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Abstract

We mainly consider the nonconforming finite elements with their continuity only at the vertices or the edge midpoints of the elements of the quasi–uniform mesh. Based on the extension theorems[10,11,19], we developed a unifying and systematic way to design and analyse the domain decomposition methods for the second order self–adjoint elliptic problems with two nonoverlap subdomains in the conforming or the nonconforming discrete cases, in spite of the loss of global continuity of the nonconforming elements. A concise summary of all existing domain decomposition algorithms is presented in this paper. The results of numerical experiments are reported.

§1 Introduction

In recent ten years a considerable attention has been devoted to the use of domain decomposition methods for the numerical solution of partial differential equations. Among others, the following reasons underly the development of these methods. The equations in different subdomains may be of different type, or, more simply, they might contain different parameters. Besides, when dealing with complicated geometries, a subdivision of the entire domain by simply shaped subdomains on which special solution techniques cah be applied may increase the overall efficiency of the numerical scheme. A further important reason is that very often domain decomposition methods are well suited for computations in parallel environments.

Although the domain decomposition algorithms with two nonoverlap subdomains (substructures) don't virtually or completely have the advantages of domain

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decomposition, the study of them is the first step to establish more efficient and more complicated algorithms. In the conforming finite element discrete cases, different algorithms are presented for second order elliptic problems[1,2,4,17,21] , and their rate of convergence are bounded independently of the size of the problem, which is ensured by the so–called extension theorem[19]. It remains a problem, which the present paper will solve, to design and analyse the domain decomposition algorithms in the nonconforming discrete cases. We consider the nonconforming finite elements which are only continuous at the vertices, or the edge midpoints of the elements of the quasi–uniform mesh[5,7,14,20]. It is noted that the extension theorem for those nonconforming elements has already been established [10,11], which play a key role in the analysis of domain decomposition methods. Based on the extension theorems in the discrete cases[10,11,19], a unifying and systematic way to design and analyse the domain decomposition methods with two substructures are developed, for the second order self–adjoint elliptic problems discretized by finite elements, which may include the nonconforming finite elements. In addition, a brief survey of all existing algorithms, some of which are discussed in detail, is presented in this paper.

An outline of the paper is as follows: In Sect.2, the extension theorem in the continuous cases is stated, while its counterpart in the finite element discrete cases is presented in Sect.3. The convergence analysis and matrix analysis are given in Sect.4 and Sect.5. Finally, in Sect.6, we describe the results of numerical calculations which show that the theoretical estimates are fully realized in practice.

We point out that in the multi–subdomain nonoverlap cases, many domain decomposition methods are developed[10,12,13], which can be viewed as the extension of the algorithms in the present paper.

§2 The Extension Theorem in the Continuous Cases

Let $\Omega \subset \Re^2$ be an open bounded domain. Consider the second order self–adjoint elliptic problem in the weak formulation

$$u \in H_0^1(\Omega): \ a(u,v) = (f,v), \ \forall\, v \in H_0^1(\Omega) \qquad (2.1)$$

where, $(f,v) = \int_\Omega fv$, $a(u,v) = \int_\Omega \Big[\sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0(x)uv\Big]$, $f \in H^{-1}(\Omega)$, $a_0(x) \geq 0$. $(a_{ij}(x))$ is a symmetric, uniformly positive definite matrix. $a_{ij}(x), a_0(x)$ are bounded and piecewise smooth on Ω. Let Ω be divided by an open curve Γ

into two open subdomains Ω_1, Ω_2, such that

$$\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma, \quad \Omega_1 \cap \Omega_2 = \emptyset$$

For $k = 1, 2$, denote

$$(f, v)_k = \int_{\Omega_k} fv, \quad a_k(u, v) = \int_{\Omega_k} \Big[\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv\Big].$$

Once the restriction to Γ of the solution u of (2.1), also known as the trace , $\lambda = r_0 u = u|_\Gamma \in H_{00}^{\frac{1}{2}}(\Gamma)$ is obtained,

$$u_k = u|_{\Omega_k} \in H_\Gamma^1(\Omega_k) = \{w \in H^1(\Omega_k) : \ w = 0, \text{ on } \partial\Omega_k \backslash \Gamma\},$$

satisfies independently

$$\begin{cases} a_k(u_k, v) = (f, v)_k, & \forall\, v \in H_0^1(\Omega_k) \\ u_k = \lambda, & on\ \Gamma \end{cases} \tag{2.2}$$

But λ is unknown. It follows from the Green formula that λ should satisfy the following interface condition

$$\frac{\partial u_1(\lambda)}{\partial \mathrm{n}_1} = -\frac{\partial u_2(\lambda)}{\partial \mathrm{n}_2}, \text{ on } \Gamma. \tag{2.3}$$

Here, $\frac{\partial}{\partial \mathrm{n}_k} = \sum_{i,j=1}^{2} a_{ij} \nu_i^k \frac{\partial}{\partial x_j}$, $(\nu_1^k, \nu_2^k)^T$ is the unit outward normal vector. It is easy to see that the solution u_k of (2.2) can be expressed as

$$u_k = R_k \lambda + T_k f, \tag{2.4}$$

where $T_k f$ is the solution of

$$T_k f \in H_0^1(\Omega_k) : \ a_k(T_k f, v) = (f, v)_k, \ \forall\, v \in H_0^1(\Omega_k), \tag{2.5}$$

and $R_k \lambda \in H_\Gamma^1(\Omega_k)$ is the harmonic extension function on Ω_k determined by λ, which satisfies

$$\begin{cases} a_k(R_k \lambda, v) = 0, & \forall\, v \in H_0^1(\Omega_k) \\ R_k \lambda = \lambda, & \text{on } \Gamma \end{cases} \tag{2.6}$$

Correspondingly, $R_k : H_{00}^{\frac{1}{2}}(\Gamma) \rightarrow H_\Gamma^1(\Omega_k)$ is called the harmonic extension operator. The substitution of (2.4) into (2.3) yields

$$\frac{\partial}{\partial \mathbf{n}_1}(R_1 - R_2)\lambda = \frac{\partial}{\partial \mathbf{n}_1}(T_2 - T_1)f, \text{ on } \Gamma. \tag{2.7}$$

For the right hand side of (2.7), $\mu = \dfrac{\partial}{\partial \mathbf{n}_1}(T_2 - T_1)f$, is independent of λ, and can be obtained by solving (2.5) previously and independently, (2.7) is equivalent to the operator equation

$$S\lambda = \mu, \tag{2.8}$$

where $S = S_1 + S_2,\ S_k\cdot = \dfrac{\partial}{\partial \mathbf{n}_k}(R_k\cdot), k = 1,2$. S is called the Steklov–Poincaré operator. We know that

1) $S : H_{00}^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma),\ S_k : H_{00}^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ are symmetric, positive definite operators ;

2) The inverse operator of S_k, denoted $S_k^{-1} : H^{-\frac{1}{2}}(\Gamma) \rightarrow H_{00}^{\frac{1}{2}}(\Gamma)$, exists. For any $\mu \in H^{-\frac{1}{2}}(\Gamma), \lambda = S_k^{-1}\mu$ can be obtained by first solving the Neumann–Dirichlet mixed problem on Ω_k:

$$u_k \in H_\Gamma^1(\Omega_k) : \ a_k(u_k, v) = \int_\Gamma \mu v, \ \forall\ v \in H_\Gamma^1(\Omega_k)$$

and then setting $\lambda = u|_\Gamma$.

With the trace theorem and a well–known priori inequality[15], we obtain

Theorem 2.1.[15] *(The extension theorem in the continuous cases) There exist two constants $\tilde{\sigma}, \tilde{\tau}$, such that*

$$\tilde{\sigma} a_1(R_1\lambda, R_1\lambda) \leq a_2(R_2\lambda, R_2\lambda) \leq \tilde{\tau} a_1(R_1\lambda, R_1\lambda), \ \forall\ \lambda \in H_{00}^{\frac{1}{2}}(\Gamma) \tag{2.9}$$

§3 The Extension Theorem in the Discrete Cases

In what follows, we assume that Ω is a polygonal domain. Let $\Omega_h = \{e\}$ be a quasi–uniform mesh of Ω, such that $e \cap \Gamma = \emptyset,\ \forall\ e \in \Omega_h$, where e, a triangle or a quadrilateral, represents the typical element in Ω_h. Let V_h be the finite element space related to the mesh Ω_h. Of course, V_h can be the space of the conforming elements. Also, V_h can be the space of the Wilson's elements[20], Carey membrane elements[5], etc., which are only continuous at the vertices of the elements of the mesh. Besides, V_h can be the space of Crouzeix–Raviart elements[7], quartic rectangular elements[14], etc., which are only continuous at the edge midpoints of the elements of the mesh. For simplicity of the exposition, an interpolation point x of V_h is briefly called a node, and we assume that

$$v(x) = 0, \ \forall \text{ node } x \in \partial\Omega, \ \forall\, v \in V_h$$

All the nodes on Γ are labelled $\{\xi_j\}_{j=1}^m$, which are ordered in some way. For $k = 1, 2$, we introduce the following notations

$$A(u,v) = \sum_{e\subset\Omega} \int_e \Big[\sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\Big],$$

$$A_k(u,v) = \sum_{e\subset\Omega_k} \int_e \Big[\sum_{i,j=1}^{2} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + a_0 uv\Big],$$

$$V_h^k = \{v \in V_h : \ v(x) = 0, \ \forall \text{ node } x \in \Omega\backslash\overline{\Omega}_k\},$$

$$V_h^{k,0} = \{v \in V_h : \ v(x) = 0, \ \forall \text{ node } x \in \Omega\backslash\Omega_k\}.$$

The finite element discrete problem corresponding to (2.1) is

$$u_h \in V_h : \ A(u_h, v) = (f, v), \ \forall\, v \in V_h \tag{3.1}$$

Let $r_0^h : V_h \longrightarrow \Re^m$ be the discrete trace operator, such that

$$\forall\, v \in V_h, \ r_0^h v \in \Re^m : \ \big(r_0^h v\big)(j) = v(\xi_j), \ j = 1, 2, \cdots, m,$$

where $\big(r_0^h v\big)(j)$ represents the jth element of $r_0^h v$. We define the discrete harmonic extension operator $R_k^h : \Re^m \longrightarrow V_h^k$ as follows

$$\forall\, \lambda \in \Re^m, \ R_k^h\lambda \in V_h^k : \ \begin{cases} A_k(R_k^h\lambda, v) = 0, & \forall\, v \in V_h^{k,0} \\ r_0^h R_k^h \lambda = \lambda \end{cases}$$

Theorem 3.1.[10,11,19] *(The extension theorem in the discrete cases) There exist constants σ, τ, c, C, independent of the quasi–uniform mesh parameter h, such that*

$$\sigma = \sup_{\lambda\in\Re^m\backslash\{0\}} \frac{A_1(R_1^h\lambda, R_1^h\lambda)}{A_2(R_2^h\lambda, R_2^h\lambda)}, \quad \tau = \sup_{\lambda\in\Re^m\backslash\{0\}} \frac{A_2(R_2^h\lambda, R_2^h\lambda)}{A_1(R_1^h\lambda, R_1^h\lambda)}, \tag{3.2}$$

$$c\|\lambda_h\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)} \leq A_k(R_k^h\lambda, R_k^h\lambda) \leq C\|\lambda_h\|^2_{H^{\frac{1}{2}}_{00}(\Gamma)}, \quad k = 1, 2, \tag{3.3}$$

where λ_h is the piecewise linear continuous function on Γ interpolated by $\{\xi_j\}_{j=1}^m$ with λ, such that

$$\lambda_h(\xi_j) = \lambda_j, \; j = 1, 2, \cdots, m, \quad \lambda_h(\nu) = 0, \; \forall \text{ endpoint } \nu \text{ of } \Gamma.$$

Furthermore, for any $\lambda \in \Re^m$, define $T_k^h : \Re^m \to V_h^k$ as follows

$$\begin{cases} T_2^h\lambda \in V_h^2 \\ A_2(T_2^h\lambda, v) = -A_1(R_1^h\lambda, R_1^h r_0^h v), \forall\, v \in V_h^2 \end{cases} \tag{3.4}$$

$$\begin{cases} T_1^h\lambda \in V_h^1 \\ A_1(T_1^h\lambda, v) = -A_2(R_2^h\lambda, R_2^h r_0^h v), \forall\, v \in V_h^1 \end{cases} \tag{3.5}$$

Corrollary 3.2. *Let Ω_h be quasi-uniform, σ, τ the same as those in Theorem 3.1. Then we have*

$$\frac{1}{\tau}A_1(R_1^h\lambda, R_1^h\lambda) \leq A_2(T_2^h\lambda, T_2^h\lambda) \leq \sigma A_1(R_1^h\lambda, R_1^h\lambda), \;\; \forall\, \lambda \in \Re^m \tag{3.6}$$

$$\frac{1}{\sigma}A_2(R_2^h\lambda, R_2^h\lambda) \leq A_1(T_1^h\lambda, T_1^h\lambda) \leq \tau A_2(R_2^h\lambda, R_2^h\lambda), \;\; \forall\, \lambda \in \Re^m \tag{3.7}$$

Proof It follows from the substitution of v in (3.4) with $T_2^h\lambda$, the Schwarz inequality and (3.2) that

$$\begin{aligned} A_2(T_2^h\lambda, T_2^h\lambda) &= -A_1(R_1^h\lambda, R_1^h r_0^h T_2^h\lambda) \\ &\leq \left(A_1(R_1^h\lambda, R_1^h\lambda)\right)^{\frac{1}{2}} \left(A_1(R_1^h r_0^h T_2^h\lambda, R_1^h r_0^h T_2^h\lambda)\right)^{\frac{1}{2}} \\ &\leq \left(A_1(R_1^h\lambda, R_1^h\lambda)\right)^{\frac{1}{2}} \left(\sigma A_2(R_2^h r_0^h T_2^h\lambda, R_2^h r_0^h T_2^h\lambda)\right)^{\frac{1}{2}} \\ &= \left(A_1(R_1^h\lambda, R_1^h\lambda)\right)^{\frac{1}{2}} \left(\sigma A_2(T_2^h\lambda, T_2^h\lambda)\right)^{\frac{1}{2}}, \end{aligned}$$

which implies that the right hand side of (3.6) is true.

On the other hand, taking $v = R_2^h\lambda$ in (3.4), using (3.2), we get

$$\begin{aligned} A_1(R_1^h\lambda, R_1^h\lambda) = -A_2(T_2^h\lambda, R_2^h\lambda) &\leq \left(A_2(T_2^h\lambda, T_2^h\lambda)\right)^{\frac{1}{2}} \left(A_2(R_2^h\lambda, R_2^h\lambda)\right)^{\frac{1}{2}} \\ &\leq \left(A_2(T_2^h\lambda, T_2^h\lambda)\right)^{\frac{1}{2}} \left(\tau A_1(R_1^h\lambda, R_1^h\lambda)\right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, the left hand side of (3.6) is proved.

Similarly, (3.7) can be established. □

For $k = 1, 2$, for any $\lambda \in \Re^m$, let $\rho_k : \Re^m \longrightarrow V_h^k$ be the linear operator, s.t.

$$\rho_k \lambda \in V_h^k, \ \Big(\rho_k \lambda\Big)(x) = \begin{cases} 0, & \forall \text{ node } x \in \Omega_k \\ \lambda_j, & x = \xi_j, \ j = 1, 2, \cdots, m \end{cases}$$

Let $\{\phi_i\}$ be the set of the standard bases of V_h. Let $\{\phi_i^k\}$ be the subset of $\{\phi_i\}$ related to $\Omega_k, k = 1, 2$. Let $\{\phi_i^\Gamma\}$ be the subset of $\{\phi_i\}$ related to Γ. Suppose that the nodes are ordered in the way: first those on $\Omega_k, k = 1, 2$ and then $\{\xi_j\}_{j=1}^m$. Let

$$u_h = \sum_i \alpha_i^1 \phi_i^1 + \sum_j \alpha_j^2 \phi_j^2 + \sum_{k=1}^m \alpha_k^\Gamma \phi_k^\Gamma$$

be the solution of (3.1). Denote $U_1 = (\alpha_i^1)$, $U_2 = (\alpha_j^2)$, $U_3 = (\alpha_k^\Gamma)$. (3.1) can then be rewriten as

$$KU = \begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \tag{3.8}$$

where

$$\begin{aligned} &K = \Big(A(\phi_i, \phi_j)\Big), \qquad K_{kk} = \Big(A(\phi_i^k, \phi_j^k)\Big), \ \ K_{k3} = \Big(A(\phi_i^k, \phi_j^\Gamma)\Big), \\ &K_{33}^{(k)} = \Big(A_k(\phi_i^\Gamma, \phi_j^\Gamma)\Big), \ \ K_{33} = K_{33}^{(1)} + K_{33}^{(2)}, \quad k = 1, 2. \end{aligned}$$

By block Gaussian elimination, (3.8) reduces to

$$S^h U_3 = \tilde{b}_3, \tag{3.9}$$

where, $\qquad \tilde{b}_3 = b_3 - K_{13}^T K_{11}^{-1} b_1 - K_{23}^T K_{22}^{-1} b_2,$

$$S^h = K_{33} - K_{13}^T K_{11}^{-1} K_{13} - K_{23}^T K_{22}^{-1} K_{23} = S_1^h + S_2^h, \tag{3.10}$$

$$S_k^h = K_{33}^{(k)} - K_{k3}^T K_{kk}^{-1} K_{k3}, \quad k = 1, 2. \tag{3.11}$$

Note that in the equality

$$\begin{bmatrix} K_{kk} & K_{k3} \\ K_{k3}^T & K_{33}^{(k)} \end{bmatrix} \left\{ \begin{bmatrix} K_{kk} & K_{k3} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \lambda \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ S_k^h \lambda \end{bmatrix}, \quad k = 1, 2, \tag{3.12}$$

the first matrix corresponds to $\frac{\partial}{\partial \mathrm{n}_k}$, while the part in the braces corresponds to the harmonic extension function $R_k^h\lambda$ for $\lambda \in \Re^m$ (cf.[18]). Therefore, S_k^h, S^h are, respectively, the discrete counterparts of S_k, S. S^h, S_k^h are called, respectively, the capacitance matrices of Ω, Ω_k with regard to Γ. Sometimes, S^h is called the Schur complement. S^h, S_k^h are symmetric, positive definite matrices and spectrally equivalent, which is implied by

Theorem 3.3. *Let Ω_h be quasi-uniform. Then there exist two constants c, C, independent of h, such that*

$$c\lambda^T S_k^h \lambda \le \lambda^T S^h \lambda \le C \lambda^T S_k^h \lambda, \ \forall\ \lambda \in \Re^m, \ k = 1, 2. \tag{3.13}$$

Proof It follows from the definitions of R_k^h, S_k^h that

$$\lambda^T S_k^h \mu = A_k(R_k^h \lambda, R_k^h \mu), \ k = 1, 2, \ \forall\ \lambda, \mu \in \Re^m \tag{3.14}$$

Therefore, by Theorem 3.1, we eventually get (3.13). □

For $k = 1, 2$, define $D_k^h : V_h \to \Re^m$ as

$$\forall\ v \in V_h,\ D_k^h v \in \Re^m : \ \left(D_k^h v\right)(j) = A_k(v, \phi_j^\Gamma) - (f, \phi_j^\Gamma)_k, \ j = 1, 2, \cdots, m.$$

$\forall\ v \in V_h$, let $v^k \in V_h^k$, such that $\quad v^k(x) = \begin{cases} v(x), & \forall \text{ node } x \in \overline{\Omega}_k \\ 0, & \forall \text{ node } x \in \Omega \backslash \overline{\Omega}_k \end{cases}$

With the above notations, definitions and theorems, we can state and analyse all the algorithms in two–subdomain nonoverlap cases, which are presented in the following sections.

§4 The Iterative Substructuring Algorithms

Algorithm 4.1.[17] (Marini–Quarteroni scheme)

Step 1. Let $g^0 \in \Re^m$ be any initial guess, and set n:=1.

Step 2. Solve the Dirichlet subproblem posed on Ω_1:

$$\begin{cases} u_1^n \in V_h^1 \\ A_1(u_1^n, v) = (f, v)_1, \ \forall\ v \in V_h^{1,0} \\ r_0^h u_1^n = g^{n-1} \end{cases}$$

Step 3. Solve the Dirichlet–Neumann subproblem set on Ω_2:

$$\begin{cases} u_2^n \in V_h^2 \\ A_2(u_2^n, v) = (f, v)_2, \ \forall\, v \in V_h^{2,0} \\ A_2(u_2^n, \rho_2\lambda) = (f, \rho_1\lambda)_1 + (f, \rho_2\lambda)_2 - A_1(u_1^n, \rho_1\lambda), \ \forall\, \lambda \in \Re^m \end{cases}$$

Step 4. Choose the relaxation factor $\theta_n \in (0,1)$. Calculate

$$g^n = \theta_n r_0^h u_2^n + (1-\theta_n) g^{n-1}.$$

Set n:=n+1, return to Step 2 until some reasonable stopping criterion is satisfied.

Theorem 4.1. *Let u_h be the solution of (3.1). u_1^n, u_2^n, g^n are generated by Algorithm 4.1. Denote $\varepsilon_k^n = u_k^n - u_h^k$, $k = 1, 2$, $\mu^n = g^n - r_0^h u_h = g^n - U_3$. Then*

1)

$$\frac{1}{\tau} A_1(\varepsilon_1^n, \varepsilon_1^n) \le A_2(\varepsilon_2^n, \varepsilon_2^n) \le \sigma A_1(\varepsilon_1^n, \varepsilon_1^n). \tag{4.1}$$

2) There exists a constant $\theta^* \in (0,1]$, such that

$$A_1(\varepsilon_1^{n+1}, \varepsilon_1^{n+1}) \le \kappa(\theta_n) A_1(\varepsilon_1^n, \varepsilon_1^n), \tag{4.2}$$

where $\kappa(\theta_n) < 1, \ \forall\, \theta_n \in (0, \theta^*)$.

3) There exists the optimal relaxation factor θ^{opt}, such that

$$\kappa(\theta^{opt}) = \min_{\theta \in (0,\theta^*)} \kappa(\theta).$$

4) Algorithm 4.1 is equivalent to the following iterative method for the capacitance equation (3.9):

$$S_2^h(g^{n+1} - g^n) = \theta_{n+1}(\tilde{b}_3 - S^h g^n). \tag{4.3}$$

Proof It is easy to see that $\varepsilon_k^{n+1} \in V_h^k$ satisfies

$$\begin{cases} A_1(\varepsilon_1^{n+1}, v) = 0, \ \forall\, v \in V_h^{1,0} \\ r_0^h \varepsilon_1^{n+1} = \mu^n \end{cases} \tag{4.4}$$

$$\begin{cases} A_2(\varepsilon_2^{n+1}, v) = 0, \ \forall\, v \in V_h^{2,0} \\ A_2(\varepsilon_2^{n+1}, \rho_2\lambda) = -A_1(\varepsilon_1^{n+1}, \rho_1\lambda), \ \forall\, \lambda \in \Re^m \end{cases} \tag{4.5}$$

$$\mu^{n+1} = \theta_{n+1} r_0^h \varepsilon_2^{n+1} + (1 - \theta_{n+1})\mu^n. \tag{4.6}$$

(4.4) and (4.5) yield

$$A_2(\varepsilon_2^{n+1}, v) = -A_1(\varepsilon_1^{n+1}, R_1^h r_0^h v), \ \forall\, v \in V_h^2 \tag{4.7}$$

which indicates that $\varepsilon_2^{n+1} = T_2^h r_0^h \varepsilon_1^{n+1}$. By (3.6), we get (4.1).

On the other hand, it follows from (4.4) and (4.6) that

$$\varepsilon_1^{n+1} = \theta_n R_1^h r_0^h \varepsilon_2^n + (1-\theta_n)\varepsilon_1^n.$$

Hence

$$\begin{aligned} A_1(\varepsilon_1^{n+1}, \varepsilon_1^{n+1}) = \; & \theta_n^2 A_1(R_1^h r_0^h \varepsilon_2^n, R_1^h r_0^h \varepsilon_2^n) + (1-\theta_n)^2 A_1(\varepsilon_1^n, \varepsilon_1^n) \\ & + 2\theta_n(1-\theta_n) A_1(R_1^h r_0^h \varepsilon_2^n, \varepsilon_1^n). \end{aligned} \tag{4.8}$$

By (3.2) and (4.1), we obtain

$$A_1(R_1^h r_0^h \varepsilon_2^n, R_1^h r_0^h \varepsilon_2^n) \leq \sigma A_2(R_2^h r_0^h \varepsilon_2^n, R_2^h r_0^h \varepsilon_2^n) \leq \sigma^2 A_1(\varepsilon_1^n, \varepsilon_1^n). \tag{4.9}$$

The substitution of v in (4.7) with ε_2^{n+1} gives

$$A_1(R_1^h r_0^h \varepsilon_2^n, \varepsilon_1^n) = -A_2(\varepsilon_2^n, \varepsilon_2^n). \tag{4.10}$$

(4.1) implies that

$$A_2(\varepsilon_2^n, \varepsilon_2^n) \geq \frac{1}{\tau} A_1(\varepsilon_1^n, \varepsilon_1^n). \tag{4.11}$$

If $0 < \theta_n < 1$, then (4.2) results from the substitution of (4.9), (4.10) and (4.11) into (4.8). Furthermore, $\kappa(\theta_n)$ has the following expression

$$\kappa(\theta_n) = \frac{1}{\tau}\Big(\theta_n^2(\sigma^2\tau + \tau + 2) - 2\theta_n(\tau+1) + \tau\Big). \tag{4.12}$$

An elementary calculation indicates that if and only if

$$0 < \theta_n < \theta^* = \min\Big(1, \frac{2(\tau+1)}{\sigma^2\tau + \tau + 2}\Big),$$

then $0 \leq \kappa(\theta_n) < 1$. Besides, the optimal relaxation factor

$$\theta^{opt} = \frac{\tau + 1}{\sigma^2\tau + \tau + 2}, \tag{4.13}$$

satisfies

$$\kappa(\theta^{opt}) = \frac{\sigma^2\tau^2 - 1}{\tau(\sigma^2\tau + \tau + 2)} = \min_{\theta \in (0,\theta^*)} \kappa(\theta). \tag{4.14}$$

By (4.4), (4.5) and (3.12), we have $r_0^h \varepsilon_2^{n+1} = -[S_2^h]^{-1} S_1^h \mu^n$, which combining (4.6) yields (4.3). □

Algorithm 4.2.[2] (Trace averaging method introduced by Glowinski)

Step 1. Let $g^0 \in \Re^m$ be any initial guess. Set n:=1. Let $\rho \in (0,1)$ be the given relaxation factor.

Step 2. For $k = 1,2$, solve in parallel $u_k^n \in V_h^k$, such that

$$\begin{cases} A_k(u_k^n, v) = (f, v)_k, \ \forall\, v \in V_h^{k,0} \\ r_0^h u_k^n = g^{n-1} \end{cases}$$

Step 3. For $k = 1,2$, solve in parallel $w_k^n \in V_h^k$, such that

$$A_k(w_k^n, v) = \frac{1}{2}\sum_{i=1}^{2}\Big(A_i(u_i^n, \rho_i r_0^h v) - (f, \rho_i r_0^h v)_i\Big), \ \forall\, v \in V_h^k$$

Step 4. Calculate $g^n = g^{n-1} - \rho(r_0^h w_1^n + r_0^h w_2^n)$, set n:=n+1 and then go to Step 2.

Now, lets investigate Algorithm 4.2. Suppose that u_h is the solution of (3.1) and u_1^n, u_2^n, g^n are generated by Algorithm 4.2. Then

$$\varepsilon_k^n = u_k^n - u_h^k, \ k = 1,2, \ \mu^n = g^n - r_0^h u_h = g^n - U_3$$

satisfy

$$\begin{cases} A_k(\varepsilon_k^n, v) = 0, \ \forall\, v \in V_h^{k,0} \\ r_0^h \varepsilon_k^n = \mu^{n-1} \end{cases} \tag{4.15}$$

$$A_k(w_k^n, v) = \frac{1}{2}\sum_{i=1}^{2} A_i(\varepsilon_i^n, \rho_i r_0^h v), \ \forall\, v \in V_h^k \tag{4.16}$$

$$\mu^{n+1} = \mu^n - \rho(r_0^h w_1^n + r_0^h w_2^n). \tag{4.17}$$

(4.15) and (4.16) give

$$A_k(w_k^n, v) = \frac{1}{2}\sum_{i=1}^{2} A_i(\varepsilon_i^n, R_i^h r_0^h v), \ \forall\, v \in V_h^k \tag{4.18}$$

It follows from (4.15), (4.17) and (4.18) that

$$\varepsilon_1^{n+1} = \varepsilon_1^n - \rho(w_1^n + R_1^h r_0^h w_2^n), \quad \varepsilon_2^{n+1} = \varepsilon_2^n - \rho(w_2^n + R_2^h r_0^h w_1^n). \tag{4.19}$$

By (4.15) and the substitution of v in (4.18) with ε_k^n, we get

$$\sum_{i=1}^{2} A_i(\varepsilon_i^n, \varepsilon_i^n) = \sum_{i=1}^{2} A_i(w_i^n, \varepsilon_i^n) \le \Big(\sum_{i=1}^{2} A_i(\varepsilon_i^n, \varepsilon_i^n)\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{2} A_i(w_i^n, w_i^n)\Big)^{\frac{1}{2}}.$$

Hence

$$\sum_{i=1}^{2} A_i(\varepsilon_i^n, \varepsilon_i^n) \le \sum_{i=1}^{2} A_i(w_i^n, w_i^n). \tag{4.20}$$

Taking $v = w_k^n$ in (4.18) yields

$$\sum_{i=1}^{2} A_i(w_i^n, w_i^n) = \frac{1}{2}A_1(\varepsilon_1^n, w_1^n + R_1^h r_0^h w_2^n) + \frac{1}{2}A_2(\varepsilon_2^n, w_2^n + R_2^h r_0^h w_1^n). \tag{4.21}$$

The substitution of v in (4.18) with $R_1^h r_0^h w_2^n$ for $k=1$ and $R_2^h r_0^h w_1^n$ for $k=2$ gives

$$A_1(w_1^n, R_1^h r_0^h w_2^n) = \frac{1}{2}\Big(A_1(\varepsilon_1^n, R_1^h r_0^h w_2^n) + A_2(\varepsilon_2^n, w_2^n)\Big),$$

$$A_2(w_2^n, R_2^h r_0^h w_1^n) = \frac{1}{2}\Big(A_2(\varepsilon_2^n, R_2^h r_0^h w_1^n) + A_1(\varepsilon_1^n, w_1^n)\Big).$$

Summing up the above two equalities and applying (4.21), we obtain

$$\sum_{i=1}^{2} A_i(w_i^n, w_i^n) = A_1(w_1^n, R_1^h r_0^h w_2^n) + A_2(w_2^n, R_2^h r_0^h w_1^n). \tag{4.22}$$

It follows from (4.19), (4.21), (4.22) and Theorem 3.1 that

$$\begin{aligned}\sum_{i=1}^{2} A_i(\varepsilon_i^{n+1}, \varepsilon_i^{n+1}) &= \sum_{i=1}^{2} A_i(\varepsilon_i^n, \varepsilon_i^n) - 4\rho \sum_{i=1}^{2} A_i(w_i^n, w_i^n) + \rho^2\Big\{3\sum_{i=1}^{2} A_i(w_i^n, w_i^n) \\ &\quad + A_1(R_1^h r_0^h w_2^n, R_1^h r_0^h w_2^n) + A_2(R_2^h r_0^h w_1^n, R_2^h r_0^h w_1^n)\Big\} \\ &\le \sum_{i=1}^{2} A_i(\varepsilon_i^n, \varepsilon_i^n) - \Big(4\rho - (3+\beta)\rho^2\Big)\sum_{i=1}^{2} A_i(w_i^n, w_i^n),\end{aligned}$$

where $\beta = \max(\sigma, \tau)$. Furthermore, by (4.20), we see that if $4\rho - (3+\beta)\rho^2 \ge 0$, then

$$\sum_{i=1}^{2} A_i(\varepsilon_i^{n+1}, \varepsilon_i^{n+1}) \le \eta(\rho)\sum_{i=1}^{2} A_i(\varepsilon_i^n, \varepsilon_i^n),$$

where $\eta(\rho) = 1 - \Big(4 - (3+\beta)\rho\Big)\rho$. Obviously, we have

$$0 \le \eta(\rho) < 1, \quad \forall\, \rho \in (0, \rho^*) = (0, \frac{4}{3+\beta})$$

$$\eta(\rho^{opt}) = \min_{\rho\in(0,\rho^*)} \eta(\rho) = 1 - \frac{4}{3+\beta}, \quad \text{where} \quad \rho^{opt} = \frac{2}{3+\beta}. \tag{4.23}$$

By now, we get the following theorem.

Theorem 4.2. 1) *Let u_h be the solution of (3.1). u_1^n, u_2^n are generated by Algorithm 4.2. Let ε^n be the iterative error, s.t. $\varepsilon^n = u_i^n - u_h^i$ on Ω_i. Then, there exists a constant $\rho^* \in (0,1)$, such that*

$$A(\varepsilon^{n+1}, \varepsilon^{n+1}) \le \eta(\rho) A(\varepsilon^n, \varepsilon^n), \tag{4.24}$$

and $\eta(\rho) < 1$, $\forall\ \rho \in (0, \rho^*)$. Besides, the optimal relaxation factor ρ^{opt} exists.

2) Algorithm 4.2 is equivalent to the following iterative method for (3.9)

$$g^{n+1} = g^n + \frac{1}{2}\rho\Big([S_1^h]^{-1} + [S_2^h]^{-1}\Big)(\tilde{b}_3 - S^h g^n). \tag{4.25}$$

Algorithm 4.3[21] (Parallel iterative method)

Step 1. Let $g^0 \in \Re^m$ be any initial guess. Set n:=0. Choose the relaxation factors $\theta_1, \theta_2 \in (0,1)$.

Step 2. For $k = 1, 2$, solve in parallel the Dirichlet subproblems on Ω_k

$$\begin{cases} u_k^{2n} \in V_h^k \\ A_k(u_k^{2n}, v) = (f, v)_k, \ \forall\ v \in V_h^{k,0} \\ r_0^h u_k^{2n} = g^n \end{cases}$$

$D_j u_i^{2n} = ((a_j(u_j^{2n}, \varphi_i^\Gamma))$

Step 3. Compute $d^n = \theta_1 D_1^h u_1^{2n} - (1-\theta_1) D_2^h u_2^{2n}$.

Step 4. For $k = 1, 2$, solve simultaneously the mixed subproblems on Ω_k

$$\begin{cases} u_k^{2n+1} \in V_h^k \\ A_k(u_k^{2n+1}, v) = (f, v)_k, \ \forall\ v \in V_h^{k,0} \\ A_k(u_k^{2n+1}, \phi_i^\Gamma) = (f, \phi_i^\Gamma)_k + (-1)^{k+1}\Big(d^n\Big)(i), \ i = 1, 2, \cdots, m \end{cases}$$

Step 5. Calculate $g^{n+1} = \theta_2 r_0^h u_1^{2n+1} + (1-\theta_2) r_0^h u_2^{2n+1}$, set n:=n+1, and go back to Step 2 for the next iteration.

Theorem 4.3. *1) Let ε^n be the error of Algorithm 4.3, where $\varepsilon^n = u_i^n - u_h^i$ on Ω_i. Then, if*

$$\theta_1 \in (\theta_1^*, 1) = (\frac{\tau^2\sigma - \sigma}{\tau^2\sigma + \sigma + 2}, 1), \quad \theta_2 \in (\theta_2^*, 1) = (\frac{\sigma^2\tau - \tau}{\sigma^2\tau + \tau + 2}, 1),$$

then

$$\kappa_1(\theta_1) = \frac{1}{\sigma}\Big\{(1-\theta_1)^2(\sigma\tau^2+\sigma+2) - 2(1-\theta_1)(\sigma+1)+\sigma\Big\} \in [0,1),$$

$$\kappa_2(\theta_2) = \frac{1}{\tau}\Big\{(1-\theta_2)^2(\tau\sigma^2+\tau+2) - 2(1-\theta_2)(\tau+1)+\tau\Big\} \in [0,1),$$

satisfy

$$A(\varepsilon^{n+2},\varepsilon^{n+2}) \le \kappa_1(\theta_1)\kappa_2(\theta_2)A(\varepsilon^n,\varepsilon^n).$$

2)

$$\theta_1^{opt} = \frac{\tau^2\sigma+1}{\tau^2\sigma+\sigma+2}, \quad \theta_2^{opt} = \frac{\sigma^2\tau+1}{\sigma^2\tau+\tau+2}, \tag{4.26}$$

satisfy

$$\begin{cases} \kappa_1(\theta_1^{opt}) = \dfrac{\sigma^2\tau^2-1}{\sigma(\tau^2\sigma+\sigma+2)} = \min\limits_{\theta\in(\theta_1^*,1)} \kappa_1(\theta) \\ \kappa_2(\theta_2^{opt}) = \dfrac{\sigma^2\tau^2-1}{\tau(\sigma^2\tau+\tau+2)} = \min\limits_{\theta\in(\theta_2^*,1)} \kappa_2(\theta) \end{cases} \tag{4.27}$$

3) Algorithm 4.3 is equivalent to the following iterative method for (3.9)

$$g^{n+1} = g^n + \Big\{\theta_2(1-\theta_1)[S_1^h]^{-1} + \theta_1(1-\theta_2)[S_2^h]^{-1}\Big\}(\tilde{b}_3 - S^h g^n). \tag{4.28}$$

As before, Theorem 4.3 can be proved by Theorem 3.1 and Corrollary 3.2. For details, refer to [10,13].

On the whole, the above iterative algorithms converge geometrically, independently of h. Besides, the optimal relaxation factors exist. In the special case that the equation (2.1) and the domain Ω are symmetric in respect of Γ, $\sigma = \tau = 1$, then $\kappa(\theta^{opt}) = 0$ in (4.14), $\eta(\rho^{opt}) = 0$ in (4.23), $\kappa_1(\theta_1^{opt}) = \kappa_2(\theta_2^{opt}) = 0$ in (4.27), which imply that only one iteration is needed to obtained the solution of (3.1). And, they are equivalent to the simple iterative method for the capacitance equation (3.9) which have been preconditioned, respectively, by

$$\frac{1}{\theta}S_2^h,\ \frac{2}{\rho}\Big([S_1^h]^{-1}+[S_2^h]^{-1}\Big)^{-1},\ \Big\{\theta_2(1-\theta_1)[S_1^h]^{-1}+\theta_1(1-\theta_2)[S_2^h]^{-1}\Big\}^{-1}. \tag{4.29}$$

Theorem 3.3 yields that all the preconditioners in (4.29) are equivalent to S^h and the relaxation factors can be selected so that the spectral radius of the iterative matrices of (4.3), (4.25) and (4.28) are less than one.

§5 Preconditioners Based on Domain Decomposition

Generally, the capacitance matrix S^h is dense and its condition number is $O(h^{-1})$ (cf.[6],(3.3),(3.14)), hence, to solve (3.9), the iterative methods are peferable to the direct methods. It is well–known that the simple iterative method converges much slowly than the conjugate gradient method (CG). Consequently, we may use one of the conditioners in (4.29) or any others implied in the iterative methods just like Algorithm 4.1–4.3 to accelerate the CG method for solving (3.9), i.e., to apply the preconditioned conjugate gradient method (PCG) to solving (3.9).

Let $S_0^h(\Gamma) \subset H_0^1(\Gamma)$ be the piecewise linear continuous finite element space with $\{\xi_j\}_{j=1}^m$ as the interpolation points. We note that the discrete operator l_0 defined on $S_0^h(\Gamma)$ by

$$\langle l_0 v, w\rangle = \int_\Gamma \frac{dv}{ds}\frac{dw}{ds}ds, \quad \forall\, w \in S_0^h(\Gamma)$$

is a finite–dimensional approximation to the Laplace operator $-\dfrac{d^2}{ds^2}$. Here, s is the arc length along Γ. Let $l_0^{\frac{1}{2}}$ be the positive square root of the symmetric, positive definite matrix l_0. It follows from [3] and Theorem 3.1 that there exists two constants c, C , independent of h, such that

$$c\lambda^T S^h\lambda \le \langle l_0^{\frac{1}{2}}\lambda_h, \lambda_h\rangle \le C\lambda^T S^h\lambda, \quad \forall\, \lambda \in \Re^m \tag{5.1}$$

By now, there has been some investigation on the $m \times m$ matrix $l_0^{\frac{1}{2}}$. For $j = 1, 2, \cdots, m$, denote

$$E_j = \sqrt{\frac{2}{m+1}}\Big(\sin\frac{j\pi}{m+1},\ \sin\frac{2j\pi}{m+1},\ \cdots, \sin\frac{mj\pi}{m+1}\Big)^T,\ E = (E_1, E_2, \cdots, E_m),$$

$$\xi_j = 4\sin^2\frac{j\pi}{2(m+1)},\ \alpha_j^D = 2\sqrt{\xi_j},\ \alpha_j^G = 2\sqrt{\xi_j + \xi_j^2/4},\ \alpha_j^J = \sqrt{\xi_j(6-\xi_j)/6},$$

$$D = E\ diag(\alpha_1^D, \alpha_2^D, \cdots, \alpha_m^D)\ E^T, \tag{5.2}$$

$$G = E\ diag(\alpha_1^G, \alpha_2^G, \cdots, \alpha_m^G)\ E^T, \tag{5.3}$$

$$J = E\ diag(\alpha_1^J, \alpha_2^J, \cdots, \alpha_m^J)\ E^T. \tag{5.4}$$

We point out that $l_0^{\frac{1}{2}}$ becomes J when $\{\xi_j\}_{j=1}^m$ are equally spaced on Γ (cf.[3]).

Theorem 5.1.[3,8,9] *Let S^h, D, G, J be defined by (3.10), (5.2), (5.3) and (5.4), respectively. If Ω_h is quasi–uniform, then D, G, J are spectrally equivalent to S^h.*

The above theorem indicates that even if S^h is the capacitance matrix in the nonconforming cases, D, G, J are still efficient preconditioners to apply PCG method to solving (3.9). In each iteration, $D^{-1}g, G^{-1}g, J^{-1}g$ can be obtained by FFT, while the number of iterations is independent of the size of the problems.

By Theorem 3.3, it is easy to see that

$$Q = \begin{bmatrix} K_{11} & 0 & K_{13} \\ 0 & K_{22} & K_{23} \\ K_{13}^T & K_{23}^T & K_{33}^{(1)} + K_{23}^T K_{22}^{-1} K_{23} \end{bmatrix} \tag{5.5}$$

is spectrally equivalent to the stiff matrix K in (3.8), thus Q can be used as a preconditioner to apply PCG method to solving (3.8). In each iteration, $Q^{-1}g$ can be calculated through first solving simultaneously the Dirichlet subproblems on Ω_1 and Ω_2 and then solving a Dirichlet–Neumann subproblem on Ω_1. Consequently, it is equivalent to the so–called substructure partitioned method[4]. In fact, Q is the matrix representation of the preconditioner developed in [4].

By now, we have surveyed all existing algorithms in two–subdomain nonoverlap cases.

§6 Numerical Experiments

Example 1. We use the Crouzeix–Raviart elements to discretize

$$\begin{cases} \Delta u = f, & \text{in } \Omega = (0,2) \times (0,1) \cup (0,1) \times [1,2) \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{6.1}$$

Ω is divided into $\Omega_1 = (0,1) \times (1,2)$, $\Omega_2 = (0,2) \times (0,1)$. We triangulate Ω to get the fine mesh Ω_h so that each element $e \in \Omega_h$ is an isosceles right triangle with h as its diameter. In Ω_1 and Ω_2, there are, respectively, 16, 32 elements, 28, 54 nodes for $h = 1/2$, 64,128 elements, 104, 204 nodes for $h = 1/4$, 256, 512 elements, 400, 792 nodes for $h = 1/8$. In the following tables, n is the number of iterations, and $n = 0$ represents the initial situation. ε^n is the error after n iterations.

$$\|\varepsilon^n\|_A = \sqrt{A(\varepsilon^n, \varepsilon^n)}, \qquad \rho_n = \sqrt[n]{\|\varepsilon^n\|_A / \|\varepsilon^0\|_A},$$
$$\|\varepsilon^n\|_\infty = \|\varepsilon^n\|_{L^\infty(\Omega)}, \qquad \delta_n = \sqrt[n]{\|\varepsilon^n\|_\infty / \|\varepsilon^0\|_\infty}.$$

We apply Algorithm 4.1 to solving (6.1), where a procedure is built up to generate a sequence of nonconforming discrete harmonic functions on Ω_1 and Ω_2

Table 1: ρ_n, δ_n vs. h for Example 1

n		2	4	6	8	10
ρ_n	$h=1/2$	0.0179	0.0187	0.0188	0.0189	0.0190
	$h=1/4$	0.0020	0.0119	0.0196	0.0243	0.0275
	$h=1/8$	0.0045	0.0205	0.0309	0.0370	0.0409
δ_n	$h=1/2$	0.1285	0.1345	0.1359	0.1365	0.1369
	$h=1/4$	0.0414	0.0985	0.1308	0.1481	0.1587
	$h=1/8$	0.0611	0.1285	0.1633	0.1815	0.1926

Table 2: The errors when $h = 1/8$ for Example 1

n	2	4	6	8	10
$\lVert\varepsilon^n\rVert_A$	$0.335 \cdot 10^1$	$0.646 \cdot 10^{-2}$	$0.212 \cdot 10^{-4}$	$0.708 \cdot 10^{-7}$	$0.242 \cdot 10^{-9}$
$\lVert\varepsilon^n\rVert_\infty$	$0.112 \cdot 10^1$	$0.391 \cdot 10^{-1}$	$0.214 \cdot 10^{-2}$	$0.119 \cdot 10^{-3}$	$0.669 \cdot 10^{-5}$

with the same nodal values on Γ. This allows us to compute , at each iteration, two constants σ_n, τ_n suggested by (3.2), which combining (4.13) leads to the sequence of approximate values θ_n of the optimal relaxation factor θ^{opt}. We point out that the evaluation of θ_n doesn't require the solution of any additional problem in our algorithm (for details, cf.[17]). The main experimental results are listed in Table 1 and Table 2, which coincide with our theoretical analysis.

Example 2. We use the Carey membrane elements to discretize

$$\begin{cases} -\triangle u + u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{6.2}$$

Here, Ω and its subdivision Ω_1, Ω_2, the fine mesh Ω_h are the same as those in Example 1. In Ω_1, Ω_2, there are, respectively, 29, 55 nodes for $h = 1/2$, 105, 205 nodes for $h = 1/4$, 401, 793 nodes for $h = 1/8$. As before, Algorithm 4.1 is applied to solving (6.2) adaptively, since the intermediate values of the iterations can be used to calculate the approximate value of the optimal relaxation factor θ^{opt}. The numerical results are listed in Table 3 and Table 4, which show an impressive reduction of the initial error after very few iterations.

Table 3: ρ_n, δ_n vs. h for Example 2

n		2	4	5	6	7
ρ_n	$h = 1/2$	0.0002	0.0021	0.0041	0.0019	0.0053
	$h = 1/4$	0.0001	0.0005	0.0007	0.0015	0.0036
	$h = 1/8$	0.0002	0.0016	0.0021	0.0028	0.0054
δ_n	$h = 1/2$	0.0035	0.0277	0.0324	0.0428	0.0441
	$h = 1/4$	0.0087	0.0209	0.0255	0.0395	0.0604
	$h = 1/8$	0.0091	0.0364	0.0442	0.0559	0.0763

Table 4: The errors when $h = 1/8$ for Example 2

n	2	4	5	6	7
$\|\varepsilon^n\|_A$	$0.760 \cdot 10^{-1}$	$0.164 \cdot 10^{-5}$	$0.780 \cdot 10^{-8}$	$0.687 \cdot 10^{-10}$	$0.897 \cdot 10^{-11}$
$\|\varepsilon^n\|_\infty$	$0.105 \cdot 10^{0}$	$0.555 \cdot 10^{-3}$	$0.441 \cdot 10^{-4}$	$0.628 \cdot 10^{-5}$	$0.226 \cdot 10^{-5}$

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