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Abstract

A simple domain decomposition algorithm is proposed to solve the algebraic system of equations arising from the discretization of symmetric elliptic problems via nonconforming finite elements which are only continuous at the midpoints of the elements' edges of the quasi–uniform mesh. Theoretical analysis is given and the results of numerical experiments are reported.

§1 Introduction

The idea of domain decomposition is not new, but it obtains much attention with the present development of parallel computers[1,2,15].

There has been much progress in the study of nonoverlap domain decomposition methods, or substructuring methods to solve the following elliptic problems via the conforming finite elements[1,2,17]

$$u \in H_0^1(\Omega): \ a(u,v) = (f,v), \ \forall\, v \in H_0^1(\Omega) \tag{1.1}$$

where $a(u,v) = \int_\Omega \left[\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x)uv \right]$, $(f,v) = \int_\Omega fv$, $f \in H^{-1}(\Omega)$, $\Omega \subset \Re^2$ is a bounded polygonal open domain , $a_0(x) \geq \alpha > 0$, $a_0(x)$, $a_{ij}(x)$, $i,j = 1,2$ are piecewise smooth and bounded functions in Ω, (a_{ij}) is a symmetric, uniformly positive definite matrix in Ω.

On the other hand, some recent developments indicate that better results can be obtained in the nonconforming discrete case, although the nonconforming elements lack the global continuity. Oswald [16] has constructed a hierachical basis multilevel method for two dimensional Crouzeix–Raviart elements with O(k) as the bound of the condition number of the hierarchical discretization, where k is the number of

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refinement levels. Gu, et al. [5–10] have developed a series of nonoverlap domain decomposition methods with nonconforming elements, which are as efficient as their counterparts in the conforming cases, and even easier in implementation.

The aim of the present paper is to extend the trace averaging domain decomposition algorithm[1] to solve (1.1), which is discretized by the nonconforming finite elements. Let $\Omega_h = \{e\}$ be a quasi–uniform mesh of Ω with h as its mesh parameter. Let V^h be the piecewise linear nonconforming Crouzeix–Raviart element space[4], or the piecewise quartic nonconforming rectangular element space[13]. The properties of V^h are well known. For example, every function in V^h is continuous at the midpoint of each edge of e, $\forall\ e \in \Omega_h$, which is called the edge midpoint for conciseness. And, the optimal error estimates can be obtained, if V^h is used for the velocity field and piecewise constant elements for the pressure[4,13]. The nonconforming finite element discrete problem for (1.1) is

$$u_h \in V_0^h : \ A(u_h, v_h) = (f, v_h), \quad \forall\ v_h \in V_0^h \tag{1.2}$$

where
$$V_0^h = \{v_h \in V^h : \ v_h(x) = 0, \ \forall \text{ edge midpoint } x \in \partial\Omega\},$$

$$A(u,v) = \sum_{e\subset\Omega} \int_e \left[\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv \right].$$

Suppose that $\{\Omega_i\}_{i=1}^N \triangleq \Omega_H$ is a nonoverlap subdivision of Ω, which is, in fact, a quasi–uniform mesh of Ω with H as its mesh parameter. Ω_i is either a quadrilater or a triangle, referred to as a subdomain. Besides, the common boundary point of more than two subdomains that lies inside Ω is called the internal cross point. Assume that Ω_h can be obtained by refining Ω_H, so that Ω_H, Ω_h form a two–level triangulation on Ω. Generally, $H \gg h$, therefore, Ω_H, Ω_h are called the coarse mesh and the fine mesh, respectively. If all the subdomains are colored by two colors, red and black, such that no pair of subdomains which have a common edge have the same color, then such a subdivision of Ω is called the red–black subdivision. We note that the red–black subdivision is of coarse not always possible. To solve (1.2), [7] introduced an iterative substructuring method for the red–black subdivisions, while the present approach is applicable for general subdivisions. In each iteration of our algorithm, the Dirichlet subproblems set on Ω_i are solved simultaneously, which is then followed by the parallel solving of the Neumann subproblems posed on Ω_i . For it is unnecessary and in fact impossible to calculate the values at the internal cross points in the nonconforming discrete cases, the internal cross points are not handled, which plus the simple way to interchange the information between the subdomains leads to the high parallelism of our algorithm. Another remarkable feature of the algorithm is that it is easy to implement. Based on the essential estimates[5,12], it is proved to be geometrically convergent. Virtually, it is equivalent to the simple iterative method applied to the preconditioned capacitance equation. Also, the preconditioner implied in the iteration is easily invertible and the condition number of the preconditioned capacitance matrix is no more than $O((1+\ln\frac{H}{h})max(1+H^{-2}, 1+\ln\frac{H}{h}))$.

This paper is organized as follows: Sect.2 describes the domain decomposition algorithm. Sect.3 gives the convergence analysis and matrix analysis. Finally, Sect.4 presents numerical experiments to show the efficiency of our algorithm.

We point out that, by the idea of [1] and the estimations[5,11,12], a domain decomposition algorithm is developed and analysed in [6] to solve (1.1) discretized by the Wilson elements[18], or the Carey membrane elements[3], etc., which are only continuous at the mesh nodes.

§2 Domain Decomposition Algorithm

Denote $\Gamma = \bigcup_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$. Let $\{\xi_j\}_{j=1}^{m}$ be the set of the edge midpoints on Γ (ordered in some way), $\{\phi_i^{\Gamma}\}$ be the set of the corresponding bases and m_i be the number of the edge midpoints on $\partial\Omega_i \backslash \partial\Omega$ (ordered in some way). The discrete trace operator $r_0^h : V_0^h \rightarrow \Re^m$ is defined as follows

$$\forall\, v \in V_0^h,\ r_0^h v \in \Re^m :\ \left(r_0^h v\right)(j) = v(\xi_j),\ j = 1, 2, \cdots, m.$$

Let $\beta_i = \{k : \Omega_k \in \Omega_H, \text{ meas}(\partial\Omega_k \cap \partial\Omega_i) > 0\}$. For $j \in \beta_i$, the open common edge of Ω_i and Ω_j is labeled Γ_{ij}, and for $v \in V_0^h$, $\overline{v}_{ij}$, the piecewise linear continuous function on Γ_{ij}, satisfies

$$\overline{v}_{ij}(\xi_k) = v(\xi_k),\ \forall\, \xi_k \in \Gamma_{ij},\ \ \overline{v}_{ij}(\nu) = 0,\ \forall \text{ endpoint } \nu \text{ of } \Gamma_{ij}.$$

For $i = 1, 2, \cdots, N$, define a discrete trace averaging operator from Γ to $\partial\Omega_i \backslash \partial\Omega$ to be the matrix $R_i \in \Re^{m_i \times m}$ so that

$$(R_i)_{lj} = \begin{cases} \dfrac{1}{2}, & \text{if the } l\text{th edge midpoint on } \partial\Omega_i \backslash \partial\Omega \text{ is } \xi_j \\ 0, & \text{else} \end{cases}$$

where $(R_i)_{lj}$ represents the (l,j)th element of R_i. Finally we introduce the following notations:

$$V_i^h = \{v \in V_0^h : \quad v(x) = 0, \ \forall \text{ interpolation point } x \in \overline{\Omega} \backslash \overline{\Omega}_i\},$$

$$V_{i,0}^h = \{v \in V_0^h : \quad v(x) = 0, \ \forall \text{ interpolation point } x \in \overline{\Omega} \backslash \Omega_i\},$$

$$A_i(u,v) = \sum_{e \subset \Omega_i} \int_e \left[\sum_{k,j=1}^{2} a_{kj} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_j} + a_0 uv \right], \quad (f,v)_i = \int_{\Omega_i} fv.$$

Here, an interpolation point $x \in \overline{\Omega}$ is related to the definition of V^h.

Now we describe the trace averaging domain decomposition algorithm (DDA) to solve the discrete problem (1.2).

DDA 2.1. Choose arbitrarily $\lambda^0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)^T \in \Re^m$. Give a relaxation factor $\rho \in (0,1)$. Set n:=0.

Step 1 For $i = 1, 2, \cdots, N$, solve in parallel

$$\begin{cases} u_i^n \in V_i^h \\ A_i(u_i^n, \theta) = (f, \theta)_i , & \forall \, \theta \in V_{i,0}^h \\ u_i^n(\xi_j) = \lambda_j^n, & \forall \, \xi_j \in \partial\Omega_i \backslash \partial\Omega \end{cases}$$

Step 2 For $l = 1, 2, \cdots, m$, calculate in parallel

$$d_l^n = \frac{1}{2}\left[A_i(u_i^n, \phi_l^\Gamma) - (f, \phi_l^\Gamma)_i\right] + \frac{1}{2}\left[A_j(u_j^n, \phi_l^\Gamma) - (f, \phi_l^\Gamma)_j\right], \quad \text{if } \xi_l \in \partial\Omega_i \cap \partial\Omega_j$$

Step 3 For $i = 1, 2, \cdots, N$, solve in parallel

$$\begin{cases} \delta_i^n \in V_i^h \\ A_i(\delta_i^n, \theta) = 0 , & \forall \, \theta \in V_{i,0}^h \\ A_i(\delta_i^n, \phi_l^\Gamma) = d_l^n, & \forall \, \xi_l \in \partial\Omega_i \backslash \partial\Omega \end{cases}$$

Step 4 For $l = 1, 2, \cdots, m$, calculate in parallel

$$\lambda_l^{n+1} = \lambda_l^n - \frac{1}{2}\rho\left(\delta_i^n(\xi_l) + \delta_j^n(\xi_l)\right), \quad \text{if } \xi_l \in \partial\Omega_i \cap \partial\Omega_j$$

Set n:=n+1 , return to Step 1 until some reasonable stopping criterion is satisfied.

§3 Theoretical Analysis

From now on, c and C will denote generic positive constants which are independent of h, H and the Ω_i.

Lemma 3.1.[5,12] *Let $v \in V_0^h$ be the discrete harmonic function on Ω_i, i.e. v satisfies*

$$A_i(v,\theta) = 0, \quad \forall\, \theta \in V_{i,0}^h$$

Then

$$A_i(v,v) \le c \sum_{j\in\beta_i} \|\overline{v}_{ij}\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})}.$$

Lemma 3.2.[5,12] *For $v \in V_i^h$, $j \in \beta_i$, we have*

$$\|\overline{v}_{ij}\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} \le c(1+\ln\frac{H}{h})\Big\{(1+\ln\frac{H}{h})|v|^2_{1,\Omega_i,h} + (1+H^{-2})\|v\|^2_{L^2(\Omega_i)}\Big\},$$

where $|v|_{1,\Omega_i,h} \triangleq \Big(\sum_{e\subset\Omega_i} |v|^2_{H^1(e)}\Big)^{\frac{1}{2}}$.

Theorem 3.3. *Let u_h be the solution of (1.2), $u^n \in V_0^h$ the approximate solution obtained by DDA 2.1, where $u^n = u_i^n$ on Ω_i . $\varepsilon^n = u^n - u_h$ is the error. If $0 < \rho < \frac{2}{\sigma}$, then*

$$A(\varepsilon^{n+1}, \varepsilon^{n+1}) \le k(\rho) A(\varepsilon^n, \varepsilon^n), \tag{3.1}$$

where $k(\rho) = 1 - 2\rho + \sigma\rho^2 < 1$, *and σ is a postive constant so that*

$$1 \le \sigma \le c(1+\ln\frac{H}{h})\max(1+H^{-2}, 1+\ln\frac{H}{h}). \tag{3.2}$$

Furthermore, $k(\rho_{opt}) = 1 - \frac{1}{\sigma} = \min\limits_{0<\rho<\frac{2}{\sigma}} k(\rho)$, *where* $\rho_{opt} = \frac{1}{\sigma}$.

Proof For $i = 1, 2, \cdots, N$, define the linear operator $\Pi_i : V_0^h \to V_i^h$ as follows

$$\forall\, v \in V_0^h,\ \Pi_i v \in V_i^h:\ \Big(\Pi_i v\Big)(x) = \begin{cases} 0, & \forall \text{ interpolation point } x \in \Omega_i \\ v(\xi_j), & \forall\, x = \xi_j \in \partial\Omega_i \backslash \partial\Omega \end{cases}$$

For the solution u_h of (1.2), let $u_h^i \in V_i^h$, which satisfies

$$u_h^i(x) = u_h(x),\ \forall \text{ interpolation point } x \in \overline{\Omega}_i.$$

Denote $\varepsilon_i^n = u_i^n - u_h^i,\ \mu^n = \lambda^n - r_0^h u_h$. It is easy to see that

$$\begin{cases} \varepsilon_i^n \in V_i^h \\ A_i(\varepsilon_i^n, \theta) = 0, \qquad \forall\, \theta \in V_{i,0}^h \\ \varepsilon_i^n(\xi_j) = (\mu^n)(j), \quad \forall\, \xi_j \in \partial\Omega_i \backslash \partial\Omega \end{cases} \tag{3.3}$$

$$\begin{cases} \delta_i^n \in V_i^h \\ A_i(\delta_i^n, \theta) = \frac{1}{2}\sum_{j=1}^{N} A_j(\varepsilon_j^n, \Pi_j\theta), \quad \forall\, \theta \in V_i^h \end{cases} \tag{3.4}$$

$$\mu^{n+1} = \mu^n - \frac{1}{2}\rho \sum_{j=1}^{N} r_0^h \delta_j^n. \tag{3.5}$$

Let

$$\psi_i^n = \frac{1}{\rho}(\varepsilon_i^n - \varepsilon_i^{n+1}). \tag{3.6}$$

It follows from (3.3), (3.5) and (3.6) that

$$\Pi_i \psi_i^n = \Pi_i\Big(\frac{1}{2}\sum_{j=1}^{N} \delta_j^n\Big). \tag{3.7}$$

For $j \in \beta_i$, let $\omega_{ij}^n \in V_i^h$ satisfy

$$\begin{cases} A_i(\omega_{ij}^n, \theta) = 0, & \forall\, \theta \in V_{i,0}^h \\ \omega_{ij}^n(\xi_k) = \delta_j^n(\xi_k), & \forall\, \xi_k \in \Gamma_{ij} \\ \omega_{ij}^n(x) = 0, & \forall \text{ edge midpoint } x \in \partial\Omega_i \backslash \Gamma_{ij} \end{cases}$$

Then, Lemma 3.1 and Lemma 3.2 lead to

$$\begin{aligned} A_i(\omega_{ij}^n, \omega_{ij}^n) &\le c\|\overline{(\omega_{ij}^n)}_{ij}\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} = c\|\overline{(\delta_j^n)}_{ij}\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} \\ &\le c(1 + \ln \tfrac{H}{h})\Big\{(1 + \ln \tfrac{H}{h})|\delta_j^n|^2_{1,\Omega_j,h} + (1 + H^{-2})\|\delta_j^n\|^2_{L^2(\Omega_j)}\Big\} \\ &\le \sigma A_j(\delta_j^n, \delta_j^n), \end{aligned} \tag{3.8}$$

where σ is a constant defined by (3.2).

On the other hand, by (3.6), (3.3) and (3.5), we obtain

$$\psi_i^n = \frac{1}{2}\delta_i^n + \frac{1}{2}\sum_{j\in\beta_i} \omega_{ij}^n. \tag{3.9}$$

Furthermore, it follows from the triangle inequality and (3.8) that

$$A_i(\psi_i^n,\psi_i^n) \le c\Big\{A_i(\delta_i^n,\delta_i^n) + \sum_{j\in\beta_i} A_i(\omega_{ij}^n,\omega_{ij}^n)\Big\} \le \sigma\Big\{A_i(\delta_i^n,\delta_i^n) + \sum_{j\in\beta_i} A_j(\delta_j^n,\delta_j^n)\Big\}.$$

$$\sum_{i=1}^N A_i(\psi_i^n,\psi_i^n) \le \sigma \sum_{i=1}^N A_i(\delta_i^n,\delta_i^n). \tag{3.10}$$

By taking $\theta=\varepsilon_i^n$ in (3.4), summing up with i, and applying (3.3) , we have

$$\begin{aligned}\sum_{i=1}^N A_i(\delta_i^n,\varepsilon_i^n) &= \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N A_j(\varepsilon_j^n,\Pi_j\varepsilon_i^n)\\ &= \sum_{j=1}^N A_j\Big(\varepsilon_j^n,\Pi_j(\frac{1}{2}\sum_{i=1}^n \varepsilon_i^n)\Big) = \sum_{j=1}^n A_j(\varepsilon_j^n,\varepsilon_j^n),\end{aligned}$$

$$\sum_{i=1}^N A_i(\varepsilon_i^n,\varepsilon_i^n) = \sum_{i=1}^N A_i(\delta_i^n,\varepsilon_i^n) \le \Big(\sum_{i=1}^N A_i(\varepsilon_i^n,\varepsilon_i^n)\Big)^{\frac{1}{2}}\Big(\sum_{i=1}^N A_i(\delta_i^n,\delta_i^n)\Big)^{\frac{1}{2}},$$

which implies

$$\sum_{i=1}^N A_i(\varepsilon_i^n,\varepsilon_i^n) \le \sum_{i=1}^N A_i(\delta_i^n,\delta_i^n). \tag{3.11}$$

It follows from the substitution of θ in (3.3) with $\psi_i^n - \Pi_i\psi_i^n$ that

$$\sum_{i=1}^N A_i(\varepsilon_i^n,\psi_i^n) = \sum_{i=1}^N A_i(\varepsilon_i^n,\Pi_i\psi_i^n).$$

Furthermore, by taking $\theta=\delta_i^n$ in (3.4) and using (3.7) and (3.3), we get

$$\begin{aligned}\sum_{i=1}^N A_i(\delta_i^n,\delta_i^n) &= \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N A_j(\varepsilon_j^n,\Pi_j\delta_i^n)\\ &= \sum_{j=1}^N A_j\Big(\varepsilon_j^n,\Pi_j(\frac{1}{2}\sum_{i=1}^N \delta_i^n)\Big) = \sum_{j=1}^N A_j(\varepsilon_j^n,\psi_j^n).\end{aligned} \tag{3.12}$$

If $0 < \rho < \frac{2}{\sigma}$, then (3.12), 3.10) and (3.11) lead to

$$\begin{aligned}\sum_{i=1}^{N} A_i(\varepsilon_i^{n+1}, \varepsilon_i^{n+1}) &= \sum_{i=1}^{N} A_i(\varepsilon_i^n - \rho\psi_i^n, \varepsilon_i^n - \rho\psi_i^n) \\ &= \sum_{i=1}^{N} A_i(\varepsilon_i^n, \varepsilon_i^n) - 2\rho \sum_{i=1}^{N} A_i(\delta_i^n, \delta_i^n) + \rho^2 \sum_{i=1}^{N} A_i(\psi_i^n, \psi_i^n) \\ &\le \sum_{i=1}^{N} A_i(\varepsilon_i^n, \varepsilon_i^n) - (2\rho - \sigma\rho^2) \sum_{i=1}^{N} A_i(\delta_i^n, \delta_i^n) \\ &\le (1 - 2\rho + \sigma\rho^2) \sum_{i=1}^{N} A_i(\varepsilon_i^n, \varepsilon_i^n),\end{aligned}$$

which completes the proof of (3.1), and hence the theorem. □

Theorem 3.3 indicates that if $0 < \rho < \frac{2}{\sigma}$, then DDA 2.1 converges geometrically with the convergence factor $k(\rho) = 1 - 2\rho + \sigma\rho^2$, dependent on H, h. Now, we analyse DDA 2.1 from algebraic viewpoint. Let $\{\phi_i\}$ be the set of the bases of V_0^h. ϕ_j^I represents basis of V_0^h in $\bigcup_{k=1}^{N} \Omega_k$. Suppose $u_h = \sum_i u_i^I \phi_i^I + \sum_{j=1}^{m} u_j^\Gamma \phi_j^\Gamma$ is the solution of (1.2). Then $u_j^\Gamma = u_h(\xi_j)$, and (1.2) can be written as

$$KU = \begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{I\Gamma}^T & K_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_I \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_I \\ F_\Gamma \end{bmatrix}, \qquad (3.13)$$

where $\quad U_I = (u_i^I),\ \ U_\Gamma = (u_j^\Gamma),\ \ F_I = \Big((f, \phi_i^I)\Big),\ \ F_\Gamma = \Big((f, \phi_j^\Gamma)\Big),$

$$K = \Big(A(\phi_i, \phi_j)\Big),\ \ K_{II} = \Big(A(\phi_i^I, \phi_j^I)\Big),\ \ K_{I\Gamma} = \Big(A(\phi_i^I, \phi_j^\Gamma)\Big),\ \ K_{\Gamma\Gamma} = \Big(A(\phi_i^\Gamma, \phi_j^\Gamma)\Big).$$

With the block Gaussian elimination, we obtain

$$SU_\Gamma = \tilde{F}_\Gamma\ , \qquad (3.14)$$

where $\tilde{F}_\Gamma = F_\Gamma - K_{I\Gamma}^T K_{II}^{-1} F_I$, $S = K_{\Gamma\Gamma} - K_{I\Gamma}^T K_{II}^{-1} K_{I\Gamma}$ is the Schur complement or the capacitance matrix of Ω with regard to Γ. In the same way ,we can define the capacitance matrix of Ω_i with regard to $\partial\Omega_i \backslash \partial\Omega$, which is labeled S_i. It is easy to see that DDA 2.1 is essentially equivalent to the following iterative method for (3.14):

$$\lambda^{n+1} = \lambda^n - \rho \sum_{i=1}^{N} R_i^T S_i^{-1} R_i S \lambda^n + \rho \sum_{i=1}^{N} R_i^T S_i^{-1} R_i \tilde{F}_\Gamma. \qquad (3.15)$$

Obviously, (3.15) can be viewed as the simple iterative method for (3.14) preconditioned by

$$Q = \Big(\rho \sum_{i=1}^{N} R_i^T S_i^{-1} R_i\Big)^{-1}. \tag{3.16}$$

Note that, Q is an efficient preconditioner of the capacitance matrix S, because it can be inversed easily in parallel and the condition number of $Q^{-1}S$ is bounded by $O\big((1+\ln\frac{H}{h})\max(1+H^{-2}, 1+\ln\frac{H}{h})\big)$, which is implied by the following theorem.

Theorem 3.4. *Let S, Q be those in (3.14) and (3.16), then* $\forall\ \lambda \in \Re^m\backslash\{0\}$

$$\rho \le \frac{(SQ^{-1}S\lambda, \lambda)}{(S\lambda, \lambda)} \le c\rho(1+\ln\frac{H}{h})\max(1+H^{-2}, 1+\ln\frac{H}{h}). \tag{3.17}$$

Proof Let ε_i, δ_i be generated by DDA 2.1 with $f = 0$ and $\lambda^n = \lambda$. A more iteration yields $\bar{\varepsilon}_i, \bar{\delta}_i$. Denote $\psi_i = \dfrac{1}{\rho}(\varepsilon_i - \bar{\varepsilon}_i)$. It is easy to see that

$$(S\lambda, \lambda) = \sum_{i=1}^{N} A_i(\varepsilon_i, \varepsilon_i), \quad (SQ^{-1}S\lambda, \lambda) = \rho\sum_{i=1}^{N} A_i(\varepsilon_i, \psi_i).$$

Thus, by (3.12) and (3.11), we get the left hand side of (3.17).

On the other hand, it follows from (3.12) and (3.10) that

$$\begin{aligned}\sum_{i=1}^{N} A_i(\delta_i, \delta_i) = \sum_{i=1}^{N} A_i(\varepsilon_i, \psi_i) &\le \Big(\sum_{i=1}^{N} A_i(\varepsilon_i, \varepsilon_i)\Big)^{\frac{1}{2}} \Big(\sum_{i=1}^{N} A_i(\psi_i, \psi_i)\Big)^{\frac{1}{2}} \\ &\le \Big(\sum_{i=1}^{N} A_i(\varepsilon_i, \varepsilon_i)\Big)^{\frac{1}{2}} \Big(\sigma\sum_{i=1}^{N} A_i(\delta_i, \delta_i)\Big)^{\frac{1}{2}},\end{aligned}$$

which implies the right hand side of (3.17) holds, and hence the theorem. □

Remark 3.5. It follows from the properties of Q defined by (3.16) that

$$\begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{I\Gamma}^T & Q + K_{I\Gamma}^T K_{II}^{-1} K_{I\Gamma} \end{bmatrix}$$

is an efficient preconditioner of the stiff matrix in (3.13).

§4 Numerical Experiments

We consider the following model problem on the unit square Ω :

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{4.1}$$

which is discretized by the Crouzeix–Raviart nonconforming elements[4].

Example 1. Ω is divided into three nonoverlap subdomains:

$$\Omega_1 = (0, 0.5) \times (0, 1), \quad \Omega_2 = (0.5, 1) \times (0.5, 1), \quad \Omega_3 = (0.5, 1) \times (0, 0.5),$$

which share an internal cross point , but do not satisfy the red–black subdivision. We triangulate Ω to get the fine mesh Ω_h so that each element $e \in \Omega_h$ is an isosceles right triangle with h as its diameter. When $h = 1/4$, there are 32, 16, and 16 elements, 54, 28, and 28 nodes in Ω_1, Ω_2, Ω_3, respectively. When $h = 1/8$, there are 128, 64, and 64 elements, 204, 104, and 104 nodes in Ω_1, Ω_2, Ω_3, respectively. In Table 1, n represents the number of iterations, ε^n the error function as that in Theorem 3.1, ρ the relaxation factor, and $\delta_n = \sqrt[n-1]{E_n/E_1}$ the average reduction factor, where $E_n = A(\varepsilon^n, \varepsilon^n)$.

Table 1 illustrates that DDA 2.1 converges faster when $h = 1/4$ than $h = 1/8$, which is in accordance with our theoretical analysis. We also observe that the optimal relaxation factor ρ_{opt} depends on h.

Example 2. We subdivide Ω into four subdomains:

$$\Omega_1 = (0, 0.75) \times (0, 0.75), \quad \Omega_2 = (0, 0.75) \times (0.75, 1),$$

$$\Omega_3 = (0.75, 1) \times (0.75, 1), \quad \Omega_4 = (0.75, 1) \times (0, 0.75),$$

which share an internal cross point and satisfy the red–black subdivision. The fine mesh Ω_h is the same as that in Example 1. In $\Omega_i, i = 1, 2, 3, 4$, there are, respectively, 36, 12, 4 and 12 elements, 60, 22, 8, and 22 nodes for $h = 1/4$, 144, 48, 16, and 48 elements, 228, 80, 28, and 80 nodes for $h = 1/8$. The Numerical results are listed in Table 2, from which we see that for the red–black subdivision, DDA 2.1 can perform adaptively (for details, cf.[5]) to avoid the blindness of choosing ρ, therefore, it converges much faster.

Table 1: δ_n vs. ρ, h for Example 1

n		2	6	10	20	30
$h = 1/4$	$\rho = 0.45$	0.320	0.320	0.289	0.313	0.328
	$\rho = 0.40$	0.339	0.294	0.362	0.381	0.375
	$\rho = 0.35$	0.398	0.296	0.413	0.423	0.425
	$\rho = 0.30$	0.395	0.395	0.399	0.490	0.494
	$\rho = 0.20$	0.507	0.491	0.545	0.598	0.623
$h = 1/8$	$\rho = 0.45$	0.566	0.575	0.689	0.790	0.824
	$\rho = 0.40$	0.479	0.369	0.340	0.387	0.396
	$\rho = 0.35$	0.425	0.419	0.426	0.437	0.447
	$\rho = 0.30$	0.405	0.499	0.483	0.500	0.508
	$\rho = 0.20$	0.568	0.603	0.639	0.645	0.647

Table 2: Iterative Convergence for Example 2

n		5	10	20	30
$h = 1/4$	E_n	$0.14 \cdot 10^5$	$0.15 \cdot 10^2$	$0.16 \cdot 10^{-4}$	$0.17 \cdot 10^{-10}$
	δ_n	0.253	0.253	0.253	0.253
	ρ_n	0.495	0.495	0.496	0.496
$h = 1/8$	E_n	$0.18 \cdot 10^5$	$0.20 \cdot 10^2$	$0.23 \cdot 10^{-4}$	$0.27 \cdot 10^{-10}$
	δ_n	0.254	0.254	0.255	0.255
	ρ_n	0.494	0.495	0.495	0.495

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