University of Colorado at Denver

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February 1995

UCD/CCM Report No. 44

CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

Preconditioners for Nonconforming Element Discrete Problems. I‡

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Abstract

We consider the problem of solving the algebraic system of equations arising from the discretization of self–adjoint elliptic problems via the nonconforming finite elements, which are continuous at the quasi–uniform mesh nodes. The condition number of the algebraic system is proved to be $O(h^{-2})$, where h is the mesh parameter. By substructuring, also known as nonoverlap domain decomposition, we construct a series of preconditioners. The resulting preconditioned algorithms are well suited to emerging parallel computing architectures. A basic theory for the analysis of the condition number of the preconditioned system, which determines the iterative convergence rate of the algorithm, is presented. The condition number of the preconditioned system is bounded by $O\Big((1+\ln\frac{H}{h})^2\Big)$ or $O\Big((1+\ln\frac{H}{h})^3\Big)$, which is the same as that in the conforming finite element discrete cases. Here, H is the maximum diameter of the subdomains.

§1 Introduction

It is no doubt that domain decomposition methods have become a new focus of computational mathematics due to their potential in parallel computing environments[1,6,13]. Nonoverlap domain decomposition methods (substructuring methods), with a long history from the structural analysis community[14], are methods where the global domain is partitioned into disjoint smaller subdomains. More recently, the construction of preconditioners by substructuring has been investigated for the solving of the following elliptic problems via the conforming finite element methods[2,15]

$$u \in H_0^1(\Omega): \ a(u,v) = (f,v), \ \forall\, v \in H_0^1(\Omega) \tag{1.1}$$

‡ This work was supported by the National Natural Science Foundation of China under Grant No. 19171050 and the Tsinghua University Fund for Science.

where $a(u,v) = \int_\Omega \Big[\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\Big]$, $(f,v) = \int_\Omega fv$, $f \in H^{-1}(\Omega)$, $\Omega \subset \Re^2$ is a bounded polygonal open domain, $a_{ij}(x)$, $i,j = 1,2$ are piecewise smooth and bounded functions in Ω, (a_{ij}) is a symmetric, uniformly positive definite matrix in Ω.

The aim of the present paper is to construct preconditioners for the discretization of (1.1) via the nonconforming finite elements. Let $\Omega_h = \{e\}$ be a quasi–uniform mesh of Ω with h as its mesh parameter. Let V^h be the Wilson element space[4] or the Carey membrane element space[3], etc. Every function in V^h is continuous at the vertices of each element e, $\forall\, e \in \Omega_h$. The nonconforming finite element discrete problem for (1.1) is

$$u_h \in V_0^h : \ A(u_h, v) = (f,v), \quad \forall\, v \in V_0^h \tag{1.2}$$

where $$V_0^h = \{v \in V^h : \ v(x) = 0, \ \forall \text{ interpolation point } x \in \partial\Omega\},$$

$$A(u,v) = \sum_{e\subset\Omega} \int_e \Big[\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\Big].$$

Suppose that $\{\Omega_k\}_{k=1}^N \triangleq \Omega_H$ is a nonoverlap subdivision of Ω, which is, in fact, a quasi–uniform mesh of Ω with H as its mesh parameter. Ω_i is either a quadrilater or a triangle, referred to as the subdomain. Besides, the common boundary point of more than two subdomains that lies inside Ω is called the internal cross point. Assume that Ω_h can be obtained by refining Ω_H, so that Ω_H, Ω_h form a two–level triangulation[15] on Ω. Generally, $H \gg h$, therefore, Ω_H, Ω_h are called the coarse mesh and the fine mesh respectively. If all the subdomains are colored by two colors, red and black, such that no pair of subdomains which have a common edge have the same color, then such a subdivision of Ω is called the red–black subdivision. We note that the red–black subdivision is of coarse not always possible. In the conforming discrete cases, Bramble et al.[2], Widlund [15] have developed many preconditioners which are easily inversed in parallel and can decrease the condition number of the discrete system from $O(h^{-2})$ to $O\big((1 + \ln \frac{H}{h})^k\big)$, $k = 2,3$. When handling the nonconforming discrete problem (1.2), we are faced with two difficulties: one is how to cope with the internal cross points and the other is the lack of global continuity of functions in V^h.

To overcome the former, a coarse mesh problem must be solved, which is employed the idea of Bramble, et al.[2], in order to separate the internal cross points from other mesh nodes. To overcome the latter, some essential estimates must be established. For this purpose, the conforming interpolation operator I_h is introduced to act as a bridge between the nonconforming element space V^h and the conforming element space where many fundamental inequalities have already been constructed. Therefore, for the nonconforming element space V^h, the Poincaré inequalities, the maximum norm estimate, etc., holds still, which together with the extension theorem[6,9,10] for nonconforming elements form a basic theory for the analysis of the condition number of the preconditioned system (which determines the iterative convergence rate of the algorithm). Our preconditioners are constructed for (1.2) in the red–black subdivisional case and in the general case. The preconditioned algorithms are high parallel and the condition number of the preconditioned system is either $O\Big((1+\ln\frac{H}{h})^2\Big)$ or $O\Big((1+\ln\frac{H}{h})^3\Big)$. Thus, they are very efficient.

We point out that if (1.1) is discretized by the Crouzeix–Raviart triangular elements[5], the quartic rectangular elements[11], etc., which are continuous at the edge midpoints of the elements, plenty of preconditioners are developed in [8].

In the remainder of the present paper, the notations and lemmas are given in Sect.2 and Sect.3, respectively. A series of preconditioners are constructed and analysed in Sect.4.

§2 Notations

Let $\{\xi_j\}_{j=1}^m$ be the set of the vertices of Ω_k, $k=1,2,\cdots,N$ (ordered in some way). The conforming interpolation operator $I_h: V^h \to C(\Omega)$ is defined by

$$\forall\, v\in V^h,\ I_h v\in C(\Omega) \text{ s.t.}$$

$$(I_h v)(x) = v(x),\ \forall \text{ vertice } x \text{ of } e,\ I_h v|_e \text{ is linear (bilinear)},\ \forall\, e\in\Omega_h.$$

Let $\Gamma_{ij}\not\subset\partial\Omega$ be the open edge with ξ_i,ξ_j as its endpoints which is a boundary edge of some subdomain Ω_k. Let $\beta_k=\{(i,j):\ \Gamma_{ij}\subset\partial\Omega_k,\ i,j \text{ are positive integers}\}$. And, we introduce the following notations:

$$V_k^h=\{v\in V_0^h:\ v(x)=0,\ \forall \text{ interpolation point } x\in\Omega\backslash\overline{\Omega}_k\},$$

$$V_{k,0}^h=\{v\in V_0^h:\ v(x)=0,\ \forall \text{ interpolation point } x\in\Omega\backslash\Omega_k\},$$

$$A_k(u,v)=\sum_{e\subset\Omega_k}\int_e\Big[\sum_{i,j=1}^2 a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\Big].$$

If $v \in V^h$ satisfies $A_k(v,w) = 0, \ \forall\ w \in V^h_{k,0}$, then v is called a discrete harmonic function on Ω_k. For each $v \in V^h$, $\overline{v}_{ij}$ denotes the piecewise linear continuous function on $\bigcup_{k=1}^{N} \partial\Omega_k$, which satisfies

$$\overline{v}_{ij}(x) = \begin{cases} v(x), & \forall \text{ interpolation point } x \in \overline{\Gamma}_{ij} \\ 0, & \forall \text{ interpolation point } x \in \bigcup_{k=1}^{N} \partial\Omega_k \backslash \overline{\Gamma}_{ij} \end{cases}$$

And let $\hat{v}_{ij} \in V^h$ be the discrete harmonic function on $\Omega_k,\ k = 1, 2, \cdots, N$, s.t.

$$\hat{v}_{ij}(x) = \overline{v}_{ij}(x), \ \forall \text{ interpolation point } x \in \bigcup_{k=1}^{N} \partial\Omega_k.$$

Besides, v_{ij} denotes the vector of the values of v at the interpolation points on Γ_{ij}. Let $I_H : V^h \to C(\Omega)$ be the coarse mesh interpolation operator, s.t.

$$\forall\ v \in V^h,\ I_H v \in C(\Omega) : \begin{cases} (I_H v)(\xi_j) = v(\xi_j),\ j = 1, 2, \cdots, m \\ I_H v|_{\Omega_k} \text{ is linear (bilinear)},\ k = 1, 2, \cdots, N \end{cases}$$

Let $S_0^h(\Gamma_{ij}) \subset H_0^1(\Gamma_{ij})$ be the piecewise linear continuous finite element space. We note that the operator $l_{\Gamma_{ij}}$ defined on $S_0^h(\Gamma_{ij})$ by

$$\langle l_{\Gamma_{ij}} v, w \rangle_{\Gamma_{ij}} = \int_{\Gamma_{ij}} \frac{dv}{ds} \frac{dw}{ds} ds,\ \forall\ w \in S_0^h(\Gamma_{ij})$$

is a finite–dimensional approximation to the Laplace operator $-\dfrac{d^2}{ds^2}$. Here, s is the arc length along Γ_{ij}. Let $l^{\frac{1}{2}}_{\Gamma_{ij}}$ be the positive square root of the symmetric, positive definite matrix $l_{\Gamma_{ij}}$.

§3 Lemmas

From now on, c and C, with or without subscript, will denote generic positive constants independent of h, H.

Lemma 3.1. *For the quasi-uniform mesh $\Omega_h = \{e\}$, we have*

$$\max_{e \subset \Omega_k} \|v - I_h v\|_{L^\infty(e)} \le c|v|_{1,\Omega_k,h},\ \forall\ v \in V^h \tag{3.1}$$

$$\|v - I_h v\|_{L^2(\Omega_k)} \le ch|v|_{1,\Omega_k,h} \le c\|v\|_{L^2(\Omega_k)}, \ \forall\, v \in V^h \tag{3.2}$$

$$|v - I_h v|_{1,\Omega_k,h} \le c|v|_{1,\Omega_k,h}, \ \forall\, v \in V^h \tag{3.3}$$

where $|v|_{1,\Omega_k,h} \triangleq \Big(\sum_{e\subset\Omega_k} |v|^2_{H^1(e)}\Big)^{\frac{1}{2}}$.

The proof of Lemma 3.1 is trivial, which has been given in [6,10] via the inverse inequalities and the interpolation error estimates[4]. And based on Lemma 3.1, Lemma 3.3[2], the Poincaré inequality in $H_0^1(\Omega)$, we can easily show the following two lemmas.

Lemma 3.2.[6,10] *(Maximum norm estimate in the nonconforming space V^h)*

$$\max_{e\subset\Omega_k} \|v\|^2_{L^\infty(e)} \le c\Big\{H^{-2}\|v\|^2_{L^2(\Omega_k)} + (1 + \ln\frac{H}{h})|v|^2_{1,\Omega_k,h}\Big\}, \ \forall\, v \in V^h$$

Lemma 3.3[6,10] *(Poincaré inequalities in the nonconforming space V^h)*

1) $\|v\|^2_{L^2(\Omega_k)} \le c\Big\{H^2|v|^2_{1,\Omega_k,h} + H^{-2}|\int_{\Omega_k} v|^2\Big\}, \ \forall\, v \in V^h$

2) $H^{-2}\|v\|^2_{L^2(\Omega_k)} \le c|v|^2_{1,\Omega_k,h}, \ \forall\, v \in V^h_{k,0}$

3) if $v \in V^h$, $v(x) = 0$, $\forall$ *interpolation point* $x \in \Gamma$, *where* $\Gamma \subset \partial\Omega_k$ *has at least two interpolation points, then*

$$H^{-2}\|v\|^2_{L^2(\Omega_k)} \le c|v|^2_{1,\Omega_k,h}.$$

Lemma 3.4. *1) If* $v \in V^h$ *is a discrete harmonic function on* Ω_k, *then*

$$A_k(v,v) \le c\|I_h v\|^2_{\frac{1}{2},\partial\Omega_k}.$$

2) If $v \in V^h$ *satisfies* $\int_{\Omega_k} v = 0$ *or* $v(x) = 0$, $\forall$ *interpolation point* $x \in \Gamma$, *where* $\Gamma \subset \partial\Omega_k$ *has at least two interpolation points, then*

$$A_k(v,v) \ge c\|I_h v\|^2_{\frac{1}{2},\partial\Omega_k}.$$

Proof. 1) has been shown by Theorem 3[9] or Theorem 3.3[10]. Thus we need only prove 2). It follows from Lemma 3.3 1) or Lemma 3.3 3) that

$$\|v\|^2_{L^2(\Omega_k)} \le cH^2|v|^2_{1,\Omega_k,h}.$$

Therefore, by (3.2), (3.3) and the trace theorem, we obtain

$$\begin{aligned}\|I_h v\|^2_{\frac{1}{2},\partial\Omega_k} &\le c\|I_h v\|^2_{H^1(\Omega_k)} = c\Big\{\|I_h v\|^2_{L^2(\Omega_k)} + |I_h v|^2_{1,\Omega_k,h}\Big\} \\ &\le c\Big\{\|v\|^2_{L^2(\Omega_k)} + |v|^2_{1,\Omega_k,h}\Big\} \le c|v|^2_{1,\Omega_k,h} \le cA_k(v,v).\end{aligned}$$

Here, we assume that $H \leq c_0$, where c_0 is some positive constant, e.g., $c_0 = 1$, hence c is independent of H. By now, Lemma 3.4 follows. □

Lemma 3.5. *Let $v \in V^h$ be a discrete harmonic function on Ω_k.*

1) If $v(x) = 0$, $\forall\ x \in \{\xi_j\} \bigcap \partial\Omega_k$, then

$$A_k(v,v) \leq c \sum_{(i,j)\in\beta_k} \langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{v}_{ij}, \overline{v}_{ij} \rangle_{\Gamma_{ij}}.$$

2) If v is linear on Γ_{ij}, $\forall\ (i,j) \in \beta_k$, then

$$A_k(v,v) \leq c \sum_{(i,j)\in\beta_k} \Big(v(\xi_i) - v(\xi_j)\Big)^2.$$

Proof. Let's show 1) first. Note that $I_h v = \sum\limits_{(i,j)\in\beta_k} \overline{v}_{ij}$ on $\partial\Omega_k$. Therefore, by Lemma 3.4 1) and the triangle inequality, we have

$$\begin{aligned} A_k(v,v) \leq c\|I_h v\|^2_{\frac{1}{2},\partial\Omega_k} &\leq c \sum_{(i,j)\in\beta_k} \|\overline{v}_{ij}\|^2_{\frac{1}{2},\partial\Omega_k} \\ &\leq c \sum_{(i,j)\in\beta_k} \|\overline{v}_{ij}\|^2_{H^{\frac{1}{2}}_{00}(\Gamma_{ij})} \leq c \sum_{(i,j)\in\beta_k} \langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{v}_{ij}, \overline{v}_{ij} \rangle_{\Gamma_{ij}}, \end{aligned}$$

where the last two inequalities are because of the equivalence[2] of $\|w\|^2_{\frac{1}{2},\partial\Omega_k}$, $\|w\|^2_{H^{\frac{1}{2}}_{00}(\Gamma_{ij})}$ and $\langle l^{\frac{1}{2}}_{\Gamma_{ij}} w_{ij}, w_{ij} \rangle_{\Gamma_{ij}}$ for $w \in C(\partial\Omega_k)$ with $w = 0$ on $\partial\Omega_k \backslash \Gamma_{ij}$.

Next we prove 2). Let $\tilde{v}$ be the linear (bilinear) function on Ω_k, s.t. $\tilde{v} = v$ on $\partial\Omega_k$. Denote $\beta = \dfrac{1}{\text{meas}(\Omega_k)} \displaystyle\int_{\Omega_k} \tilde{v}$. $v - \beta$ is also a discrete harmonic function on Ω_k. Hence, it follows from Lemma 3.4 1) that

$$A_k(v,v) = A_k(v-\beta, v-\beta) \leq c\|I_h(v-\beta)\|^2_{\frac{1}{2},\partial\Omega_k} = c\|v-\beta\|^2_{\frac{1}{2},\partial\Omega_k}.$$

By the trace theorem and the Poincaré inequality in $H^1(\Omega_k)$, we obtain

$$\|v-\beta\|^2_{\frac{1}{2},\partial\Omega_k} = \|\tilde{v}-\beta\|^2_{\frac{1}{2},\partial\Omega_k} \leq c\|\tilde{v}-\beta\|^2_{H^1(\Omega_k)} \leq c|\tilde{v}-\beta|^2_{H^1(\Omega_k)} = c|\tilde{v}|^2_{H^1(\Omega_k)}.$$

A simple calculation yields

$$|\tilde{v}|^2_{H^1(\Omega_k)} \leq c \sum_{(i,j)\in\beta_k} \Big(v(\xi_i) - v(\xi_j)\Big)^2.$$

With the above inequalities, we end the proof of Lemma 3.5 2). □

Lemma 3.6. *1) If $v \in V^h$ satisfies $v(p) = 0$ for some point $p \in \overline{\Omega}_k$, then*

$$\max_{e \subset \Omega_k} \|v\|^2_{L^\infty(e)} \le c\Big(1 + \ln \frac{H}{h}\Big) A_k(v, v).$$

2)
$$\sum_{(i,j)\in\beta_k} \Big(v(\xi_i) - v(\xi_j)\Big)^2 \le c\Big(1 + \ln \frac{H}{h}\Big) A_k(v, v), \quad \forall\, v \in V^h$$

Proof. Obviously, 2) is the direct corollary of 1). It suffices to show 1). Denote $\beta = \dfrac{1}{\mathrm{meas}(\Omega_k)} \displaystyle\int_{\Omega_k} v$. Lemma 3.3 1) indicates that

$$H^{-2}\|v - \beta\|^2_{L^2(\Omega_k)} \le c|v - \beta|^2_{1,\Omega_k,h} = c|v|^2_{1,\Omega_k,h} \le cA_k(v, v).$$

Furthermore, by Lemma 3.2, we obtain

$$\max_{e \subset \Omega_k} \|v - \beta\|^2_{L^\infty(e)} \le c\Big(1 + \ln \frac{H}{h}\Big) A_k(v, v),$$
$$|\beta|^2 = |\beta - v(p)|^2 \le \max_{e \subset \Omega_k} \|v - \beta\|^2_{L^\infty(e)}.$$

Hence, applying the triangle inequality leads to Lemma 3.6 1). □

Lemma 3.7. *Let Γ_h be the quasi-uniform mesh of the interval $\Gamma = [0, H]$. Suppose that $v(x)$ is the piecewise linear continuous function on Γ_h and $v(0) = 0$. Then*

$$\int_\Gamma \frac{(v(x))^2}{x} dx \le c\Big(1 + \ln \frac{H}{h}\Big) \|v\|^2_{L^\infty(\Gamma)}.$$

The proof of Lemma 3.7 is trivial, which is omitted here.

Lemma 3.8. *Let $v \in V^h$ s.t. $v(x) = 0$, $\forall\, x \in \{\xi_j\} \bigcap \partial\Omega_k$. If v_L is a linear (bilinear) function on Ω_k or v_L is discrete harmonic on Ω_k and linear on Γ_{ij}, $\forall\, (i,j) \in \beta_k$, then*

$$\langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{v}_{ij}, \overline{v}_{ij} \rangle_{\Gamma_{ij}} \le c\Big(1 + \ln \frac{H}{h}\Big)^2 A_k(v + v_L, v + v_L), \ \forall\, (i,j) \in \beta_k \tag{3.4}$$

Proof. Let's show (3.4) in the case that $v_L = 0$. By the definition[2,12,15] of the norm of $H^{\frac{1}{2}}_{00}(\Gamma_{ij})$, we have

$$\begin{aligned} \langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{v}_{ij}, \overline{v}_{ij} \rangle_{\Gamma_{ij}} &\le c\|\overline{v}_{ij}\|^2_{H^{\frac{1}{2}}_{00}(\Gamma_{ij})} \\ &\le c\Big\{ \|I_h v\|^2_{\frac{1}{2},\partial\Omega_k} + \int_{\Gamma_{ij}} \Big[\frac{(\overline{v}_{ij}(x))^2}{|x - \xi_i|} + \frac{(\overline{v}_{ij}(x))^2}{|x - \xi_j|}\Big] ds(x) \Big\}. \end{aligned} \tag{3.5}$$

Denote $\beta = \frac{1}{\text{meas}(\Omega_k)} \int_{\Omega_k} v$. Applying Lemma 3.2, Lemma 3.4 2), and Lemma 3.3 1) leads to

$$\begin{aligned}
\|I_h v\|^2_{\frac{1}{2},\partial\Omega_k} &\le 2\{|\beta|^2 + \|I_h v - \beta\|^2_{\frac{1}{2},\partial\Omega_k}\} \\
&\le 2\{\max_{e\subset\Omega_k} \|v-\beta\|^2_{L^\infty(e)} + \|I_h(v-\beta)\|^2_{\frac{1}{2},\partial\Omega_k}\} \\
&\le c\{H^{-2}\|v-\beta\|^2_{L^2(\Omega_k)} + (1+\ln\tfrac{H}{h})|v-\beta|^2_{1,\Omega_k,h} + A_k(v-\beta, v-\beta)\} \\
&\le c(1+\ln\tfrac{H}{h})A_k(v,v).
\end{aligned}$$

On the other hand, with Lemma 3.7 and Lemma 3.6 1), we obtain that

$$\begin{aligned}
\int_{\Gamma_{ij}} \frac{(\overline{v}_{ij}(x))^2}{|x-\xi_i|} ds(x) &\le c\Big(1+\ln\tfrac{H}{h}\Big)\|\overline{v}_{ij}\|^2_{L^\infty(\Gamma_{ij})} \\
&\le c\Big(1+\ln\tfrac{H}{h}\Big)\max_{e\subset\Omega_k}\|v\|^2_{L^\infty(e)} \le c\Big(1+\ln\frac{H}{h}\Big)^2 A_k(v,v).
\end{aligned}$$

$$\int_{\Gamma_{ij}} \frac{(\overline{v}_{ij}(x))^2}{|x-\xi_j|} ds(x) \le c\Big(1+\ln\frac{H}{h}\Big)^2 A_k(v,v).$$

The substitution of the above inequalities into (3.5) implies that (3.4) is true in the case that $v_L = 0$.

To prove (3.4) in the general case, Let $v_\perp \in V_k^h$, s.t.

$$\begin{cases} A_k(v_\perp, w) = 0, & \forall\, w \in V_k^h,\ w(x) = 0,\ \forall\, x \in \{\xi_j\}\bigcap\partial\Omega_k \\ v_\perp(x) = v_L(x), & \forall\, x \in \{\xi_j\}\bigcap\partial\Omega_k \end{cases}$$

Then
$$(v + v_L - v_\perp)(x) = (v_L - v_\perp)(x) = 0, \quad \forall\, x \in \{\xi_j\}\bigcap\partial\Omega_k$$

$$A_k(v_\perp, v + v_L - v_\perp) = A_k(v_\perp, v_L - v_\perp) = 0. \tag{3.6}$$

Hence, $A_k(v+v_L-v_\perp, v+v_L-v_\perp) \le A_k(v+v_L, v+v_L)$. Furthermore, the special case of (3.4) proved above gives

$$\begin{aligned}
\langle l^{\frac{1}{2}}_{\Gamma_{ij}}\overline{v}_{ij}, \overline{v}_{ij}\rangle_{\Gamma_{ij}} &\le 2\langle l^{\frac{1}{2}}_{\Gamma_{ij}}\overline{(v_L-v_\perp)}_{ij}, \overline{(v_L-v_\perp)}_{ij}\rangle_{\Gamma_{ij}} \\
&\quad +2\langle l^{\frac{1}{2}}_{\Gamma_{ij}}\overline{(v+v_L-v_\perp)}_{ij}, \overline{(v+v_L-v_\perp)}_{ij}\rangle_{\Gamma_{ij}} \\
&\le 2\langle l^{\frac{1}{2}}_{\Gamma_{ij}}\overline{(v_L-v_\perp)}_{ij}, \overline{(v_L-v_\perp)}_{ij}\rangle_{\Gamma_{ij}} \\
&\quad +c\Big(1+\ln\tfrac{H}{h}\Big)^2 A_k(v+v_L-v_\perp, v+v_L-v_\perp) \\
&\le 2\langle l^{\frac{1}{2}}_{\Gamma_{ij}}\overline{(v_L-v_\perp)}_{ij}, \overline{(v_L-v_\perp)}_{ij}\rangle_{\Gamma_{ij}} \\
&\quad +c\Big(1+\ln\tfrac{H}{h}\Big)^2 A_k(v+v_L, v+v_L).
\end{aligned}$$

Thus to complete the proof of the lemma we need only show that

$$\langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{(v_L - v_\perp)_{ij}}, \overline{(v_L - v_\perp)_{ij}} \rangle_{\Gamma_{ij}} \leq c\Big(1 + \ln\frac{H}{h}\Big)^2 A_k(v + v_L, v + v_L). \tag{3.7}$$

Since $v_L - v_\perp$ vanishes at the vertices of Ω_k, applying inequality (3.5) and the subsequent arguments gives

$$\langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{(v_L - v_\perp)_{ij}}, \overline{(v_L - v_\perp)_{ij}} \rangle_{\Gamma_{ij}} \leq c\Big(1+\ln\frac{H}{h}\Big)^2 A_k(v_L - v_\perp, v_L - v_\perp) + \Theta_1 + \Theta_2, \tag{3.8}$$

where $\Theta_1 = c\int_{\Gamma_{ij}} \frac{|\overline{(v_L - v_\perp)_{ij}}(x)|^2}{|x - \xi_i|} ds(x), \quad \Theta_2 = c\int_{\Gamma_{ij}} \frac{|\overline{(v_L - v_\perp)_{ij}}(x)|^2}{|x - \xi_j|} ds(x).$
Since $v_\perp$ is orthogonal to $v_L - v_\perp$ in the $A_k(\cdot,\cdot)$ –inner product, we have in view of Lemma 3.6 2) that

$$\begin{aligned} A_k(v_L - v_\perp, v_L - v_\perp) &\leq A_k(v_L, v_L) \\ &\leq c \sum_{(i,j)\in\beta_k} \Big(v_L(\xi_i) - v_L(\xi_j)\Big)^2 \\ &\leq c \sum_{(i,j)\in\beta_k} \Big((v + v_L)(\xi_i) - (v + v_L)(\xi_j)\Big)^2 \\ &\leq c\Big(1 + \ln\tfrac{H}{h}\Big) A_k(v + v_L, v + v_L), \end{aligned} \tag{3.9}$$

where the second inequality holds by a simple calculation if v_L is linear (bilinear) on Ω_k, or by Lemma 3.5 2) if v_L is discrete harmonic on Ω_k and linear on $\Gamma_{ij} \subset \partial\Omega_k$, $\forall\, (i,j) \in \beta_k$.

The triangle inequality leads to

$$\Theta_1 \leq \Theta_{11} + \Theta_{12}, \tag{3.10}$$

where $\Theta_{11} = c\int_{\Gamma_{ij}} \frac{|v_L(x) - v_L(\xi_i)|^2}{|x - \xi_i|} ds(x)$, $\Theta_{12} = c\int_{\Gamma_{ij}} \frac{|\overline{(v_\perp)}_{ij}(x) - v_\perp(\xi_i)|^2}{|x - \xi_i|} ds(x).$

(3.6) implies that $A_k(v_\perp, v_\perp) \leq A_k(v + v_L, v + v_L)$. Furthermore, it follows from the linearity of v_L on Γ_{ij}, Lemma 3.6 and Lemma 3.7 that

$$\begin{aligned} \Theta_{11} &\leq c\Big(v_L(\xi_j) - v_L(\xi_i)\Big)^2 \\ &= c\Big[(v + v_L)(\xi_j) - (v + v_L)(\xi_i)\Big]^2 \\ &\leq c\Big(1 + \ln\tfrac{H}{h}\Big) A_k(v + v_L, v + v_L), \end{aligned}$$

$$
\begin{aligned}
\Theta_{12} &\le c\Big(1+\ln\tfrac{H}{h}\Big)\|\overline{(v_\perp)}_{ij} - v_\perp(\xi_i)\|^2_{L^\infty(\Gamma_{ij})} \\
&\le c\Big(1+\ln\tfrac{H}{h}\Big)\max_{e\subset\Omega_k}\|v_\perp - v_\perp(\xi_i)\|^2_{L^\infty(e)} \\
&\le c\Big(1+\ln\tfrac{H}{h}\Big)^2 A_k(v_\perp, v_\perp) \le c\Big(1+\ln\tfrac{H}{h}\Big)^2 A_k(v+v_L, v+v_L).
\end{aligned}
$$

In view of (3.10), we have $\Theta_1 \le c\Big(1+\ln\frac{H}{h}\Big)^2 A_k(v+v_L, v+v_L)$.

By symmetry, we also have $\Theta_2 \le c\Big(1+\ln\frac{H}{h}\Big)^2 A_k(v+v_L, v+v_L)$.

Thus, (3.7) follows from (3.8) and (3.9). □

§4 Preconditioners

Let $\{\phi_i\}_1^M$ be the set of the bases of V_0^h. Let $u_h = \sum_{i=1}^{M} u_i\phi_i$ be the solution of (1.2). Then, (1.2) can be rewritten as

$$\mathcal{A}U = F, \tag{4.1}$$

where $U = (u_i) \in \Re^M$, $F = ((f,\phi_i)) \in \Re^M$, $\mathcal{A} = (A(\phi_i,\phi_j)) \in \Re^{M\times M}$. Obviously, $\mathcal{A}$ is symmetric and positive definite. In addition, the condition number of $\mathcal{A}$ is proportional to h^{-2} which is implied by the following theorem.

Theorem 4.1.[6] *If Ω_h is quasi-uniform, then*

$$cond(\mathcal{A}) = O(h^{-2}).$$

As in the conforming cases, we can show Theorem 4.1 by the Rayleigh theorem, the inverse inequality and Lemma 3.3 2).

When h is small enough, $\mathcal{A}$ is rather ill–conditioned. The preconditioned conjugate gradient method (PCG)[2] is preferable to solve (1.2). For this purpose, another symmetric, positive definite matrix $\mathcal{B}$, which is called the preconditioner, must be constructed to satisfy

(1) $\mathcal{B}^{-1}b$ is easier to obtain than $\mathcal{A}^{-1}b$ in parallel computing environments;

(2) $\mathcal{B}$ is spectrally close to $\mathcal{A}$ in the sense that $cond(\mathcal{B}^{-1}\mathcal{A})$ should not be large.

And they are equivalent to find a bilinear form $B(\cdot,\cdot)$, s.t.

(1) For a given function g, the solution v of the following problem is easier in parallel to obtain than that of (1.2)

$$v \in V_0^h:\ B(v,w) = (g,w),\ \forall\, w \in V_0^h \tag{4.2}$$

(2) There exist two constants λ_0, λ_1, $\frac{\lambda_1}{\lambda_0}$ is as small as possible, s.t.

$$\lambda_0 B(w,w) \le A(w,w) \le \lambda_1 B(w,w), \ \forall\ w \in V_0^h \tag{4.3}$$

$B(\cdot,\cdot)$ is also called the preconditioner for convenience. To this end, we decompose $w \in V_0^h$ into $w = w_p + w_H$, $w_H = w_E + w_V$, s.t.

$$\begin{cases} w_p \in V_{1,0}^h \oplus V_{2,0}^h \oplus \cdots \oplus V_{N,0}^h \\ A(w_p, v) = A(w,v), \ \forall\ v \in V_{1,0}^h \oplus V_{2,0}^h \oplus \cdots \oplus V_{N,0}^h \end{cases} \tag{4.4}$$

$$\begin{cases} w_H \in V_0^h \\ A(w_H, v) = 0, \quad \forall\ v \in V_{1,0}^h \oplus V_{2,0}^h \oplus \cdots \oplus V_{N,0}^h \\ w_H(x) = w(x), \quad \forall \text{ interpolation point } x \in \bigcup_{k=1}^{N} \partial\Omega_k \end{cases} \tag{4.5}$$

$$\begin{cases} w_V \in V_0^h \\ A(w_V, v) = 0, \qquad \forall\ v \in V_{1,0}^h \oplus V_{2,0}^h \oplus \cdots \oplus V_{N,0}^h \\ w_V(\xi_j) = w_H(\xi_j), \quad j = 1,2,\cdots,m \\ w_V|_{\Gamma_{ij}} \text{ is linear}, \qquad \forall\ \Gamma_{ij} \end{cases} \tag{4.6}$$

$$\begin{cases} w_E \in V_0^h \\ A(w_E, v) = 0, \qquad \forall\ v \in V_{1,0}^h \oplus V_{2,0}^h \oplus \cdots \oplus V_{N,0}^h \\ w_E(x) = w_H(x) - w_V(x), \quad \forall \text{ interpolation point } x \in \bigcup_{k=1}^{N} \partial\Omega_k \end{cases} \tag{4.7}$$

Obviously, w_H, w_E, w_V are discrete harmonic on Ω_k, $k = 1,2,\cdots,N$, respectively. In addition, we have

$$A(w,w) = A(w_p, w_p) + A(w_H, w_H). \tag{4.8}$$

Note that w_p can be obtained by solving previously independent subproblems (4.4). Thus, the key to constructing the preconditioner $B(\cdot,\cdot)$ is to find some proper form to replace $A(w_H, w_H) = \sum_{k=1}^{N} A_k(w_H, w_H)$ in (4.8).

Theorem 4.2. *The bilinear form*

$$B_1(v,w) = A(v_p, w_p) + \sum_{\Gamma_{ij}} \langle l_{\Gamma_{ij}}^{\frac{1}{2}} \overline{(v_E)_{ij}}, \overline{(w_E)_{ij}} \rangle_{\Gamma_{ij}} + A(I_H v, I_H w), \tag{4.9}$$

satisfies

$$c\Big(1 + \ln \frac{H}{h}\Big)^{-2} B_1(w,w) \le A(w,w) \le C B_1(w,w), \ \forall\ w \in V_0^h \tag{4.10}$$

Proof. By the proof of Lemma 3.5 2), we see that

$$A_k(w_V, w_V) \le cA_k(I_H w, I_H w).$$

Furthermore, Lemma 3.5 1) leads to

$$\begin{aligned} A_k(w_H, w_H) &\le 2A_k(w_E, w_E) + 2A_k(w_V, w_V) \\ &\le c\Big\{ \sum_{(i,j)\in\beta_k} \langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{(w_E)}_{ij}, \overline{(w_E)}_{ij}\rangle_{\Gamma_{ij}} + A_k(I_H w, I_H w)\Big\}, \end{aligned}$$

which implies that the right hand side of (4.10) holds.

On the other hand, it follows from Lemma 3.6 2) and Lemma 3.8 that

$$A_k(I_H w, I_H w) \le c \sum_{(i,j)\in\beta_k} \Big(w_H(\xi_i) - w_H(\xi_j)\Big)^2 \le c\Big(1 + \ln\frac{H}{h}\Big) A_k(w_H, w_H), \quad (4.11)$$

$$\langle l^{\frac{1}{2}}_{\Gamma_{ij}} \overline{(w_E)}_{ij}, \overline{(w_E)}_{ij}\rangle_{\Gamma_{ij}} \le c\Big(1 + \ln\frac{H}{h}\Big)^2 A_k(w_H, w_H), \ \forall\ (i,j) \in \beta_k$$

Thus, the left hand side of (4.10) is true. □

Now, we outline the steps for obtaining the solution of

$$w \in V_0^h: \ B_1(w, v) = (g, v), \ \forall\ v \in V_0^h \quad (4.12)$$

Algorithm 4.1.

1. Find w_p by solving in parallel the homogeneous Dirichlet subproblems.
2. Find $\overline{(w_E)}_{ij}$ by solving in parallel one–dimensional equations on each Γ_{ij}.
3. Find the values of w_V at $\{\xi_j\}_{j=1}^m$ by solving the coarse mesh problem.
4. Find w_H by solving in parallel Dirichlet subproblems (4.5) since the values of w_H at the interpolation points on $\bigcup_{k=1}^{N} \partial\Omega_k \backslash \partial\Omega$ follows from Step 2 and Step 3.
5. Set $w = w_p + w_H$.

Besides, we see that Step 2 and Step 3 can be performed simultaneously. Hence, (4.12) is easily solved in parallel. Theorem 4.2 indicates that $B_1(\cdot,\cdot)$ can decrease the condition number of the system (1.2) from $O(h^{-2})$ to $O\big((1+\ln\frac{H}{h})^2\big)$. Therefore, $B_1(\cdot,\cdot)$ is an efficient preconditioner.

Suppose $\Gamma_{ij} \subset \partial\Omega_k$. Let $\{\phi_l^k\}$ be the set of the bases of $V_{k,0}^h$. Let $\{\phi_n^{\Gamma_{ij}}\}$ be the set of the bases related to the interpolation points on Γ_{ij}. Denote

$$\begin{aligned} K &= \big(A_k(\phi_l^k, \phi_n^k)\big), & K_\Gamma &= \big(A_k(\phi_l^k, \phi_n^{\Gamma_{ij}})\big), \\ K_{\Gamma\Gamma} &= \big(A_k(\phi_l^{\Gamma_{ij}}, \phi_n^{\Gamma_{ij}})\big), & S_{ij}^k &= K_{\Gamma\Gamma} - K_\Gamma^T K^{-1} K_\Gamma. \end{aligned}$$

We call S_{ij}^k the capacitance matrix of Ω_k concerning Γ_{ij}. Lemma 3.4 indicates that $\langle l_{\Gamma_{ij}}^{\frac{1}{2}}\overline{(w_E)_{ij}}, \overline{(w_E)_{ij}}\rangle_{\Gamma_{ij}}$ is equivalent to $A_k(\widehat{(w_E)}_{ij}, \widehat{(w_E)}_{ij})$. Thus, we have

Corollary 4.3. *If the second term of the right hand side of (4.9) is replaced by*

$$\sum_{\Gamma_{ij}} A_k(\widehat{(v_E)}_{ij}, \widehat{(w_E)}_{ij}) = \sum_{\Gamma_{ij}} (v_E)_{ij}^T S_{ij}^k (w_E)_{ij}, \tag{4.13}$$

then (4.10) is still true.

Remark 4.1. If the second term of the right hand side of (4.9) is replaced by (4.13), only Step 2 of Algorithm 4.1 is different. In this case, Step 2 is: Find $\widehat{(w_E)}_{ij}$ or $(w_E)_{ij}$ by solving simultaneously the discrete harmonic subproblems with the homogeneous Dirichlet condition on $\partial\Omega_k\backslash\Gamma_{ij}$ and the Neumann condition on Γ_{ij}.

Remark 4.2. In [6,7], three matrices D, G, J are introduced, which are all inverted easily by FFT and spectrally equivalent to the capacitance matrix. For Γ_{ij}, they are denoted D_{ij}, G_{ij}, J_{ij}, respectively. Therefore, if the second term of the right hand side of (4.9) is replaced by

$$\sum_{\Gamma_{ij}} (v_E)_{ij}^T D_{ij} (w_E)_{ij}, \quad \sum_{\Gamma_{ij}} (v_E)_{ij}^T G_{ij} (w_E)_{ij}, \text{ or } \sum_{\Gamma_{ij}} (v_E)_{ij}^T J_{ij} (w_E)_{ij},$$

the only difference is that Step 2 of Algorithm 4.1 can be performed by FFT.

Theorem 4.4. *Suppose that $\{\Omega_k\}_{k=1}^N$ satisfies the red–black subdivision. Denote $N_R = \Big\{k:\ \Omega_k$ is red, $1 \le k \le N\Big\}$, $N_B = \Big\{k:\ \Omega_k$ is black, $1 \le k \le N\Big\}$, Then the bilinear form*

$$B_2(v,w) = A(v_p, w_p) + \Big(1 + \ln\frac{H}{h}\Big) \sum_{k\in N_R} A_k(v_H - I_H v, w_H - I_H w) + A(I_H v, I_H w), \tag{4.14}$$

satisfies

$$c\Big(1+\ln\frac{H}{h}\Big)^{-2} B_2(w,w) \le A(w,w) \le C\Big(1+\ln\frac{H}{h}\Big) B_2(w,w), \ \forall\, w \in V_0^h \tag{4.15}$$

Proof. Let Ω_k be a black subdomain, $\Gamma_{ij} \subset \partial\Omega_k$. The red subdomain which has the common edge Γ_{ij} with Ω_k is denoted $\Omega_{k_{ij}}$. Since $w_H - I_H w$ is discrete harmonic on Ω_k, it follows from Lemma 3.5 1), Lemma 3.8 that

$$\begin{aligned} A_k(w_H - I_H w, w_H - I_H w) &\le c \sum_{(i,j)\in\beta_k} \langle l_{\Gamma_{ij}}^{\frac{1}{2}} \overline{(w_H - I_H w)_{ij}}, \overline{(w_H - I_H w)_{ij}} \rangle_{\Gamma_{ij}} \\ &\le c\Big(1+\ln\tfrac{H}{h}\Big)^2 \sum_{(i,j)\in\beta_k} A_{k_{ij}}(w_H - I_H w, w_H - I_H w). \end{aligned}$$

Therefore, the right hand side of (4.15) follows from the triangle inequality.

On the other hand, by (4.11) and the triangle inequality, we see that the left hand side of (4.15) is true. □

Theorem 4.5. *If the assumption of Theorem 4.4 is satisfied, and if the second term of the right hand side of (4.14) is replaced by* $\left(1+\ln\frac{H}{h}\right)\sum\limits_{k\in N_R} A_k(v_E, w_E)$, *then (4.15) holds still.*

The proof of Theorem 4.5 is similar to that of Theorem 4.4, and omitted here. $B_2(\cdot,\cdot)$ can reduce the condition number of the system (1.2) to $O\left((1+\ln\frac{H}{h})^3\right)$, and its preconditioning algorithm is different from Algorithm 4.1 only in Step 2, where solving in parallel the subproblems on the red subdomains takes the place of solving simultaneously the one–dimensional equations on each Γ_{ij}. Hence, $B_2(\cdot,\cdot)$ is also an efficient preconditioner.

Remark 4.3. The term $A(I_H v, I_H w)$ in $B_1(\cdot,\cdot)$ and $B_2(\cdot,\cdot)$ can be replaced by $\sum\limits_{\Gamma_{ij}}\left(v(\xi_i)-v(\xi_j)\right)\left(w(\xi_i)-w(\xi_j)\right)$ or $A(v_V, w_V)$.

Remark 4.4. As in [2,6], It is not difficult to write out the matrix representations of $B_1(\cdot,\cdot)$,and $B_2(\cdot,\cdot)$.

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