h^2|v|_{1,\Omega,h}^2\right\}. \end{aligned}$$

Hence, 1) is right.

In the same manner, 2) and 3) can be proved. \square

Theorem 3.2. (Maximum norm estimate in the nonconforming space V_h)

$$\max_{e \in \Omega_h} \|v\|_{L^\infty(e)}^2 \leq c\left\{d^{-2}\|v\|_{L^2(\Omega)}^2 + \left(1 + \ln \frac{d}{h}\right)|v|_{1,\Omega,h}^2\right\}, \quad \forall v \in V_h$$

Proof. $\forall v \in V_h$, $I_h v \in S^h(\Omega) \subset H^1(\Omega)$. It follows from Lemma 3.3^[2] and Theorem 2.1 that

$$\begin{aligned} \|I_h v\|_{L^\infty(\Omega)}^2 &\leq c\left\{d^{-2}\|I_h v\|_{L^2(\Omega)}^2 + \left(\ln \frac{d}{h}\right)|I_h v|_{H^1(\Omega)}^2\right\} \\ &\leq c\left\{d^{-2}\|v\|_{L^2(\Omega)}^2 + d^{-2}\|v - I_h v\|_{L^2(\Omega)}^2\right. \\ &\quad \left.+ \left(\ln \frac{d}{h}\right)|v|_{1,\Omega,h}^2 + \left(\ln \frac{d}{h}\right)|v - I_h v|_{1,\Omega,h}^2\right\} \\ &\leq c\left\{d^{-2}\|v\|_{L^2(\Omega)}^2 + \left(\ln \frac{d}{h}\right)|v|_{1,\Omega,h}^2\right\}, \\ \max_{e \in \Omega_h} \|v\|_{L^\infty(e)}^2 &\leq 2\left(\max_{e \in \Omega_h} \|v - I_h v\|_{L^\infty(e)}^2 + \|I_h v\|_{L^\infty(\Omega)}^2\right) \\ &\leq c\left\{d^{-2}\|v\|_{L^2(\Omega)}^2 + \left(1 + \ln \frac{d}{h}\right)|v|_{1,\Omega,h}^2\right\}. \end{aligned}$$

By now, the theorem is proved. \square

Theorem 3.3. (*Extension theorem*) Let $\Gamma \subset \partial\Omega$ be an open boundary, $\{\xi_j\}_1^m$ ($m \geq 2$) the set of the interpolation points on Γ (ordered in some way). Suppose that Ω_h is quasi-uniform. Give $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T \in \mathfrak{R}^m$. If $w_h \in V_h$ satisfies

$$\begin{cases} A(w_h, v) = 0, & \forall v \in V_h^0 \\ w_h(\xi_j) = \lambda_j, & j = 1, 2, \dots, m \\ w_h(x) = 0, & \forall \text{ interpolation point } x \in \partial\Omega \setminus \Gamma \end{cases} \quad (3.1)$$

then

$$c \|\lambda_h\|_{H_{00}^{\frac{1}{2}}(\Gamma)}^2 \leq A(w_h, w_h) \leq C \|\lambda_h\|_{H_{00}^{\frac{1}{2}}(\Gamma)}^2, \quad (3.2)$$

where λ_h , the piecewise linear continuous function on Γ , satisfies

$$\lambda_h(\xi_j) = \lambda_j, \quad j = 1, 2, \dots, m, \quad \lambda_h(\nu) = 0, \quad \forall \text{ endpoint } \nu \text{ of } \Gamma.$$

Proof. Construct the harmonic function $w \in H^1(\Omega)$, which satisfies

$$\begin{cases} a(w, v) = 0, & \forall v \in H_0^1(\Omega) \\ w = \lambda_h, & \text{on } \Gamma \\ w = 0, & \text{on } \partial\Omega \setminus \Gamma \end{cases} \quad (3.3)$$

It follows from the priori estimate of the elliptic problems^[7,15,16] that even if Ω is concave, there exists $\varepsilon \geq \frac{1}{2}$, such that $w \in H^{\varepsilon+1}(\Omega)$. In addition, we note that w_h is the nonconforming finite element approximation of w .

Theorem 1.9 indicates that

$$\begin{aligned} A(w_h, w_h) &\leq 2\{A(w, w) + A(w - w_h, w - w_h)\} \\ &\leq c\{a(w, w) + |w - w_h|_{1,\Omega,h}^2\} \\ &\leq c\{\|w\|_{H^1(\Omega)}^2 + h^{2\varepsilon}\|w\|_{H^{\varepsilon+1}(\Omega)}^2\}. \end{aligned}$$

Furthermore, with the priori estimate of the elliptic problems and the inverse inequality of fractional order implied by the interpolation theorem of Sobolev Spaces^[1,15], we have

$$\begin{aligned} A(w_h, w_h) &\leq c\{\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + h^{2\varepsilon}\|w\|_{H^{\varepsilon+\frac{1}{2}}(\partial\Omega)}^2\} \\ &\leq c\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \leq c\|w\|_{H_{00}^{\frac{1}{2}}(\Gamma)}^2 = c\|\lambda_h\|_{H_{00}^{\frac{1}{2}}(\Gamma)}^2, \end{aligned} \quad (3.4)$$

in the above last inequality, we have applied the fact that $w = 0$ on $\partial\Omega \setminus \Gamma$ yields that $\|w\|_{H^{\frac{1}{2}}(\partial\Omega)}$ is equivalent to $\|w\|_{H_{00}^{\frac{1}{2}}(\Gamma)}$ (cf.[2]).

On the other hand, it follows from the construction of I_h that

$$I_h w_h \in H^1(\Omega), \quad I_h w_h = 0, \quad \text{on } \partial\Omega \setminus \Gamma, \quad I_h w_h = \lambda_h, \quad \text{on } \Gamma.$$

With Theorem 2.1, the Poincaré inequality and the trace theorem, we obtain

$$\begin{aligned} A(w_h, w_h) &\geq c|w_h|_{1,\Omega,h}^2 \geq c|I_h w_h|_{H^1(\Omega)}^2 \geq c\|I_h w_h\|_{H^1(\Omega)}^2 \\ &\geq c\|I_h w_h\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \geq c\|I_h w_h\|_{H_{00}^{\frac{1}{2}}(\Gamma)}^2 = c\|\lambda_h\|_{H_{00}^{\frac{1}{2}}(\Gamma)}^2, \end{aligned}$$

which combining (3.4) indicates that (3.2) is true. \square

Lemma 3.4. *Suppose that $d \gg h$ and there exists a positive constant β , such that $\sup_{B_r \subset \Omega} r \geq \beta d$. Let ν_1 be a corner point of Ω . v , the piecewise linear continuous function on $\partial\Omega$, satisfies*

$$v(x) = 0, \quad \forall \text{ interpolation point } x \in \partial\Omega, \quad v(\nu) = 0, \quad \forall \text{ corner point } \nu \text{ of } \Omega, \quad \nu \neq \nu_1.$$

Then, there exists a positive constant c , independent of d , such that

$$\|v\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq c|v(\nu_1)|. \quad (3.5)$$

Proof. Without loss of generality, we assume that Ω is a triangle. Let $\Gamma_1, \Gamma_2, \Gamma_3$ be the three edges of Ω with ν_1, ν_2, ν_3 as their opposite corner points. The definition of the Sobolev space $H^{\frac{1}{2}}(\partial\Omega)$ is (cf.[15])

$$\begin{aligned} \|v\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 &\triangleq \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} v^2 ds + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^2} ds(x) ds(y) \\ &= \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} v^2 ds + \sum_{i,j=1}^3 \int_{\Gamma_i} \int_{\Gamma_j} \frac{|v(x) - v(y)|^2}{|x - y|^2} ds(x) ds(y). \end{aligned}$$

Since $d \gg h$, $\sup_{B_r \subset \Omega} r \geq \beta d$, a simple calculus calculation yields

$$\begin{aligned} \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} v^2 ds &\leq c|v(\nu_1)|^2, \\ \int_{\Gamma_1} \int_{\Gamma_1} &= 0, \quad \int_{\Gamma_1} \int_{\Gamma_2} = \int_{\Gamma_2} \int_{\Gamma_1} \leq c|v(\nu_1)|^2, \\ \int_{\Gamma_2} \int_{\Gamma_2} &\leq 2|v(\nu_1)|^2, \quad \int_{\Gamma_2} \int_{\Gamma_3} = \int_{\Gamma_3} \int_{\Gamma_2} < 2|v(\nu_1)|^2, \\ \int_{\Gamma_3} \int_{\Gamma_3} &\leq 2|v(\nu_1)|^2, \quad \int_{\Gamma_3} \int_{\Gamma_1} = \int_{\Gamma_1} \int_{\Gamma_3} \leq c|v(\nu_1)|^2. \end{aligned}$$

Here, we omit the integrands for conciseness, and c is independent of d, h .

By now, (3.5) is established. \square

Lemma 3.5. *Let Γ_h be the quasi-uniform mesh of the interval $\Gamma = [0, L]$. Suppose that $v(x)$ is the piecewise linear continuous function on Γ_h , and $v(0) = 0$. Then, there exists a constant c , independent of L , such that*

$$\int_{\Gamma} \frac{(v(x))^2}{x} dx \leq \left(1 + \ln \frac{L}{h}\right) \|v\|_{L^\infty(\Gamma)}^2. \quad (3.6)$$

Proof. Suppose that x_1 is the interpolation point closest to the origin. It follows from the quasi-uniformness of Γ_h that $ch \leq x_1 \leq Ch$. Furthermore, the linearity of $v(x)$ on $[0, x_1]$ yields

$$\int_0^{x_1} \frac{(v(x))^2}{x} dx = \left(\frac{v(x_1) - v(0)}{x_1 - 0}\right)^2 \int_0^{x_1} x dx \leq \frac{1}{2} \|v\|_{L^\infty(\Gamma)}^2.$$

$$\int_{x_1}^L \frac{(v(x))^2}{x} dx \leq \|v\|_{L^\infty(\Gamma)}^2 \int_{x_1}^L \frac{dx}{x} \leq c \left(\ln \frac{L}{h}\right) \|v\|_{L^\infty(\Gamma)}^2.$$

Thus, we get (3.6). \square

Theorem 3.6. *Suppose that $d \gg h$ and there exists a positive constant β , such that $\sup_{B_r \subset \Omega} r \geq \beta d$. Let $\{\nu_k\}_1^J$ be the set of the corner points of Ω . $\Gamma_{ij} \subset \partial\Omega$ denotes the edge of Ω with ν_i, ν_j as its endpoints. Suppose that V_h is the nonconforming finite element space of the second kind. Then*

$$\|v^{ij}\|_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})}^2 \leq c \left(1 + \ln \frac{d}{h}\right) \left\{ \left(1 + \ln \frac{d}{h}\right) |v|_{1,\Omega,h}^2 + (1 + d^{-2}) \|v\|_{L^2(\Omega)}^2 \right\},$$

where v^{ij} , the piecewise linear continuous function on Γ_{ij} , satisfies

$$v^{ij}(x) = v(x), \quad \forall \text{ interpolation point } x \in \Gamma_{ij}, \quad v^{ij}(\nu_i) = v^{ij}(\nu_j) = 0,$$

and the positive constant c is independent of d .

Proof. Let v^B be the piecewise linear continuous function on $\partial\Omega$, such that $v^B = v^{ij}$, on Γ_{ij} , $\forall \Gamma_{ij} \subset \partial\Omega$. Let $v_k, k = 1, 2, \dots, J$ be the piecewise linear continuous function on $\partial\Omega$, such that

$$v_k(\nu_k) = (I_h v)(\nu_k), \quad v_k(\nu_j) = 0, \quad j = 1, 2, \dots, k-1, k+1, \dots, J,$$

$$v_k(x) = 0, \quad \forall \text{ interpolatoin point } x \in \partial\Omega.$$

It is easy to see that $I_h v = v^B + \sum_{k=1}^J v_k$, on $\partial\Omega$. It follows from the definition of the Sobolev space $H_{00}^{\frac{1}{2}}(\Gamma_{ij})$ that

$$\|v^{ij}\|_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})}^2 \leq c \left\{ \|v^B\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + \int_{\Gamma_{ij}} \left(\frac{|v^{ij}(x)|^2}{|x-\nu_i|} + \frac{|v^{ij}(x)|^2}{|x-\nu_j|} \right) ds(x) \right\}. \quad (3.7)$$

With the trace theorem, Lemma 3.4, Lemma 3.3^[2] and Theorem 2.1, we have

$$\begin{aligned} \|v^B\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 &= \|I_h v - \sum_{k=1}^J v_k\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \\ &\leq c \left\{ \|I_h v\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 + \sum_{k=1}^J \|v_k\|_{H^{\frac{1}{2}}(\partial\Omega)}^2 \right\} \\ &\leq c \left\{ \|I_h v\|_{H^1(\Omega)}^2 + \sum_{k=1}^J |(I_h v)(\nu_k)|^2 \right\} \\ &\leq c \left\{ \|I_h v\|_{H^1(\Omega)}^2 + \|I_h v\|_{L^\infty(\Omega)}^2 \right\} \\ &\leq \left\{ \left(1 + \ln \frac{d}{h}\right) \|I_h v\|_{H^1(\Omega)}^2 + (1 + d^{-2}) \|I_h v\|_{L^2(\Omega)}^2 \right\}, \\ &\leq \left\{ \left(1 + \ln \frac{d}{h}\right) |v|_{1,\Omega,h}^2 + (1 + d^{-2}) \|v\|_{L^2(\Omega)}^2 \right\}. \end{aligned} \quad (3.8)$$

It follows from Lemma 3.5, Lemma 3.3^[2] and Theorem 2.1 that

$$\begin{aligned} \int_{\Gamma_{ij}} \left(\frac{|v^{ij}(x)|^2}{|x-\nu_i|} + \frac{|v^{ij}(x)|^2}{|x-\nu_j|} \right) ds(x) &\leq c \left(1 + \ln \frac{d}{h}\right) \|v^{ij}\|_{L^\infty(\Gamma_{ij})}^2 \\ &\leq c \left(1 + \ln \frac{d}{h}\right) \|I_h v\|_{L^\infty(\Omega)}^2 \\ &\leq c \left(1 + \ln \frac{d}{h}\right) \left\{ \left(\ln \frac{d}{h}\right) \|I_h v\|_{H^1(\Omega)}^2 + d^{-2} \|I_h v\|_{L^2(\Omega)}^2 \right\} \\ &\leq c \left(1 + \ln \frac{d}{h}\right) \left\{ \left(\ln \frac{d}{h}\right) |v|_{1,\Omega,h}^2 + d^{-2} \|v\|_{L^2(\Omega)}^2 \right\}. \end{aligned} \quad (3.9)$$

The substitution of (3.8) and (3.9) into (3.7) indicates Theorem 3.6 holds. \square

Remark 3.1. Although we deal with the Dirichlet form $a(u, v) = \int_{\Omega} \nabla u \nabla v$ in this paper, all the above conclusions are true for the general form

$$a(u, v) = \int_{\Omega} \left[\sum_{i,j=1}^2 a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x) uv \right],$$

where $a_{ij}(x), a_0(x)$ are bounded, piecewise smooth on Ω , $a_0(x) \geq 0$, (a_{ij}) is a symmetric, uniformly positive definite matrix on Ω .

Remark 3.2. Besides [4,14,21], the nonconforming finite element spaces of the first kind may include other elements, which are continuous at the vertices of the elements of the mesh, if (1.9) is true.

Remark 3.3. Besides [6,13], the nonconforming finite element spaces of the second kind may include other elements, which are continuous at the midpoints, even the Gaussian quadrature points, of the edges of the elements of the mesh, if (1.8) and (2.6) are established.

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