University of Colorado at Denver

An Iterative Substructuring Method with Nonconforming Elements

Jin-sheng Gu and Xian-cheng Hu

February 1995

UCD/CCM Report No. 45

CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

the introduction of a Lagrangian multiplier[17] or a small scale overlap subdomain in the neighborhood[18] at each internal cross point. But, in the nonconforming discrete cases, it is unnecessary and in fact impossible to calculate the values at the internal cross points. Therefore, we can extend two–subdomain algorithms. The algorithm developed to solve (1.2) in this paper doesn't handle the internal cross points, which plus the simple way to interchange the information between the subdomains leads to its high parallelism. In each iteration of our algorithm, the Dirichlet subproblems set on Ω_i are solved simultaneously, which is then followed by the parallel solving of the Neumann subproblems posed on Ω_i . Another remarkable feature of the algorithm is that it is easy to implement. Based on the essential estimates[6,13], it is proved to be geometrically convergent. Virtually, it is equivalent to the simple iterative method applied to the preconditioned capacitance equation. Also, the preconditioner implied in the iteration is easily invertible and the condition number of the preconditioned capacitance matrix is no more than $O((1+\ln\frac{H}{h})max(1+H^{-2},1+\ln\frac{H}{h}))$.

If the red–black subdivisions are not satisfied, a simple domain decomposition method is introduced in [8], but it can not choose the relaxation factor adaptively, while the present approach is an adaptive algorithm.

This paper is organized as follows: Sect.2 describes the domain decomposition algorithm. Sect.3 and Sect.4 gives the convergence analysis and the matrix analysis, respectively. Finally, Sect.5 presents numerical experiments to illustrate the efficiency of our algorithm.

§2 Domain Decomposition Method

Denote $\Gamma = \bigcup\limits_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$, let $\{\xi_j\}_{j=1}^{m}$ be the set of the edge midpoints on Γ (ordered in some way), and $\{\phi_i^\Gamma\}$ be the set of the corresponding bases. The discrete trace operator $r_0^h : V_h^0 \to \Re^m$ is defined as follows

$$\forall\ v \in V_h^0,\ r_0^h v \in \Re^m :\ \left(r_0^h v\right)(j) = v(\xi_j),\ j = 1, 2, \cdots, m.$$

Let $\beta_i = \{k:\ \Omega_k \in \Omega_H,\ \text{meas}(\partial\Omega_k \cap \partial\Omega_i) > 0\}$. For $j \in \beta_i$, the open common edge of Ω_i and Ω_j is labeled Γ_{ij}. For $v \in V_h^0$, $\overline{v}_{ij}$, the piecewise linear continuous function on Γ_{ij}, satisfies

$$\overline{v}_{ij}(\xi_k) = v(\xi_k),\ \forall\ \xi_k \in \Gamma_{ij},\ \overline{v}_{ij}(\nu) = 0,\ \forall \text{ endpoint } \nu \text{ of } \Gamma_{ij}.$$

An Iterative Substructuring Method with Nonconforming Elements‡

Jinsheng Gu and Xiancheng Hu

Abstract

When the red–black subdivisions are satisfied, an iterative substructuring method is proposed to solve the algebraic system of equations arising from the discretization of symmetric elliptic problems via nonconforming finite elements which are only continuous at the quasi–uniform mesh nodes. Theoretical analysis is given and the results of numerical experiments are reported.

§1 Introduction

It is known that much progress has been made in the study of nonoverlap domain decomposition methods, or substructuring methods to solve the following elliptic problems via the conforming finite elements[1,2,17]

$$u \in H_0^1(\Omega): \ a(u,v) = (f,v), \ \forall \ v \in H_0^1(\Omega) \tag{1.1}$$

where $a(u,v) = \int_\Omega \left[\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x)uv \right]$, $(f,v) = \int_\Omega fv$, $f \in H^{-1}(\Omega)$, $\Omega \subset \Re^2$ is a bounded polygonal open domain , $a_0(x) \geq \alpha > 0$, $a_0(x)$, $a_{ij}(x)$, $i,j = 1,2$ are piecewise smooth and bounded functions in Ω, (a_{ij}) is a symmetric, uniformly positive definite matrix in Ω.

On the other hand, some recent developments indicate that better results can be obtained in the nonconforming discrete case, although the nonconforming elements lack the global continuity. Oswald [16] has constructed a hierachical basis multilevel method for two dimensional Crouzeix–Raviart elements with $O(k)$ as the bound of the condition number of the hierarchical discretization, where k is

‡ This work was supported by the National Natural Science Foundation of China under Grant No. 19171050 and the Tsinghua University Fund for Science.

the number of refinement levels. Gu, et al. [6–11] have developed a series of nonoverlap domain decomposition methods with nonconforming elements, which are as efficient as their counterparts in the conforming cases, and even easier in implementation.

The aim of the present paper is to extend a two–subdomain nonoverlap domain decomposition algorithm[6,9] to the multi–subdomain cases to solve (1.1), which is discretized by the nonconforming finite elements. Let $\Omega_h = \{e\}$ be a quasi–uniform mesh of Ω with h as its mesh parameter. Let V_h be the piecewise linear nonconforming Crouzeix–Raviart element space[5], or the piecewise quartic nonconforming rectangular element space[14]. The properties of V_h are well known. For example, every function in V_h is continuous at the midpoint of each edge of e, $\forall\ e \in \Omega_h$, which is called the edge midpoint for conciseness. And, the optimal error estimates can be obtained for the Stokes problem[5,14], if V_h is used for the velocity field and piecewise constant elements for the pressure. The nonconforming finite element discrete problem for (1.1) is

$$(1.2) \qquad u_h \in V_h^0 : \ A(u_h, v_h) = (f, v_h), \quad \forall\ v_h \in V_h^0$$

where $\qquad V_h^0 = \{v_h \in V_h : \ v_h(x) = 0, \ \forall \text{ edge midpoint } x \in \partial\Omega\},$

$$A(u,v) = \sum_{e\subset\Omega} \int_e \left[\sum_{i,j=1}^{2} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0 uv \right].$$

Suppose that $\{\Omega_i\}_{i=1}^{N} \triangleq \Omega_H$ is a nonoverlap subdivision of Ω, which is, in fact, a quasi–uniform mesh of Ω with H as its mesh parameter. Ω_i is either a quadrilater or a triangle, referred to as a subdomain. Besides, the common boundary point of more than two subdomains that lies inside Ω is called the internal cross point. Assume that Ω_h can be obtained by refining Ω_H, so that Ω_H, Ω_h form a two–level triangulation on Ω. Generally, $H \gg h$, therefore, Ω_H, Ω_h are called the coarse mesh and the fine mesh, respectively. If all the subdomains are colored by two colors, red and black, such that no pair of subdomains which have a common edge have the same color, then such a subdivision of Ω is called the red–black subdivision. We note that the red–black subdivision is of coarse not always possible. In the conforming finite element discrete cases, we note that even if the red–black subdivision is satisfied, all the two–subdomain nonoverlap algorithms[6,9] can't be extended to the multi–subdomain cases for the existence of the internal cross points[17,18]. Several algorithms are developed in [17,18], by

And, we introduce the following notations:

$$
\begin{aligned}
V_h^i &= \{v \in V_h^0 : \ v(x) = 0, \ \forall \text{ interpolation point } x \in \overline{\Omega}\backslash\overline{\Omega}_i\}, \\
V_h^{i,0} &= \{v \in V_h^0 : \ v(x) = 0, \ \forall \text{ interpolation point } x \in \overline{\Omega}\backslash\Omega_i\}, \\
A_i(u,v) &= \sum_{e \subset \Omega_i} \int_e \left[\sum_{k,j=1}^{2} a_{kj} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_j} + a_0 uv \right], \quad (f,v)_i = \int_{\Omega_i} fv.
\end{aligned}
$$

Here, an interpolation point $x \in \overline{\Omega}$ is related to the definition of V^h.

Let $\Upsilon = \{k : \ \Omega_k \in \Omega_H, \ \Omega_k \text{ is red}\}$, $B = \{k : \ \Omega_k \in \Omega_H, \ \Omega_k \text{ is black}\}$. Obviously, we have $\Upsilon \bigcup B = \{1, 2, \cdots, N\}$, and $\beta_k \subset \Upsilon$ for $k \in B$, while $\beta_i \subset B$ for $i \in \Upsilon$. In addition, we introduce the following notations.

$$
\begin{aligned}
\Omega_\Upsilon &= \bigcup_{k\in\Upsilon} \Omega_k, & (w,v)_\Upsilon &= \sum_{k\in\Upsilon} (w,v)_k, \\
V_h^\Upsilon &= \bigcup_{k\in\Upsilon} V_h^k, & A_\Upsilon(w,v) &= \sum_{k\in\Upsilon} A_k(w,v), \\
V_h^{\Upsilon,0} &= \bigcup_{k\in\Upsilon} V_h^{k,0}, & D_\Upsilon(v) &= \Big((A_\Upsilon(v,\phi_i^\Gamma) - (f,\phi_i^\Gamma)_\Upsilon) \Big).
\end{aligned}
$$

Correspondingly, we have $\Omega_B, V_h^B, V_h^{B,0}, (\cdot,\cdot)_B, A_B(\cdot,\cdot), D_B(\cdot)$. Now, the iterative substructuring method for the solving of (1.2) is stated as follows

Algorithm 2.1. Choose properly the relaxation factors $\theta_1, \theta_2 \in (0,1)$. Let $g^0 \in \Re^m$ be any initial guess. Suppose that $g^n \in \Re^m$ is obtained. The procedure to get $g^{n+1} \in \Re^m$ consists of the following four steps:

Step 1. For $i = 1, 2, \cdots, N$, solve in parallel

$$
\begin{cases}
u_i^{2n} \in V_h^i \\
A_i(u_i^{2n}, v) = (f,v)_i, & \forall\, v \in V_h^{i,0} \\
u_i^{2n}(\xi_j) = \Big(g^n\Big)(j), & \forall\, \xi_j \in \partial\Omega_i \backslash \partial\Omega
\end{cases}
$$

Step 2. Calculate $u_\Upsilon^{2n} = \sum_{k\in\Upsilon} u_k^{2n}, \quad u_B^{2n} = \sum_{k\in B} u_k^{2n},$

$$
d^n = \theta_1 D_\Upsilon(u_\Upsilon^{2n}) - (1-\theta_1) D_B(u_B^{2n}).
$$

Step 3. For $i = 1, 2, \cdots, N$, solve in parallel

$$
\begin{cases}
i \in \Upsilon \\
u_i^{2n+1} \in V_h^i \\
A_i(u_i^{2n+1}, v) = (f,v)_i, & \forall\, v \in V_h^{i,0} \\
A_i(u_i^{2n+1}, \phi_j^\Gamma) = (f, \phi_j^\Gamma)_i + \Big(d^n\Big)(j), & \forall\, \xi_j \in \partial\Omega_i \backslash \partial\Omega
\end{cases}
$$

$$\begin{cases} i \in B \\ u_i^{2n+1} \in V_h^i \\ A_i(u_i^{2n+1}, v) = (f, v)_i, & \forall\, v \in V_h^{i,0} \\ A_i(u_i^{2n+1}, \phi_j^\Gamma) = (f, \phi_j^\Gamma)_i - \Big(d^n\Big)(j), & \forall\, \xi_j \in \partial\Omega_i \backslash \partial\Omega \end{cases}$$

Step 4. Calculate $u_\Upsilon^{2n+1} = \sum_{k \in \Upsilon} u_k^{2n+1}, \quad u_B^{2n+1} = \sum_{k \in B} u_k^{2n+1},$

$$g^{n+1} = \theta_2 r_0^h u_\Upsilon^{2n+1} + (1 - \theta_2) r_0^h u_B^{2n+1}.$$

§3 Convergence Analysis

Lemma 3.1.[6,13] *Let $v \in V_0^h$ be the discrete harmonic function on Ω_i, i.e. v satisfies*

$$A_i(v, \theta) = 0, \quad \forall\, \theta \in V_{i,0}^h$$

Then there exists a positive constant c, independent of h, H, Ω_i, such that

$$A_i(v, v) \le c \sum_{j \in \beta_i} ||\overline{v}_{ij}||^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})}.$$

Lemma 3.2.[6,13] *For $v \in V_i^h$, $j \in \beta_i$, we have*

$$||\overline{v}_{ij}||^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} \le c(1 + \ln \frac{H}{h})\Big\{(1 + \ln \frac{H}{h})|v|^2_{1,\Omega_i,h} + (1 + H^{-2})||v||^2_{L^2(\Omega_i)}\Big\},$$

where $|v|_{1,\Omega_i,h} \triangleq \Big(\sum_{e \subset \Omega_i} |v|^2_{H^1(e)}\Big)^{\frac{1}{2}}$ *, and c is independent of h, H, Ω_i.*

Let $E_\Upsilon^h : \Re^m \to V_h^\Upsilon$, $E_B^h : \Re^m \to V_h^B$ be nonconforming discrete harmonic extension operators, which satisfies ,respectively, for any $\lambda \in \Re^m$

$$\begin{cases} E_\Upsilon^h \lambda \in V_h^\Upsilon \\ A_\Upsilon(E_\Upsilon^h \lambda, v) = 0, \ \forall\, v \in V_h^{\Upsilon,0} \\ r_0^h E_\Upsilon^h \lambda = \lambda \end{cases}, \qquad \begin{cases} E_B^h \lambda \in V_h^B \\ A_B(E_B^h \lambda, v) = 0, \ \forall\, v \in V_h^{B,0} \\ r_0^h E_B^h \lambda = \lambda \end{cases}$$

Furthermore, $\forall\, \lambda \in \Re^m$, the extension operators $T_\Upsilon^h : \Re^m \to V_h^\Upsilon$, $T_B^h : \Re^m \to V_h^B$ are defined as follows

$$\begin{cases} T_\Upsilon^h \lambda \in V_h^\Upsilon \\ A_\Upsilon(T_\Upsilon^h \lambda, v) = 0, & \forall\, v \in V_h^{\Upsilon,0} \\ A_\Upsilon(T_\Upsilon^h \lambda, \phi_i^\Gamma) = -A_B(E_B^h \lambda, \phi_i^\Gamma), & i = 1, 2, \cdots, m \end{cases}$$

$$\begin{cases} T_B^h\lambda \in V_h^B \\ A_B(T_B^h\lambda, v) = 0, & \forall\, v \in V_h^{B,0} \\ A_B(T_B^h\lambda, \phi_i^\Gamma) = -A_\Upsilon(E_\Upsilon^h\lambda, \phi_i^\Gamma), & i = 1, 2, \cdots, m \end{cases}$$

For $\lambda \in \Re^m$, it is easy to see that

$$A_\Upsilon(T_\Upsilon^h\lambda, v) = -A_B(E_B^h\lambda, v), \quad \forall\, v \in V_h \tag{3.1}$$

$$A_B(T_B^h\lambda, v) = -A_\Upsilon(E_\Upsilon^h\lambda, v), \quad \forall\, v \in V_h \tag{3.2}$$

$$A_\Upsilon(T_\Upsilon^h\lambda, v) = -A_B(E_B^h\lambda, E_B^h r_0^h v), \quad \forall\, v \in V_h^\Upsilon \tag{3.3}$$

$$A_B(T_B^h\lambda, v) = -A_\Upsilon(E_\Upsilon^h\lambda, E_\Upsilon^h r_0^h v), \quad \forall\, v \in V_h^B \tag{3.4}$$

Theorem 3.3. *(The extension theorem in multi-subdomain cases)*

$$\sigma = \sup_{\lambda \in \Re^m \setminus \{0\}} \frac{A_\Upsilon(E_\Upsilon^h\lambda, E_\Upsilon^h\lambda)}{A_B(E_B^h\lambda, E_B^h\lambda)} \le c(1 + \ln\frac{H}{h}) \max(1 + \ln\frac{H}{h}, 1 + H^{-2}), \tag{3.5}$$

$$\tau = \sup_{\lambda \in \Re^m \setminus \{0\}} \frac{A_B(E_B^h\lambda, E_B^h\lambda)}{A_\Upsilon(E_\Upsilon^h\lambda, E_\Upsilon^h\lambda)} \le c(1 + \ln\frac{H}{h}) \max(1 + \ln\frac{H}{h}, 1 + H^{-2}). \tag{3.6}$$

Proof. $\forall\, \lambda \in \Re^m$, let $v = \begin{cases} E_B^h\lambda, & \text{on } \overline{\Omega}_B \\ E_\Upsilon^h\lambda, & \text{on } \Omega_\Upsilon \end{cases}$. It follows from Lemma 3.1 and Lemma 3.2 that

$$\begin{aligned} A_i(v, v) &\le c \sum_{j \in \beta_i} \|\overline{v}_{ij}\|^2_{H_{00}^{\frac{1}{2}}(\Gamma_{ij})} \\ &\le c \sum_{j \in \beta_i} (1 + \ln\frac{H}{h}) \left\{ (1 + \ln\frac{H}{h}) |v|^2_{1,\Omega_j,h} + (1 + H^{-2}) \|v\|^2_{L^2(\Omega_j)} \right\} \\ &\le c(1 + \ln\frac{H}{h}) \max(1 + H^{-2}, 1 + \ln\frac{H}{h}) \sum_{j \in \beta_i} A_j(v, v). \end{aligned}$$

Summing up with $i \in \Upsilon$ yields

$$A_\Upsilon(v, v) \le c(1 + \ln\frac{H}{h}) \max(1 + H^{-2}, 1 + \ln\frac{H}{h}) A_B(v, v),$$

where we have applied the fact that $\beta_i \subset B$ for $i \in \Upsilon$. Therefore, (3.5) holds.

Similarly (3.6) can be established . This completes the proof. □

Corrollary 3.4. *Let σ, τ be those in Theorem 3.3, we have*

$$\frac{1}{\tau} A_\Upsilon(E_\Upsilon^h \lambda, E_\Upsilon^h \lambda) \le A_B(T_B^h \lambda, T_B^h \lambda) \le \sigma A_\Upsilon(E_\Upsilon^h \lambda, E_\Upsilon^h \lambda), \quad \forall\, \lambda \in \Re^m \tag{3.7}$$

$$\frac{1}{\sigma} A_B(E_B^h \lambda, E_B^h \lambda) \le A_\Upsilon(T_\Upsilon^h \lambda, T_\Upsilon^h \lambda) \le \tau A_B(E_B^h \lambda, E_B^h \lambda), \quad \forall\, \lambda \in \Re^m \tag{3.8}$$

Proof. Taking $v = T_B^h \lambda$ in (3.4), using the Schwarz inequality and (3.5), we obtain

$$\begin{aligned} A_B(T_B^h \lambda, T_B^h \lambda) &= -A_\Upsilon(E_\Upsilon^h \lambda, E_\Upsilon^h r_0^h T_B^h \lambda) \\ &\le \left(A_\Upsilon(E_\Upsilon^h \lambda, E_\Upsilon^h \lambda)\right)^{\frac{1}{2}} \left(A_\Upsilon(E_\Upsilon^h r_0^h T_B^h \lambda, E_\Upsilon^h r_0^h T_B^h \lambda)\right)^{\frac{1}{2}} \\ &\le \left(A_\Upsilon(E_\Upsilon^h \lambda, E_\Upsilon^h \lambda)\right)^{\frac{1}{2}} \left(\sigma A_B(T_B^h \lambda, T_B^h \lambda)\right)^{\frac{1}{2}}, \end{aligned}$$

which implies the right hand side of (3.7) holds.

On the other hand, the substitution of v in (3.4) with $E_B^h \lambda$ and (3.6) yield

$$\begin{aligned} A_\Upsilon(E_\Upsilon^h \lambda, E_\Upsilon^h \lambda) = -A_B(T_B^h \lambda, E_B^h \lambda) &\le \left(A_B(T_B^h \lambda, T_B^h \lambda)\right)^{\frac{1}{2}} \left(A_B(E_B^h \lambda, E_B^h \lambda)\right)^{\frac{1}{2}} \\ &\le \left(A_B(T_B^h \lambda, T_B^h \lambda)\right)^{\frac{1}{2}} \left(\tau A_\Upsilon(E_\Upsilon^h \lambda, E_\Upsilon^h \lambda)\right)^{\frac{1}{2}}, \end{aligned}$$

which indicates the left hand side of (3.7) is true. And we get (3.7).

(3.8) follows in the same manner. Hence the proof is complete. □

Theorem 3.5. *Let u_h be the solution of (1.2), and u^n be the approximate solution obtained by Algorithm 2.1, where $u^n = u_i^n$ in Ω_i. Denote $\varepsilon^n = u_h - u^n$. Then, we have*

1) There exist $\theta_1^*, \theta_2^* \in (0,1)$, such that if $\theta_1 \in (\theta_1^*, 1), \theta_2 \in (\theta_2^*, 1)$, then

$$A(\varepsilon^{n+2}, \varepsilon^{n+2}) \le \kappa_1(\theta_1)\kappa_2(\theta_2) A(\varepsilon^n, \varepsilon^n), \tag{3.9}$$

with $0 \le \kappa_1(\theta_1) < 1$, $0 \le \kappa_2(\theta_2) < 1$.

2) There exist the optimal relaxation factors $\theta_1^{opt}, \theta_2^{opt}$, such that

$$\kappa_1(\theta_1^{opt})\kappa_2(\theta_2^{opt}) = \min_{\theta_1 \in (\theta_1^*, 1)} \min_{\theta_2 \in (\theta_2^*, 1)} \kappa_1(\theta_1)\kappa_2(\theta_2). \tag{3.10}$$

Proof. Let $u_h^i \in V_h^i$ s.t.

$$u_h^i(x) = u_h(x), \ \forall \text{ interpolation point } x \in \overline{\Omega}_i$$

Denote

$$\begin{aligned} u_h^\Upsilon &= \sum_{k\in\Upsilon} u_h^k \in V_h^\Upsilon, & u_h^B &= \sum_{k\in B} u_h^k \in V_h^B, \\ \varepsilon_i^n &= u_i^n - u_h^i, & \mu^n &= g^n - r_0^h u_h, \\ \varepsilon_\Upsilon^n &= u_\Upsilon^n - u_h^\Upsilon, & \varepsilon_B^n &= u_B^n - u_h^B. \end{aligned}$$

It is easy to see that

$$\text{(3.11)} \qquad \begin{cases} \varepsilon_\Upsilon^{2n} \in V_h^\Upsilon \\ A_\Upsilon(\varepsilon_\Upsilon^{2n}, v) = 0, \quad \forall\, v \in V_h^{\Upsilon,0} \\ r_0^h \varepsilon_\Upsilon^{2n} = \mu^n \end{cases}$$

$$\text{(3.12)} \qquad \begin{cases} \varepsilon_B^{2n} \in V_h^B \\ A_B(\varepsilon_B^{2n}, v) = 0, \quad \forall\, v \in V_h^{B,0} \\ r_0^h \varepsilon_B^{2n} = \mu^n \end{cases}$$

$$\text{(3.13)} \qquad \left(d^n\right)(i) = \theta_1 A_\Upsilon(\varepsilon_\Upsilon^{2n}, \phi_i^\Gamma) - (1-\theta_1) A_B(\varepsilon_B^{2n}, \phi_i^\Gamma), \ i = 1, 2, \cdots, m.$$

$$\text{(3.14)} \qquad \begin{cases} \varepsilon_\Upsilon^{2n+1} \in V_h^\Upsilon \\ A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, v) = 0, & \forall\, v \in V_h^{\Upsilon,0} \\ A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \phi_i^\Gamma) = \left(d^n\right)(i), & i = 1, 2, \cdots, m \end{cases}$$

$$\text{(3.15)} \qquad \begin{cases} \varepsilon_B^{2n+1} \in V_h^B \\ A_B(\varepsilon_B^{2n+1}, v) = 0, & \forall\, v \in V_h^{B,0} \\ A_B(\varepsilon_B^{2n+1}, \phi_i^\Gamma) = -\left(d^n\right)(i), & i = 1, 2, \cdots, m \end{cases}$$

$$\text{(3.16)} \qquad \mu^{n+1} = \theta_2 r_0^h \varepsilon_\Upsilon^{2n+1} + (1-\theta_2) r_0^h \varepsilon_B^{2n+1}.$$

By (3.14), we have

$$A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \phi_i^\Gamma) = \theta_1 A_\Upsilon(\varepsilon_\Upsilon^{2n}, \phi_i^\Gamma) - (1-\theta_1) A_B(\varepsilon_B^{2n}, \phi_i^\Gamma).$$

Hence

$$\varepsilon_\Upsilon^{2n+1} = \theta_1 \varepsilon_\Upsilon^{2n} + (1-\theta_1) T_\Upsilon^h r_0^h \varepsilon_B^{2n} = \theta_1 \varepsilon_\Upsilon^{2n} + (1-\theta_1) T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n} \tag{3.17}$$

$$\begin{aligned} A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \varepsilon_\Upsilon^{2n+1}) = \;& \theta_1^2 A_\Upsilon(\varepsilon_\Upsilon^{2n}, \varepsilon_\Upsilon^{2n}) + (1-\theta_1)^2 A_\Upsilon(T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) \\ & + 2\theta_1(1-\theta_1) A_\Upsilon(\varepsilon_\Upsilon^{2n}, T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) \end{aligned} \tag{3.18}$$

It follows from (3.3) and (3.5) that

$$\begin{aligned} A_\Upsilon(T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, \varepsilon_\Upsilon^{2n}) &= -A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n}, E_B^h r_0^h \varepsilon_\Upsilon^{2n}) \\ &\le -\frac{1}{\sigma} A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) = -\frac{1}{\sigma} A_\Upsilon(\varepsilon_\Upsilon^{2n}, \varepsilon_\Upsilon^{2n}). \end{aligned} \tag{3.19}$$

(3.8) and (3.6) yield

$$A_\Upsilon(T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) \le \tau A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n}, E_B^h r_0^h \varepsilon_\Upsilon^{2n}) \le \tau^2 A_\Upsilon(\varepsilon_\Upsilon^{2n}, \varepsilon_\Upsilon^{2n}). \tag{3.20}$$

With the substitution of (3.19) and (3.20) into (3.18), we get

$$A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \varepsilon_\Upsilon^{2n+1}) \le \kappa_1(\theta_1) A_\Upsilon(\varepsilon_\Upsilon^{2n}, \varepsilon_\Upsilon^{2n}), \tag{3.21}$$

where

$$\kappa_1(\theta_1) = \frac{1}{\sigma}\Big((1-\theta_1)^2(\sigma\tau^2 + \sigma + 2) - 2(1-\theta_1)(\sigma+1) + \sigma\Big). \tag{3.22}$$

On the other hand, it follows from (3.17) that

$$E_B^h r_0^h \varepsilon_\Upsilon^{2n+1} = \theta_1 E_B^h r_0^h \varepsilon_\Upsilon^{2n} + (1-\theta_1) E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n} = \theta_1 \varepsilon_B^{2n} + (1-\theta_1) E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n},$$

$$\begin{aligned} A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) = \;& \theta_1^2 A_B(\varepsilon_B^{2n}, \varepsilon_B^{2n}) + (1-\theta_1)^2 \\ & A_B(E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) \\ & + 2\theta_1(1-\theta_1) A_B(\varepsilon_B^{2n}, E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}). \end{aligned} \tag{3.23}$$

Applying (3.6) and (3.8) gives

$$\begin{aligned} A_B(E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) &\le \tau A_\Upsilon(E_\Upsilon^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, E_\Upsilon^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) \\ &= \tau A_\Upsilon(T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}, T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) \\ &\le \tau^2 A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n}, E_B^h r_0^h \varepsilon_\Upsilon^{2n}) \\ &= \tau^2 A_B(\varepsilon_B^{2n}, \varepsilon_B^{2n}). \end{aligned} \tag{3.24}$$

By (3.3) and (3.8), we obtain

$$
\begin{aligned}
A_B(\varepsilon_B^{2n}, E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n}) &= A_B(E_B^h r_0^h \varepsilon_B^{2n}, E_B^h r_0^h T_\Upsilon^h r_0^h \varepsilon_B^{2n}) \\
&= A_\Upsilon(T_\Upsilon^h r_0^h \varepsilon_B^{2n}, T_\Upsilon^h r_0^h \varepsilon_B^{2n}) \\
&\le -\frac{1}{\sigma} A_B(E_B^h r_0^h \varepsilon_B^{2n}, E_B^h r_0^h \varepsilon_B^{2n}) \\
&= -\frac{1}{\sigma} A_B(\varepsilon_B^{2n}, \varepsilon_B^{2n}).
\end{aligned}
\tag{3.25}
$$

With the substitution of (3.24) and (3.25) into (3.23) , we get

$$
A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) \le \kappa_1(\theta_1) A_B(\varepsilon_B^{2n}, \varepsilon_B^{2n}). \tag{3.26}
$$

From (3.11), it is easy to see that

$$
\varepsilon_\Upsilon^{2n+2} = \theta_2 E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1} + (1-\theta_2) E_\Upsilon^h r_0^h \varepsilon_B^{2n+1} = \theta_2 \varepsilon_\Upsilon^{2n+1} + (1-\theta_2) E_\Upsilon^h r_0^h \varepsilon_B^{2n+1}.
$$

$$
\begin{aligned}
A_\Upsilon(\varepsilon_\Upsilon^{2n+2}, \varepsilon_\Upsilon^{2n+2}) = \; & \theta_2^2 A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \varepsilon_\Upsilon^{2n+1}) \\
& + 2\theta_2(1-\theta_2) A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_B^{2n+1}) \\
& + (1-\theta_2)^2 A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_B^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_B^{2n+1})
\end{aligned}
\tag{3.27}
$$

(3.5) and (3.7) gives

$$
\begin{aligned}
A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_B^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_B^{2n+1}) &\le \sigma A_B(E_B^h r_0^h \varepsilon_B^{2n+1}, E_B^h r_0^h \varepsilon_B^{2n+1}) \\
&= \sigma A_B(\varepsilon_B^{2n+1}, \varepsilon_B^{2n+1}) \\
&= \sigma A_B(T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\
&\le \sigma^2 A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\
&= \sigma^2 A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \varepsilon_\Upsilon^{2n+1}).
\end{aligned}
\tag{3.28}
$$

It follows from (3.4) and (3.7) that

$$
\begin{aligned}
A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_B^{2n+1}) &= A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_B^{2n+1}) \\
&= -A_B(T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, \varepsilon_B^{2n+1}) \\
&= -A_B(T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\
&\le -\frac{1}{\tau} A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\
&= -\frac{1}{\tau} A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \varepsilon_\Upsilon^{2n+1}).
\end{aligned}
\tag{3.29}
$$

With the substitution of (3.28) and (3.29) into (3.27) , we obtain

$$
A_\Upsilon(\varepsilon_\Upsilon^{2n+2}, \varepsilon_\Upsilon^{2n+2}) \le \kappa_2(\theta_2) A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \varepsilon_\Upsilon^{2n+1}), \tag{3.30}
$$

where

$$\kappa_2(\theta_2) = \frac{1}{\tau}\Big((1-\theta_2)^2(\sigma^2\tau+\tau+2) - 2(1-\theta_2)(\tau+1)+\tau\Big). \tag{3.31}$$

On the other hand, (3.12) gives

$$\varepsilon_B^{2n+2} = \theta_2 E_B^h r_0^h \varepsilon_\Upsilon^{2n+1} + (1-\theta_2)\varepsilon_B^{2n+1}$$

$$\begin{aligned} A_B(\varepsilon_B^{2n+2}, \varepsilon_B^{2n+2}) = \; & \theta_2^2 A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\ & +(1-\theta_2)^2 A_B(\varepsilon_B^{2n+1}, \varepsilon_B^{2n+1}) \\ & +2\theta_2(1-\theta_2) A_B(\varepsilon_B^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}). \end{aligned} \tag{3.32}$$

By (3.7) and (3.5) , we have

$$\begin{aligned} A_B(\varepsilon_B^{2n+1}, \varepsilon_B^{2n+1}) & = A_B(T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\ & \le \sigma A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\ & \le \sigma^2 A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}). \end{aligned} \tag{3.33}$$

(3.4) and (3.6) imply

$$\begin{aligned} A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, \varepsilon_B^{2n+1}) & = A_B(T_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\ & = -A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\ & = -A_\Upsilon(E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_\Upsilon^h r_0^h \varepsilon_\Upsilon^{2n+1}) \\ & \le -\frac{1}{\tau} A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}). \end{aligned} \tag{3.34}$$

The substitution of (3.33) and (3.34) into (3.32) reduces to

$$A_B(\varepsilon_B^{2n+2}, \varepsilon_B^{2n+2}) \le \kappa_2(\theta_2) A_B(E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}, E_B^h r_0^h \varepsilon_\Upsilon^{2n+1}). \tag{3.35}$$

It follows from (3.21), (3.26), (3.30) and (3.35) that

$$A_\Upsilon(\varepsilon_\Upsilon^{2n+2}, \varepsilon_\Upsilon^{2n+2}) \le \kappa_1(\theta_1)\kappa_2(\theta_2) A_\Upsilon(\varepsilon_\Upsilon^{2n}, \varepsilon_\Upsilon^{2n}), \tag{3.36}$$

$$A_B(\varepsilon_B^{2n+2}, \varepsilon_B^{2n+2}) \le \kappa_1(\theta_1)\kappa_2(\theta_2) A_B(\varepsilon_B^{2n}, \varepsilon_B^{2n}). \tag{3.37}$$

Similarly, we obtain that

$$A_\Upsilon(\varepsilon_\Upsilon^{2n+3}, \varepsilon_\Upsilon^{2n+3}) \le \kappa_1(\theta_1)\kappa_2(\theta_2) A_\Upsilon(\varepsilon_\Upsilon^{2n+1}, \varepsilon_\Upsilon^{2n+1}), \tag{3.38}$$

$$(3.39) \qquad A_B(\varepsilon_B^{2n+3}, \varepsilon_B^{2n+3}) \le \kappa_1(\theta_1)\kappa_2(\theta_2) A_B(\varepsilon_B^{2n+1}, \varepsilon_B^{2n+1}).$$

By (3.22) and (3.31), an elementary calculation gives

$$\kappa_1(\theta_1) < 1, \quad \forall\, \theta_1 \in (\theta_1^*, 1) = \Big(\frac{\tau^2\sigma - \sigma}{\tau^2\sigma + \sigma + 2}, 1\Big)$$

$$\kappa_2(\theta_2) < 1, \quad \forall\, \theta_2 \in (\theta_2^*, 1) = \Big(\frac{\sigma^2\tau - \tau}{\sigma^2\tau + \tau + 2}, 1\Big)$$

Furthermore, when

$$(3.40) \qquad \theta_1 = \theta_1^{opt} = \frac{\tau^2\sigma + 1}{\tau^2\sigma + \sigma + 2}, \quad \theta_2 = \theta_2^{opt} = \frac{\sigma^2\tau + 1}{\sigma^2\tau + \tau + 2},$$

$\kappa_1(\theta_1), \kappa_2(\theta_2)$ attain their minimum values, respectively,

$$(3.41) \qquad (\kappa_1)_{min} = \frac{\tau^2\sigma^2 - 1}{\sigma(\tau^2\sigma + \sigma + 2)}, \quad (\kappa_2)_{min} = \frac{\tau^2\sigma^2 - 1}{\tau(\sigma^2\tau + \tau + 2)}. \qquad \square$$

§4 Matrix Analysis

Suppose that the interpolation points are ordered in the way: first $\{\xi_j\}_{j=1}^m$, then those in Ω_Υ, and finally those in Ω_B. Let $\{\phi_i^k\}$ be the set of bases of $V_h^{k,0}$. We introduce the following notations

$$\begin{array}{lll}
\{\phi_j^I\} = \bigcup_{k=1}^N \{\phi_i^k\}, & \{\phi_j^\Upsilon\} = \bigcup_{k\in\Upsilon} \{\phi_i^k\}, & \{\phi_j^B\} = \bigcup_{k\in B} \{\phi_i^k\}, \\
A_{II} = \Big(A(\phi_i^I, \phi_j^I)\Big), & A_{\Upsilon\Upsilon} = \Big(A_\Upsilon(\phi_i^\Upsilon, \phi_j^\Upsilon)\Big), & A_{BB} = \Big(A_B(\phi_i^B, \phi_j^B)\Big), \\
A_{I\Gamma} = \Big(A(\phi_i^I, \phi_j^\Gamma)\Big), & A_{\Upsilon\Gamma} = \Big(A_\Upsilon(\phi_i^\Upsilon, \phi_j^\Gamma)\Big), & A_{B\Gamma} = \Big(A_B(\phi_i^B, \phi_j^\Gamma)\Big), \\
A_{\Gamma\Gamma} = \Big(A(\phi_i^\Gamma, \phi_j^\Gamma)\Big), & A_{\Gamma\Gamma}^\Upsilon = \Big(A_\Upsilon(\phi_i^\Gamma, \phi_j^\Gamma)\Big), & A_{\Gamma\Gamma}^B = \Big(A_B(\phi_i^\Gamma, \phi_j^\Gamma)\Big),
\end{array}$$

$$S = A_{\Gamma\Gamma} - A_{I\Gamma}^T A_{II}^{-1} A_{I\Gamma},$$
$$S^\Upsilon = A_{\Gamma\Gamma}^\Upsilon - A_{\Upsilon\Gamma}^T A_{\Upsilon\Upsilon}^{-1} A_{\Upsilon\Gamma},$$
$$S^B = A_{\Gamma\Gamma}^B - A_{B\Gamma}^T A_{BB}^{-1} A_{B\Gamma},$$

where, S, S^Υ, S^B are called the capacitance matrices of Ω, Ω_Υ, Ω_B with respect to Γ, respectively. Similarly, we can write out the capacitance matrix of the

Proof. . It is easy to see that

$$\forall\ \lambda, \mu \in \Re^m, \quad \lambda^T S^\Upsilon \mu = A_\Upsilon(E^h_\Upsilon \lambda, E^h_\Upsilon \mu), \ \lambda^T S^B \mu = A_B(E^h_B \lambda, E^h_B \mu).$$

Furthermore, by $S = S^\Upsilon + S^B$ and Theorem 3.3, we get (4.6) and (4.7). □

By now, we note that S^Υ, S^B or Q are, respectively, the efficient preconditioners for S, so is the following matrix

$$\begin{bmatrix} Q + A^T_{\Upsilon\Gamma} A^{-1}_{\Upsilon\Upsilon} A_{\Upsilon\Gamma} + A^T_{B\Gamma} A^{-1}_{BB} A_{B\Gamma} & A^T_{\Upsilon\Gamma} & A^T_{B\Gamma} \\ A_{\Upsilon\Gamma} & A_{\Upsilon\Upsilon} & 0 \\ A_{B\Gamma} & 0 & A_{BB} \end{bmatrix} \tag{4.8}$$

for the stiff matrix of (4.2).

§5 Numerical Experiments

We present, in this section, some examples of application of Algorithm 2.1 using the Crouzeix–Raviart elements. The algorithm we used is based on the idea of determining two sequences of θ^n_1, θ^n_2 converging as quickly as possible to the optimal values of the relaxation factors θ_1, θ_2, which is given by (3.40). Although the constants σ, τ (given by (3.5) and (3.6)) are not known, we can build up a procedure which generates a sequence of nonconforming discrete harmonic functions on Ω_Υ and Ω_B with the same discrete values on Γ. This allows us to compute, at each iteration, two constants σ_n, τ_n as suggested by (3.5) and (3.6). Using these constants in (3.40) gives values of θ^n_1, θ^n_2 to be used in our numerical scheme. We point out that the evaluation of θ^n_1, θ^n_2 doesn't require the solution of any additional problem in our algorithm (for details cf.[6]). With this choice of the relaxation factors, our numerical experiences show that the theoretical estimates are fully realized in practice.

The model problem is

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega = (0,1)^2 \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

Ω is divided into four subdomains:

$$\Omega_1 = (0, 0.75) \times (0.75, 1), \quad \Omega_2 = (0.75, 1)^2,$$

$$\Omega_3 = (0.75, 1) \times (0, 0.75), \quad \Omega_4 = (0, 0.75)^2.$$

Table 1: The errors when $h = 1/4$

n	0	5	10	15	20
E_n	$0.35 \cdot 10^7$	$0.45 \cdot 10^1$	$0.85 \cdot 10^{-3}$	$0.14 \cdot 10^{-5}$	$0.87 \cdot 10^{-8}$
M_n	$0.10 \cdot 10^4$	$0.83 \cdot 10^0$	$0.14 \cdot 10^{-1}$	$0.47 \cdot 10^{-3}$	$0.38 \cdot 10^{-4}$
δ_n		0.0338	0.0855	0.1303	0.1704
ρ_n		0.1699	0.2893	0.3532	0.4066

Table 2: The errors when $h = 1/8$

n	0	5	10	15	20
E_n	$0.12 \cdot 10^8$	$0.15 \cdot 10^6$	$0.56 \cdot 10^3$	$0.26 \cdot 10^1$	$0.21 \cdot 10^{-1}$
M_n	$0.10 \cdot 10^4$	$0.15 \cdot 10^3$	$0.88 \cdot 10^1$	$0.60 \cdot 10^0$	$0.41 \cdot 10^{-1}$
δ_n		0.3313	0.3296	0.3338	0.3360
ρ_n		0.6236	0.5909	0.5885	0.5879

There exists an internal cross point. We triangulate Ω to get the fine mesh Ω_h such that each element $e \in \Omega_h$ is an isosceles right triangle with h as its diameter. When $h = 1/4$, there are, respectively, 12,4,12,36 elements and 22,8,22,60 interpolation points in $\overline{\Omega}_i, i = 1,2,3,4$. When $h = 1/8$, there are, respectively, 48,16,48,144 elements and 80,28,80,228 interpolation points in $\overline{\Omega}_i, i = 1,2,3,4$. The main experimental results are listed in Table 1 and Table 2. Here, n is the number of iterations, and $n = 0$ represents the initial situation. ε^n is the error function after n iterations. And,

$$E_n = A(\varepsilon^n, \varepsilon^n), \qquad \delta_n = \sqrt[n-1]{E_n/E_1},$$
$$M_n = \|\varepsilon^n\|_{L^\infty(\Omega)}, \qquad \rho_n = \sqrt[n-1]{M_n/M_1}.$$

References

[1] Bourgat, J.F., Glowinski, R., Tallec, P.L., Vidrascu, M.(1989): Variational formulation and algorithm for trace operator in domain decomposition calculations. In: Chan, T.F., et al. (eds.) Domain Decomposition Methods. SIAM, Philadelphia. 3–16

[2] Bramble, J.H., Pasciak, J.E., Schatz, A.H.(1986): The construction of preconditioners for elliptic problems by substructuring. I. Math. Comp. 47, 103–134

[3] Carey, G.F.(1976): An analysis of finite element equations and mesh subdivision. Comput. Methods Appl. Mech. Engrg. 9, 165–179

[4] Ciarlet, P.G.(1978): The Finite Element Method for Elliptic Problems. North–holland, Amsterdam

[5] Crouzeix, M., Raviart, P.A.(1973): Conforming and nonconforming finite element methods for solving the stationary Stokes equations. RAIRO Numer. Anal. 7–R3, 33–76

[6] Gu, J.(1994): Domain Decomposition Methods with Nonconforming Finite Elements. Ph.D. thesis. (in Chinese). Department of Applied Mathematics, Tsinghua University, China

[7] Gu, J., Hu, X.(1993): Trace averaging domain decomposition method with nonconforming finite elements. Submitted

[8] Gu, J., Hu, X.(1993): A simple domain decomposition algorithm with nonconforming elements. Submitted

[9] Gu, J., Hu, X.(1994): On domain decomposition methods in two–subdomain nonoverlap cases. To appear in Chinese Journal of Numerical Mathematics and Applications. Published in USA

[10] Gu, J., Hu, X.(1993): Preconditioners for nonconforming element discrete problems. I. Submitted

[11] Gu, J., Hu, X.(1993): Preconditioners for nonconforming element discrete problems. II. Submitted

[12] Gu, J., Hu, X.(1994): On an essential estimate in the analysis of domain decomposition methods. Journal of Computational Mathematics. 12, 132–137

[13] Gu, J., Hu, X.(1994): Some estimates with nonconforming elements in the analysis of domain decomposition methods. Technical Report No. 94009. Dept. of Applied Math., Tsinghua University, China

[14] Han, H.(1984): Nonconforming elements in the mixed finite element method. Journal of Computational Mathematics. 2:3, 223–233

[15] Lions, J.L., Magenes, E.(1972): Non–homogeneous Boundary Value Problems and Applications. I. Grund. B. 181. Berlin, Springer

[16] Oswald, P.(1992): On a hierarchical basis multilevel method with nonconforming P1 elements. Numer. Math. 62, 189–212.

[17] Widlund, O.B.(1988): Iterative substructuring methods: algorithms and theory for elliptic problems in the plane. In First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, et al.(eds.), SIAM, Philadelphia, 113–128

[18] Zhang, S., Huang, H.(1991): A parallel iterative domain decomposition method for elliptic problems in multi–subdomain cases. (in Chinese). Science in China. Series

A. 12, 1233–1241

Department of Applied Mathematics, Tsinghua University, Beijing 100084, China
E–mail address: yanglp@bepc2.ihep.ac.cn

CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364

Phone: (303) 556-8442
Fax: (303) 556-8550

29. S.E. Payne, "The Fundamental Theorem of q-Clan Geometry."
30. S.E. Payne, "A Tensor Product Action on q-Clan Generalized Quadrangles with $q = 2^e$."
31. J. Mandel, "Iterative Methods for p-Version Finite Elements: Preconditioning Thin Solids."
32. D.C. Fisher, K. Fraughnaugh and L. Langley, "P_3-Connected Graphs of Minimum Size."
33. J. Mandel and R. Tezaur, "Convergence of a Substructuring Method with Lagrange Multipliers."
34. P. Vaněk, J. Mandel and M. Brezina, "Algebraic Multigrid on Unstructured Meshes."
35. P. Vaněk and J. Křížková, "Two-Level Method on Unstructured Meshes with Convergence Rate Independent of the Coarse-Space Size."
36. P. Vaněk, J. Mandel and M. Brezina, "Algebraic Multigrid by Smoothed Aggregation for Second and Fourth Order Elliptic Problems."
37. H. J. Greenberg, "Mathematical Programming Models for Environmental Quality Control."
38. W.L. Briggs and V.E. Henson, "A Table of Analytical Discrete Fourier Transforms."
39. J.S. Gu and X.C. Hu, "Some Estimates with Nonconforming Elements in the Analysis of Domain Decomposition Methods."
40. J.S. Gu and X.C. Hu, "On Domain Decomposition Methods in Two-subdomain Nonoverlap Cases."
41. J.S. Gu and X.C. Hu, "A Simple Domain Decomposition Algorithm with Nonconforming Elements."
42. J.S. Gu and X.C. Hu, "Overlapping Domain Decomposition Method for Nonselfadjoint Elliptic Problems Discretized by Crouzeix-Raviart Elements."
43. J.S. Gu and X.C. Hu, "Trace Averaging Domain Decomposition Method with Nonconforming Finite Elements."
44. J.S. Gu and X.C. Hu, "Preconditioners for Nonconforming Element Discrete Problems. I"