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EXTENSION THEOREMS FOR PLATE ELEMENTS WITH APPLICATIONS

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Abstract

The extension theorems for plate elements are established in this paper. Their applications to the analysis of nonoverlap domain decomposition methods for solving the plate bending problems are presented. Numerical results coincide with our theoretical estimates.

§1 Introduction

Consider the plate bending problem with the clamped boundary conditions

$$\begin{cases} \triangle^2 u = f, & \text{in } \Omega \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \Re^2$ is a polygonal domain and ν is the unit outward normal. The variational form of (1.1) is

$$u \in H_0^2(\Omega): \ a(u,v) = (f,v), \ \forall\, v \in H_0^2(\Omega) \tag{1.2}$$

where $a(u,v) = \int_\Omega \left[\triangle u \triangle v + (1-\gamma)(2\partial_{12}u\partial_{12}v - \partial_{11}u\partial_{22}v - \partial_{22}u\partial_{11}v\right]$, $(f,v) = \int_\Omega fv$, $\gamma \in (0, 0.5)$ is the Poisson ratio. As is well-known, the unique solvability of (1.2) for $f \in L^2(\Omega)$ follows from the continuity and coerciveness of the bilinear form $a(\cdot,\cdot)$ in $H_0^2(\Omega)$ (cf. [15,18] for details).

Suppose that $\Omega_h = \{e\}$ is a quasi–uniform mesh of Ω, i.e., Ω_h satisfies

$$\sup_{e\in\Omega_h} \inf_{B_r \supset e} r \le ch, \quad \inf_{e\in\Omega_h} \sup_{B_r \subset e} r \ge Ch, \tag{1.3}$$

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where e, a triangle, represents the typical element in Ω_h, B_r is a region bounded by the circle of radius r, $h = \max_{e\in\Omega_h} h_e$ is the mesh parameter and $h_e = \inf_{B_r \supset e} r$. Here and later, c and C denote generic positive constants independent of h. Let V_h be the Morley nonconforming finite element space[17] associated with Ω_h. Then $v \in V_h$ if and only if it has the following three properties:

(1) $v|_e$ is quadratic, $\forall\, e \in \Omega_h$;

(2) v is continuous at each vertex p of e, $\forall\, e \in \Omega_h$;

(3) $\frac{\partial v}{\partial \nu}$ is continuous at each edge midpoint m of e, $\forall\, e \in \Omega_h$.

Throughout this paper, we let p and m (with or without subscript) denote a vertex and an edge midpoint of the elements in Ω_h. The Morley element discrete problem of (1.2) is

$$u_h \in V_h^0 : \ A(u_h, v) = (f, v), \ \forall\, v \in V_h^0 \tag{1.4}$$

where $$V_h^0 = \Big\{ v \in V_h : \ v(p) = 0, \ \frac{\partial v}{\partial \nu}(m) = 0, \ \forall\, p, m \in \partial\Omega \Big\},$$

$$A(w, v) = \sum_{e\in\Omega_h} \int_e \Big[\Delta w \Delta v + (1-\gamma)(2\partial_{12} w \partial_{12} v - \partial_{11} w \partial_{22} v - \partial_{22} w \partial_{11} v \Big].$$

As far as the authors know, there have not been as satisfactory results obtained for the fourth order elliptic problems as for the second order ones. Chan et al.[5] presented the interface preconditioners for the biharmonic equations via the finite difference methods where the interface consisits of two grid lines, while Sun [21] constructed the multilevel preconditioners for the biharmonic equations via the B–spline methods. All their methods require that the domain Ω should be a rectangle. Brenner [4] proposed a two–level additive Schwarz preconditioner for nonconforming plate elements through the intergrid transfer operators. Gu [9] studied the parallel Schwarz alternating algorithm for (1.4) and found out the preconditioner implied in the algorithm by employing the idea of Widlund [7].

The purpose of this paper is to give the extension theorem for Morley elements with applications to solving (1.4) and further to point out that the extension theorems for other plate elements[4,6] hold. It is known that the extension theorems play key roles in the analysis of nonoverlap domain decomposition methods for the second order elliptic problems discretized by the finite element methods, conforming or nonoconforming[9,10,13,22]. When considering the nonoverlap domain decomposition methods for the solving of (1.4), we must establish the extension theorem correspondingly. To this end, the conforming interpolation operators introduced in [4] is modified to act as a bridge between Morley nonconforming element space and Argyris conforming element space[2]. And we estimate the error of the Morley element approximate solution of the inhomogeneous boundary value problem.

Hence the extension theorem for Morley elements is developed eventually. To illustrate its applications, we describe and analyse a nonoverlap domain decomposition algorithm with two subdomains. At each iteration of this algorithm, solving the discrete subproblem on one subdomain with the Dirichlet condition on the interface is followed by solving the discrete subproblem on another subdomain with the Neumann condition on the interface. So it is in fact the generalization of the Dirichlet–Neumann alternating method (also known as the Marini–Quarteroni algorithm[16]). Based on the extension theorem, we show that it is geometrically convergent and the convergence factor is independent of h. Numerical results are also presented to indicate that the theoretical estimate is fully realized in practice. It is more important that via the same idea as above, we eventually obtain the extension theorems for all the conforming plate elements[6] and for other nonconforming plate elements[4].

The remainder of this paper is organized as follows. In Sect.2, we describe and prove Theorem 2.4, the extension theorem for Morley elements. Its applications to the analysis of nonoverlap domain decomposition methods and numerical experiments are given in Sect.3. To conclude the paper, We point out that the extension theorems for other plate elements hold.

§2 Extension Theorem for Morley Elements

Let L_e be the linear interpolation operator on e with the vertices of e as its interpolation points. For any measurable set z , we define the mean value operator $M_z : L^2(z) \to \Re$ by $\forall\, v \in L^2(z),\ M_z v \in \Re :\ M_z v = \frac{1}{\text{meas}(z)} \int_z v.$

Lemma 2.1.[12] *If e is affine equivalent to the reference element $\hat{e}$, then*

$$\int_{\partial e} w^2 \le c\{h_e^{-1}\|w\|_{0,e}^2 + h_e|w|_{1,e}^2\},\ \forall\, w \in H^1(e)$$

Theorem 2.2. *Let $\tilde{\Gamma} \subset \partial\Omega$ be an open edge of the polygonal domain Ω. Suppose the functions g_1, g_2 defined on $\partial\Omega$ satisfy $g_1|_{\tilde{\Gamma}} \in H_{00}^{\frac{5}{2}}(\tilde{\Gamma})$, $g_2|_{\tilde{\Gamma}} \in H_{00}^{\frac{3}{2}}(\tilde{\Gamma})$, $g_1|_{\partial\Omega\setminus\tilde{\Gamma}} = g_2|_{\partial\Omega\setminus\tilde{\Gamma}} = 0$. Let $\theta \in H^3(\Omega), \theta_h \in V_h$ be respectively the solutions of the following problems:*

$$\begin{cases} a(\theta, v) = 0, & \forall\, v \in H_0^2(\Omega) \\ \theta = g_1, & on\ \partial\Omega \\ \frac{\partial\theta}{\partial\nu} = g_2, & on\ \partial\Omega \end{cases} \qquad \begin{cases} A(\theta_h, v) = 0, & \forall\, v \in V_h^0 \\ \theta_h(p) = g_1(p), & \forall\, p \in \partial\Omega \\ \frac{\partial\theta_h}{\partial\nu}(m) = g_2(m), & \forall\, m \in \partial\Omega \end{cases}$$

Then we have

$$|\theta - \theta_h|_{h,\Omega} \leq ch|\theta|_{H^3(\Omega)},$$

where $\quad |w|_{h,\Omega} \triangleq \Big(\sum_{e\subset\Omega} |w|^2_{2,e}\Big)^{\frac{1}{2}}.$

Proof. Denote $V_h^* = \{v \in V_h : \ v(p) = \theta_h(p), \ \frac{\partial v}{\partial \nu}(m) = \frac{\partial \theta_h}{\partial \nu}(m), \ \forall \ p, m \in \partial\Omega\}$. It is easy to see that

$$|\theta - \theta_h|_{h,\Omega} \leq c\Big(\inf_{v\in V_h^*} |\theta - v|_{h,\Omega} + \sup_{w\in V_h^0} \frac{|A(\theta, w)|}{|w|_{h,\Omega}}\Big), \tag{2.1}$$

(2.1) is in fact the variant of the second Strang lemma[6] in the nonhomogeneous boundary value case.

Let $w \in V_h^0$. Applying Green's formula, we obtain

$$A(\theta, w) = -\sum_{e\subset\Omega} \int_e \nabla(\triangle\theta) \cdot \nabla w + E_1(\theta, w) + E_2(\theta, w), \tag{2.2}$$

where

$$E_1(\theta, w) = (1-\gamma) \sum_{e\subset\Omega} \int_{\partial e} \frac{\partial^2\theta}{\partial\nu\partial s} \frac{\partial w}{\partial s},$$

$$E_2(\theta, w) = \sum_{e\subset\Omega} \int_{\partial e} \Big[\triangle\theta - (1-\gamma)\frac{\partial^2\theta}{\partial s^2}\Big] \frac{\partial w}{\partial \nu}.$$

Denote $\mathcal{D}(\Omega) = \{v \in C^\infty(\Omega) : \ \text{supp}v \text{ is a compact subset of } \Omega\}$. we note that θ satisfies

$$\sum_{e\subset\Omega} \int_e \nabla(\triangle\theta) \cdot \nabla v = \int_\Omega \nabla(\triangle\theta) \cdot \nabla v = -\int_\Omega \triangle\theta\triangle v = 0, \ \forall \ v \in \mathcal{D}(\Omega)$$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)$, we have

$$\sum_{e\subset\Omega} \int_e \nabla(\triangle\theta) \cdot \nabla v = 0, \ \forall \ v \in H_0^1(\Omega)$$

$$A(\theta, w) = \sum_{e\subset\Omega} \int_e \nabla(\triangle\theta) \cdot \nabla(L_e w - w) + E_1(\theta, w) + E_2(\theta, w).$$

By the Schwarz inequality, Lemma 2.1, the interpolation error estimates[6] and the inverse inequalities[6], we obtain

$$E_1(\theta, w) \leq ch|\theta|_{3,\Omega}|w|_{h,\Omega}, \tag{2.3}$$

$$E_2(\theta, w) \le ch|\theta|_{3,\Omega}|w|_{h,\Omega}. \tag{2.4}$$

Let π_h be the interpolation operator of the Morley element space V_h. Then $\pi_h\theta \in V_h^*$. The standard interpolation error estimate gives

$$\sum_{e\subset\Omega}\int_e \nabla(\triangle\theta)\cdot\triangle(w - L_e w) \le ch|\theta|_{3,\Omega}|w|_{h,\Omega},$$

$$\inf_{v\in V_h^*}|\theta - v|_{h,\Omega} \le |\theta - \pi_h\theta|_{h,\Omega} \le ch|\theta|_{3,\Omega}.$$

By (2.1), (2.2) and the subsequent arguments, we end the proof of the theorem. □

Let Γ be an open straight line which divides Ω into two open subdomains Ω_1 and Ω_2 s.t. $\Omega = \Omega_1\bigcup\Omega_2\bigcup\Gamma$, $\Omega_1\bigcap\Omega_2 = \emptyset$, $\Gamma\bigcap e = \emptyset$, $\forall\ e \in \Omega_h$. For $k = 1, 2$, denote

$$V_h^k = \{v \in V_h^0:\ v(p) = 0,\ \frac{\partial v}{\partial\nu}(m) = 0,\ \forall\ p, m \in \Omega\backslash\overline{\Omega}_k\},$$

$$V_h^{k,0} = \{v \in V_h^0:\ v(p) = 0,\ \frac{\partial v}{\partial\nu}(m) = 0,\ \forall\ p, m \in \Omega\backslash\Omega_k\},$$

$$A_k(w, v) = \sum_{e\subset\Omega_k}\int_e\Big[\triangle w\triangle v + (1-\gamma)(2\partial_{12}w\partial_{12}v - \partial_{11}w\partial_{22}v - \partial_{22}w\partial_{11}v\Big],$$

$$|w|_{h,\Omega_k} \triangleq \Big(\sum_{e\subset\Omega_k}|w|_{2,e}^2\Big)^{\frac{1}{2}}, \qquad (w, v)_k = \int_{\Omega_k} wv.$$

Notice that Brenner [4] has introduced the interpolation operator I_h^k which acts as a bridge between Morley nonconforming element space V_h^k and Argyris conforming element space $\mathcal{AR}_h^k$. Here, $v \in \mathcal{AR}_h^k$ if and only if v satisfies

(1) $v|_e$ is a fifth order polynomial, $\forall\ e \subset \Omega_k$;

(2) $\partial_\alpha v$ $(0 \le |\alpha| \le 2)$ are continuous at each vertex p of e, $\forall\ e \subset \Omega_k$;

(3) $\frac{\partial v}{\partial\nu}$ is continuous at each edge midpoint m of e, $\forall\ e \subset \Omega_k$;

(4) $\partial_\alpha v(p) = 0$, $(0 \le |\alpha| \le 2)$, $\frac{\partial v}{\partial\nu}(m) = 0$, $\forall\ p, m \in \partial\Omega_k\backslash\Gamma$.

Here $\alpha = (\alpha_1, \alpha_2) \in \mathrm{N}^2$ is the multi-index and $|\alpha| = \alpha_1 + \alpha_2$.

For our purpose, we modify I_h^k as follows: $\forall\ v \in V_h^k$, $I_h^k v \in \mathcal{AR}_h^k$ *s.t.*

1. $(I_h^k v)(p) = v(p)$,
2. $\frac{\partial(I_h^k v)}{\partial\nu}(m) = \frac{\partial v}{\partial\nu}(m)$,
3. $\Big[\partial_\alpha(I_h^k v)\Big](p) = 0, \qquad |\alpha| = 2$,
4. $\Big[\partial_\alpha(I_h^k v)\Big](p) = \begin{cases} 0, & |\alpha| = 1,\ p \in \partial\Omega_k\backslash\Gamma \\ \text{average of } \Big(\partial_\alpha v_i\Big)(p), & |\alpha| = 1,\ p \notin \partial\Omega_k\backslash\Gamma \end{cases}$

where $v|_i \triangleq v|_{e_i}$ and e_i contains p as a vertex.

Theorem 2.3. *If I_h^k is defined as above, then*

$$I_h^k v = \frac{\partial(I_h^k v)}{\partial \nu} = 0 \quad on \ \partial\Omega_k \backslash \Gamma, \quad \forall\, v \in V_h^k \tag{2.5}$$

$$\|v - I_h^k v\|_{L^2(\Omega_k)} \le ch^2 |v|_{h,\Omega_k}, \ \forall\, v \in V_h^k \tag{2.6}$$

Proof. (2.5) is implied by the definition of I_h^k. (2.6) can be obtained by modifying the proof of Lemma 5.1[4]. For completeness, we outline it as follows.

Let $v \in V_h^k$ and $e \subset \Omega_k$. Denote $w = v|_e$, $\tilde{w} = (I_h^k v)|_e$. Then

$$w - \tilde{w} = \sum_{i=1}^{3} \sum_{|\alpha|=1,2} \partial_\alpha (w - \tilde{w})(p_i) r_{\alpha,i},$$

where the functions $r_{\alpha,i}$ are the nodal basis functions corresponding to the nodal variables $(\partial_\alpha v)(p_i)$ of the Argyris element space on e.

By the standard techniques of almost affine–equivalent family of finite elements[6], we see that

$$\|r_{\alpha,i}\|_{L^2(e)} \le ch^2 \quad \text{for } |\alpha| = 1,$$

$$\|r_{\alpha,i}\|_{L^2(e)} \le ch^3 \quad \text{for } |\alpha| = 2.$$

If $|\alpha| = 2$, then

$$|\partial_\alpha (w - \tilde{w})(p_i)| = |\partial_\alpha w(p_i)| \le |v|_{W^2_\infty(e)} \le ch^{-1}|v|_{2,e}.$$

We next discuss the case that $|\alpha| = 1$.

Suppose that $p_1 \in \partial\Omega_k \backslash \Gamma$. Since Ω_h is quasi–uniform, there exists a positive integer J, independent of h, such that $e_1, e_2, \cdots, e_J = e \subset \Omega_k$, $e_1, e_2, \cdots, e_J$ contain p_1 as a common vertex, $\text{meas}(\partial e_j \bigcap \partial e_{j+1}) > 0$, $j = 1, 2, \cdots, J-1$ and $\text{meas}(\partial e_1 \bigcap (\partial\Omega_k \backslash \Gamma)) > 0$.

By Taylor's formula and the fact that $v|_{e_j}$ and $v|_{e_{j-1}}$ agree at the two endpoints of $\partial e_j \bigcap \partial e_{j-1}$, it is easy to obtain

$$\left|\frac{\partial(v|_{e_j})}{\partial s}(p_1) - \frac{\partial(v|_{e_{j-1}})}{\partial s}(p_1)\right| \le \frac{h}{2}\Big[|v|_{W^2_\infty(e_j)} + |v|_{W^2_\infty(e_{j-1})}\Big],$$

where s is the arc length along $\partial e_j \bigcap \partial e_{j-1}$. Similarly since $\frac{\partial(v|_{e_j})}{\partial \nu}$ and $\frac{\partial(v|_{e_{j-1}})}{\partial \nu}$ agree at the midpoint of $\partial e_j \bigcap \partial e_{j-1}$, we get

$$\left|\frac{\partial(v|_{e_j})}{\partial \nu}(p_1) - \frac{\partial(v|_{e_{j-1}})}{\partial \nu}(p_1)\right| \le \frac{h}{2}\Big[|v|_{W^2_\infty(e_j)} + |v|_{W^2_\infty(e_{j-1})}\Big].$$

Therefore, we have

$$|\partial_\alpha(v|_{e_j})(p_1) - \partial_\alpha(v|_{e_{j-1}})(p_1)| \le ch\Big[|v|_{W^2_\infty(e_j)} + |v|_{W^2_\infty(e_{j-1})}\Big].$$

On the other hand, let $p_1' \in \partial e_1 \bigcap(\partial\Omega_k\backslash\Gamma)$ be another endpoint of the edge $\partial e_1 \bigcap(\partial\Omega_k\backslash\Gamma)$. Since $v(p_1) = v(p_1') = 0$, there exists a point $q \in \partial e_1 \bigcap(\partial\Omega_k\backslash\Gamma)$, s.t. $\frac{\partial(v|_{e_1})}{\partial s}(q) = 0$. Obviously, $\frac{\partial(v|_{e_1})}{\partial\nu}(m_1) = 0$, where m_1 is the midpoint of the edge $\partial e_1 \bigcap(\partial\Omega_k\backslash\Gamma)$. Then

$$|\frac{\partial(v|_{e_1})}{\partial s}(p_1)| = |\frac{\partial(v|_{e_1})}{\partial s}(p_1) - \frac{\partial(v|_{e_1})}{\partial s}(q)| \le h|v|_{W^2_\infty(e_1)},$$

$$|\frac{\partial(v|_{e_1})}{\partial\nu}(p_1)| = |\frac{\partial(v|_{e_1})}{\partial\nu}(p_1) - \frac{\partial(v|_{e_1})}{\partial\nu}(m_1)| \le h|v|_{W^2_\infty(e_1)}.$$

So $\qquad |\partial_\alpha(v|_{e_1})(p_1)| \le ch|v|_{W^2_\infty(e_1)}.$

$$\begin{aligned}|\partial_\alpha(w-\tilde{w})(p_1)| &= |\partial_\alpha(v|_e)(p_1)| \\ &= |\sum_{j=2}^{J}\Big[\partial_\alpha(v|_{e_j})(p_1) - \partial_\alpha(v|_{e_{j-1}})(p_1)\Big] + \partial_\alpha(v|_{e_1})(p_1)| \\ &\le \sum_{j=2}^{J}|\partial_\alpha(v|_{e_j})(p_1) - \partial_\alpha(v|_{e_{j-1}})(p_1)| + |\partial_\alpha(v|_{e_1})(p_1)| \\ &\le ch\sum_{j=1}^{J}|v|_{W^2_\infty(e_j)} \le c\sum_{j=1}^{J}|v|_{2,e_j} \le c\sum_{e'}|v|_{2,e'},\end{aligned}$$

where $e' \subset \Omega_k$ s.t. $\partial e' \bigcap \partial e \ne \emptyset$.

If $p_1 \notin \partial\Omega_k\backslash\Gamma$, then by the same argument as above, we can easily obtain

$$|\partial_\alpha(w-\tilde{w})(p_1)| \le c\sum_{e'}|v|_{2,e'}.$$

Therefore, we have

$$\|v - I_h^k v\|_{L^2(e)} \le ch^2\sum_{e'}|v|_{2,e'}.$$

Summing up the square of the last inequality over all the element $e \subset \Omega_k$, we eventually get (2.6). □

In what follows, $\{p_i\}_{i=1}^{I}$ denotes the set of the vertices on Γ and $\{m_j\}_{j=1}^{J}$ denotes the set of the edge midpoints on Γ. Let ν_k $(k = 1, 2)$ be the unit outward normal vector of Ω_k. $r_0 : V_h \to \Re^I$ and $r_1 : V_h \to \Re^J$ denote respectively the discrete operators such that

$$\forall\, v \in V_h,\ r_0 v \in \Re^I : \ \Big(r_0 v\Big)(i) = v(p_i),\ i = 1, 2, \cdots, I;$$

$$\forall\, w \in V_h,\ r_1 w \in \Re^J : \ \Big(r_1 w\Big)(j) = \frac{\partial w}{\partial \nu_1}(m_j),\ j = 1, 2, \cdots, J.$$

Define the discrete biharmonic extension operator $E_h^k : \Re^I \times \Re^J \to V_h^k$ as follows

$$\forall\, (\lambda, \mu) \in \Re^I \times \Re^J,\ E_h^k(\lambda, \mu) \in V_h^k : \ \begin{cases} A_k(E_h^k(\lambda,\mu), v) = 0, & \forall\, v \in V_h^{k,0} \\ r_0 E_h^k(\lambda,\mu) = \lambda \\ r_1 E_h^k(\lambda,\mu) = \mu \end{cases}$$

Theorem 2.4. (Extension theorem for Morley elements) *There exist two constants σ, τ, independent of the quasi-uniform mesh parameter h, such that*

$$\sigma = \sup_{(\lambda,\mu)\in\Re^I\times\Re^J} \frac{A_1(E_h^1(\lambda,\mu), E_h^1(\lambda,\mu))}{A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu))} < \infty \tag{2.7}$$

$$\tau = \sup_{(\lambda,\mu)\in\Re^I\times\Re^J} \frac{A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu))}{A_1(E_h^1(\lambda,\mu), E_h^1(\lambda,\mu))} < \infty \tag{2.8}$$

Proof. Let $(\lambda, \mu) \in \Re^I \times \Re^J$. Denote $u_k^h = E_h^k(\lambda, \mu)$ for convenience. With the inverse inequality and Theorem 2.3, we have

$$|u_1^h - I_h^1 u_1^h|_{h,\Omega_1}^2 = \sum_{e\subset\Omega_1} |u_1^h - I_h^1 u_1^h|_{2,e}^2 \le c h^{-4} \|u_1^h - I_h^1 u_1^h\|_{L^2(\Omega_1)}^2 \le c|u_1^h|_{h,\Omega_1}^2.$$

Therefore, $I_h^1 u_1^h \in \mathcal{AR}_h^1 \subset H^2(\Omega_1)$ and the triangle inequality yield

$$|I_h^1 u_1^h|_{H^2(\Omega_1)} = |I_h^1 u_1^h|_{h,\Omega_1} \le c|u_1^h|_{h,\Omega_1}.$$

Furthermore, applying the trace theorem and the Poincaré inequality in $H^2(\Omega_1)$ gives

$$\begin{aligned} &\|I_h^1 u_1^h\|_{H_{00}^{\frac{3}{2}}(\Gamma)}^2 + \|\tfrac{\partial(I_h^1 u_1^h)}{\partial\nu_1}\|_{H_{00}^{\frac{1}{2}}(\Gamma)}^2 \le c\Big(\|I_h^1 u_1^h\|_{H^{\frac{3}{2}}(\partial\Omega_1)}^2 + \|\tfrac{\partial(I_h^1 u_1^h)}{\partial\nu_1}\|_{H^{\frac{1}{2}}(\partial\Omega_1)}^2\Big) \\ &\le c\|I_h^1 u_1^h\|_{H^2(\Omega_1)}^2 \le c|I_h^1 u_1^h|_{H^2(\Omega_1)}^2 \le c|u_1^h|_{h,\Omega_1}^2 \le cA_1(u_1^h, u_1^h). \end{aligned} \tag{2.9}$$

Construct the following continuous problem:

$$\begin{cases} a(u_2, v) = 0, & \forall\, v \in H_0^2(\Omega_2) \\ u_2 = \frac{\partial}{\partial\nu_2} u_2 = 0, & \text{on } \partial\Omega_2\backslash\Gamma \\ u_2 = I_h^1 u_1^h,\ \frac{\partial}{\partial\nu_2} u_2 = -\frac{\partial}{\partial\nu_1}(I_h^1 u_1^h), & \text{on } \Gamma \end{cases} \tag{2.10}$$

Note that u_2^h is the Morley approximation of u_2. By Theorem 2.2, we obtain

$$\begin{aligned} A_2(u_2^h, u_2^h) &\le 2\Big(A_2(u_2, u_2) + A_2(u_2 - u_2^h, u_2 - u_2^h)\Big) \\ &\le c\Big(\|u_2\|^2_{H^2(\Omega_2)} + h^2\|u_2\|^2_{H^3(\Omega_2)}\Big) \end{aligned}$$

The well–known *a priori* inequalities of the elliptic problem (2.10) yield[8,15,18]

$$\begin{aligned} \|u_2\|^2_{H^2(\Omega_2)} &\le c\Big(\|u_2\|^2_{H^{\frac{3}{2}}(\partial\Omega_2)} + \|\tfrac{\partial}{\partial\nu_2}u_2\|^2_{H^{\frac{1}{2}}(\partial\Omega_2)}\Big), \\ \|u_2\|^2_{H^3(\Omega_2)} &\le c\Big(\|u_2\|^2_{H^{\frac{5}{2}}(\partial\Omega_2)} + \|\tfrac{\partial}{\partial\nu_2}u_2\|^2_{H^{\frac{3}{2}}(\partial\Omega_2)}\Big). \end{aligned}$$

Since u_2 and $\frac{\partial}{\partial\nu_2}u_2$ are piecewise polynomial on $\partial\Omega_2$, applying the fractional order inverse inequalities implied by the interpolation theorem of Sobolev spaces[1], we see that

$$|u_2|^2_{H^3(\Omega_2)} \le ch^{-2}\Big(\|u_2\|^2_{H^{\frac{3}{2}}(\partial\Omega_2)} + \|\tfrac{\partial}{\partial\nu_2}u_2\|^2_{H^{\frac{1}{2}}(\partial\Omega_2)}\Big).$$

With above inequalities, we get

$$\begin{aligned} A_2(u_2^h, u_2^h) &\le c\Big(\|u_2\|^2_{H^{\frac{3}{2}}(\partial\Omega_2)} + \|\tfrac{\partial}{\partial\nu_2}u_2\|^2_{H^{\frac{1}{2}}(\partial\Omega_2)}\Big) \\ &\le c\Big(\|I_h^1u_1^h\|^2_{H_{00}^{\frac{3}{2}}(\Gamma)} + \|\tfrac{\partial(I_h^1u_1^h)}{\partial\nu_1}\|^2_{H_{00}^{\frac{1}{2}}(\Gamma)}\Big). \end{aligned} \tag{2.11}$$

Consequently, (2.8) follows from the arbitrariness of $(\lambda,\mu) \in \Re^I \times \Re^J$, (2.9) and (2.11). (2.7) can be established in the same manner. □

Define the discrete extension operator $T_h^k : \Re^I \times \Re^J \to V_h^k$ as follows

$$\forall\, (\lambda,\mu) \in \Re^I \times \Re^J,\ T_h^1(\lambda,\mu) \in V_h^1 :\ A_1(T_h^1(\lambda,\mu), v) = -A_2(E_h^2(\lambda,\mu), v),\ \forall\, v \in V_h^0$$

$$\forall\, (\lambda,\mu) \in \Re^I \times \Re^J,\ T_h^2(\lambda,\mu) \in V_h^2 :\ A_2(T_h^2(\lambda,\mu), v) = -A_1(E_h^1(\lambda,\mu), v),\ \forall\, v \in V_h^0$$

Corollary 2.5. *Let σ, τ be the same as those in Theorem 2.4. For any $(\lambda,\mu) \in \Re^I \times \Re^J$, we have*

$$\frac{1}{\tau}A_1(E_h^1(\lambda,\mu), E_h^1(\lambda,\mu)) \le A_2(T_h^2(\lambda,\mu), T_h^2(\lambda,\mu)) \le \sigma A_1(E_h^1(\lambda,\mu), E_h^1(\lambda,\mu)),$$

$$\frac{1}{\sigma}A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu)) \le A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu)) \le \tau A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu)).$$

§3 Applications to Domain Decomposition Analysis

Let's take the following domain decomposition algorithm as an example to illustrate the applications of Theorem 2.4.

Algorithm 3.1.

Step 1. Let $(\lambda^0, \mu^0) \in \Re^I \times \Re^J$ be given arbitrarily. Set n:=1.

Step 2. Find $u_1^n \in V_h^1$ by solving the subproblem on Ω_1:

$$\begin{cases} A_1(u_1^n, v) = (f, v)_1, \ \forall\, v \in V_h^{1,0} \\ r_0 u_1^n = \lambda^{n-1} \\ r_1 u_1^n = \mu^{n-1} \end{cases}$$

Step 3. Find $u_2^n \in V_h^2$ by solving the subproblem on Ω_2:

$$A_2(u_2^n, v) = -A_1(u_1^n, v) + (f, v), \quad \forall\, v \in V_h^2$$

Step 4. Select the relaxation factor $\theta_n \in (0,1)$ and calculate

$$\lambda^n = \theta_n r_0 u_2^n + (1-\theta_n)\lambda^{n-1}, \quad \mu^n = \theta_n r_1 u_2^n + (1-\theta_n)\mu^{n-1}.$$

Set n:=n+1, return to Step 2 until some reasonable stopping criterion is satisfied.

Theorem 3.1. *Let u_h be the solution of (1.4). Let $u_1^n, u_2^n, \lambda^n, \mu^n$ be the iteration values of Algorithm 3.1. Let $\varepsilon_k^n \in V_h^k$ s.t. $\varepsilon_k^n(p) = u_k^n(p) - u_h(p)$, $\frac{\partial \varepsilon_k^n}{\partial \nu}(m) = \frac{\partial u_k^n}{\partial \nu}(m) - \frac{\partial u_h}{\partial \nu}(m)$, $\forall\, p, m \in \overline{\Omega}_k$. Denote $\delta^n = \lambda^n - r_0 u_h$, $\eta^n = \mu^n - r_1 u_h$. Then*

1)

$$\frac{1}{\tau} A_1(\varepsilon_1^n, \varepsilon_1^n) \le A_2(\varepsilon_2^n, \varepsilon_2^n) \le \sigma A_1(\varepsilon_1^n, \varepsilon_1^n). \tag{3.1}$$

2) There exists a constant $\theta^* \in (0,1]$, such that

$$A_1(\varepsilon_1^{n+1}, \varepsilon_1^{n+1}) \le \kappa(\theta_n) A_1(\varepsilon_1^n, \varepsilon_1^n), \tag{3.2}$$

where $\kappa(\theta_n) < 1, \ \forall\, \theta_n \in (0, \theta^*)$.

3) There exists the optimal relaxation factor θ^{opt}, such that

$$\kappa(\theta^{opt}) = \min_{\theta \in (0,\theta^*)} \kappa(\theta). \tag{3.3}$$

Proof. It is easy to see that $\varepsilon_k^{n+1} \in V_h^k$ satisfies

$$\begin{cases} A_1(\varepsilon_1^{n+1}, v) = 0, \ \forall\, v \in V_h^{1,0} \\ r_0 \varepsilon_1^{n+1} = \delta^n \\ r_1 \varepsilon_1^{n+1} = \eta^n \end{cases} \tag{3.4}$$

$$A_2(\varepsilon_2^{n+1}, v) = -A_1(\varepsilon_1^{n+1}, v), \ \forall\, v \in V_h^2 \tag{3.5}$$

$$\delta^{n+1} = \theta_{n+1} r_0 \varepsilon_2^{n+1} + (1-\theta_{n+1})\delta^n, \quad \eta^{n+1} = \theta_{n+1} r_1 \varepsilon_2^{n+1} + (1-\theta_{n+1})\eta^n. \tag{3.6}$$

(3.4) and (3.5) yield

$$A_2(\varepsilon_2^{n+1}, v) = -A_1(\varepsilon_1^{n+1}, v), \ \forall\, v \in V_h^0. \tag{3.7}$$

Therefore $\varepsilon_1^{n+1} = E_h^1(\delta^n, \eta^n)$, $\varepsilon_2^{n+1} = T_h^2(\delta^n, \eta^n)$. By Corollary 2.5, we get (3.1).

On the other hand, it follows from (3.4) and (3.6) that

$$\varepsilon_1^{n+1} = \theta_n E_h^1(r_0\varepsilon_2^n, r_1\varepsilon_2^n) + (1-\theta_n)\varepsilon_1^n,$$

$$\begin{aligned} A_1(\varepsilon_1^{n+1}, \varepsilon_1^{n+1}) = \ & \theta_n^2 A_1(E_h^1(r_0\varepsilon_2^n, r_1\varepsilon_2^n), E_h^1(r_0\varepsilon_2^n, r_1\varepsilon_2^n)) \\ & +2\theta_n(1-\theta_n) A_1(E_h^1(r_0\varepsilon_2^n, r_1\varepsilon_2^n), \varepsilon_1^n) \\ & +(1-\theta_n)^2 A_1(\varepsilon_1^n, \varepsilon_1^n). \end{aligned} \tag{3.8}$$

By (2.7) and (3.1), we see that

$$A_1(E_h^1(r_0\varepsilon_2^n, r_1\varepsilon_2^n), E_h^1(r_0\varepsilon_2^n, r_1\varepsilon_2^n)) \le \sigma A_2(\varepsilon_2^n, \varepsilon_2^n) \le \sigma^2 A_1(\varepsilon_1^n, \varepsilon_1^n). \tag{3.9}$$

(3.7) gives $A_2(\varepsilon_2^{n+1}, \varepsilon_2^{n+1}) = -A_1(\varepsilon_1^{n+1}, E_h^1(r_0\varepsilon_2^{n+1}, r_1\varepsilon_2^{n+1}))$. So by (3.1), we get

$$A_1(E_h^1(r_0\varepsilon_2^n, r_1\varepsilon_2^n), \varepsilon_1^n) = -A_2(\varepsilon_2^n, \varepsilon_2^n) \le -\frac{1}{\tau} A_1(\varepsilon_1^n, \varepsilon_1^n). \tag{3.10}$$

If $0 < \theta_n < 1$, then (3.2) follows from (3.8), (3.9) and (3.10). Furthermore, $\kappa(\theta_n)$ has the following expression

$$\kappa(\theta_n) = \frac{1}{\tau}\Big(\theta_n^2(\sigma^2\tau + \tau + 2) - 2\theta_n(\tau+1) + \tau\Big). \tag{3.11}$$

An elementary calculation indicates that if and only if

$$0 < \theta_n < \theta^* = \min\Big(1, \frac{2(\tau+1)}{\sigma^2\tau + \tau + 2}\Big),$$

then $0 \le \kappa(\theta_n) < 1$. Besides, the optimal relaxation factor

$$\theta^{opt} = \frac{\tau+1}{\sigma^2\tau + \tau + 2}, \tag{3.12}$$

satisfies

$$\kappa(\theta^{opt}) = \frac{\sigma^2\tau^2 - 1}{\tau(\sigma^2\tau + \tau + 2)} = \min_{\theta \in (0,\theta^*)} \kappa(\theta). \tag{3.13}$$

So (3.3) follows from (3.13). $\square$

Algorithm 3.1 is the generalization of the so-called Dirichlet–Neumann alternative method or the Marini–Quarteroni method[16] to plate bending problems. Its essence lies in the continual correction of the initial guess of (r_0u_h, r_1u_h) by imposing $r_0(\triangle u_1^n) = r_0(\triangle u_2^n)$, $r_1(\triangle u_1^n) = r_1(\triangle u_2^n)$ at each iteration. Theorem 3.1 implies that Algorithm 3.1 converges geometrically and independently of h, which is guaranteed by the extension theorem (Theorem 2.4). In the special case that (1.1) and the domain Ω are symmetric in respect of Γ, then $\sigma = \tau = 1$ in (2.7) and (2.8), thus $\theta^{opt} = \frac{1}{2}$, $\kappa(\theta^{opt}) = 0$ in (3.3), which indicates that only one iteration is needed to obtain the solution of (1.4).

Of course, other algorithms in [10] can be generalized and their analysis can be carried out similarly, based on Theorem 2.4.

To conclude this section, we present some numerical results.

Decompose the domain $\Omega = (0, 1.5) \times (0, 1) \bigcup (0, 1) \times [1, 2)$ into subdomains: $\Omega_1 = (0, 1) \times (1, 2)$, $\Omega_2 = (0, 1.5) \times (0, 1)$. Triangulate Ω to get the fine mesh Ω_h so that each element $e \in \Omega_h$ is an isosceles right triangle with h as its diameter. When $h = 0.25$, there are 32, 48 elements and 81, 117 interpolation points in Ω_1, Ω_2 respectively. When $h = 0.125$, there are 128, 192 elements and 289, 425 interpolation points in Ω_1, Ω_2 respectively. When $h = 0.0625$, there are 512, 768 elements and 1089, 1617 interpolation points in Ω_1, Ω_2 respectively. In the above three cases, there are 7, 15 and 31 interpolation points on Γ respectively. For an edge midpoint m, if $m \in \partial e_1 \cap \partial e_2$, then the outward normal of e_1 and e_2 at m are opposite. To ensure that $\frac{\partial v}{\partial \nu}(m)$ are determined uniquely, we require that the outward (inward) normals should be chosen for the triangular elements with even (odd) numbers. In the following tables, n is the number of iterations and ε^n is the error after n iterations.

$$\|\varepsilon^n\|_A = A(\varepsilon^n, \varepsilon^n), \qquad \rho_n = \sqrt[n]{\|\varepsilon^n\|_A / \|\varepsilon^0\|_A},$$
$$\|\varepsilon^n\|_\infty = \|\varepsilon^n\|_{L^\infty(\Omega)}, \qquad \delta_n = \sqrt[n]{\|\varepsilon^n\|_\infty / \|\varepsilon^0\|_\infty}.$$

When using Algorithm 3.1 to solve (1.4), a procedure is built up to generate a sequence of the discrete biharmonic functions on Ω_1 and Ω_2 with the same values at $p, m \in \Gamma$. This allows us to compute, at each iteration, two constants σ_n, τ_n suggested by (2.7) and (2.8), which combining (3.12) gives the sequence of approximate values θ_n of the optimal relaxation factor θ^{opt}. We point out that the evaluation of θ_n doesn't require the solution of any additional problem in our algorithm (for details, cf.[16]). The main experimental results, obtained in SGI work station, are listed in Table 1 and Table 2, which coincide with our theoretical analysis.

Table 1: Error reduction factor ρ_n vs. h

n	1	5	9	13
$h = 0.2500$	0.0253	0.0343	0.0261	0.0395
$h = 0.1250$	0.0272	0.0476	0.0438	0.0558
$h = 0.0625$	0.0264	0.0513	0.0473	0.0464

Table 2: The errors $\|\varepsilon^n\|_A$ and $\|\varepsilon^n\|_\infty$ when $h = 0.0625$

n	1	5	9	13
$\|\varepsilon^n\|_A$	$0.487 \cdot 10^9$	$0.129 \cdot 10^4$	$0.532 \cdot 10^{-1}$	$0.718 \cdot 10^{-6}$
$\|\varepsilon^n\|_\infty$	$0.952 \cdot 10^4$	$0.837 \cdot 10^2$	$0.261 \cdot 10^0$	$0.459 \cdot 10^{-3}$

§4 Extension Theorems for Other Plate Elements

Let V_h be Fraeijs de Veubeke element space $\mathcal{F}_h$ or Zienkiewicz element space $\mathcal{Z}_h$ or Adini element space $\mathcal{AD}_h$ described in [4]. As in Sect.2, we can define the discrete biharmonic operator $E_h^k : \Re^M \to V_h^k$ correspondingly. Here M denotes the number of the freedoms associated with the interface Γ.

Theorem 4.1. (Extension theorem) *If V_h is one of the three nonconforming plate element spaces $\mathcal{F}_h$, $\mathcal{Z}_h$ and $\mathcal{AD}_h$, then there exist two constants $\hat{\sigma}, \hat{\tau}$, independent of the quasi-uniform mesh parameter h, such that*

$$\hat{\sigma} = \sup_{\eta \in \Re^M} \frac{A_1(E_h^1\eta, E_h^1\eta)}{A_2(E_h^2\eta, E_h^2\eta)} < \infty, \quad \hat{\tau} = \sup_{\eta \in \Re^M} \frac{A_2(E_h^2\eta, E_h^2\eta)}{A_1(E_h^1\eta, E_h^1\eta)} < \infty. \tag{4.1}$$

We can adopt the clues of the proof of Theorem 2.4 to prove Theorem 4.1 with the following points in mind:

1. The error of the nonconforming approximate solution of the inhomogeneous boundary value problem can be estimated by first getting the inequality similar to (2.1), subtracting off appropriate "conforming" parts in (2.2) and then applying the bilinear lemma[6], which can be referred to [3,6,14,20].

2. The conforming interpolation operator I_h^k must be constructed by similarly modifying the corresponding one introduced in [4] and Theorem 2.3 holds still in this case.

Theorem 4.2. (Extension theorem) *If V_h is one of the conforming plate element spaces*[6]*, then (4.1) holds.*

Since it is unnecessary to construct the conforming interpolation operator I_h^k in this case, the proof of Theorem 4.2 is much simpler than the proof of Theorem 2.4, so we omit it here.

By now, we have proved the extension theorems for plate elements, conforming or nonconforming. Their further applications will be given in the forthcoming papers.

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Appendix for the Referees

The proof of (2.1). $\forall\ v \in V_h^*$, it is easy to see that

$$\begin{aligned} c|\theta_h - v|_{h,\Omega}^2 &\le A(\theta_h - v, \theta_h - v) \\ &= A(\theta - v, \theta_h - v) + A(\theta_h, \theta_h - v) - A(\theta, \theta_h - v) \\ &\le C|\theta - v|_{h,\Omega}|\theta_h - v|_{h,\Omega} + 0 + |A(\theta, \theta_h - v)|. \end{aligned}$$

Hence

$$\begin{aligned} |\theta_h - v|_{h,\Omega} &\le c\Big\{|\theta - v|_{h,\Omega} + \frac{|A(\theta,\theta_h - v)|}{|\theta_h - v|_{h,\Omega}}\Big\} \\ &\le c\Big\{|\theta - v|_{h,\Omega} + \sup_{w\in V_h^0}\frac{|A(\theta,w)|}{|w|_{h,\Omega}}\Big\},\ \forall\ v \in V_h^* \end{aligned}$$

By the triangle inequality, we get (2.1). □

The proof of (2.3) & (2.4). For each edge F, if $F = \partial e \bigcap \partial\Omega$ for some e, it is obvious that $\frac{\partial}{\partial s}(L_e w) = 0$; if $F = \partial e_1 \bigcap \partial e_2$ for some e_1, e_2, then $\frac{\partial}{\partial s}(L_{e_1} w)|_F = -\frac{\partial}{\partial s}(L_{e_2} w)|_F$. Besides, for each e, $\int_{\partial e} \frac{\partial(w - L_e w)}{\partial s} = 0$. By the Schwarz inequality, Lemma 2.1, the interpolation error estimates[6] and the inverse inequalities[6], we obtain

$$\begin{aligned} E_1(\theta, w) &= (1-\gamma)\sum_{e\subset\Omega}\int_{\partial e}\Big[\frac{\partial^2\theta}{\partial\nu\partial s} - M_e(\frac{\partial^2\theta}{\partial\nu\partial s})\Big]\frac{\partial(w - L_e w)}{\partial s} \\ &\le (1-\gamma)\Big(\sum_{e\subset\Omega}\int_{\partial e}|\frac{\partial^2\theta}{\partial\nu\partial s} - M_e(\frac{\partial^2\theta}{\partial\nu\partial s})|^2\Big)^{\frac{1}{2}}\Big(\sum_{e\subset\Omega}\int_{\partial e}|\frac{\partial(w - L_e w)}{\partial s}|^2\Big)^{\frac{1}{2}} \\ &= (1-\gamma)\Big(\sum_{e\subset\Omega}\int_{\partial e}|\sum_{|\alpha|=2}[C_\alpha\partial_\alpha\theta - M_e(C_\alpha\partial_\alpha\theta)]|^2\Big)^{\frac{1}{2}} \\ &\quad \Big(\sum_{e\subset\Omega}\int_{\partial e}|\sum_{|\alpha|=1}C_\alpha\partial_\alpha(w - L_e w)|^2\Big)^{\frac{1}{2}} \\ &\le c\Big(\sum_{e\subset\Omega}\sum_{|\alpha|=2}\int_{\partial e}|\partial_\alpha\theta - M_e(\partial_\alpha\theta)|^2\Big)^{\frac{1}{2}}\Big(\sum_{e\subset\Omega}\sum_{|\alpha|=1}\int_{\partial e}|\partial_\alpha(w - L_e w)|^2\Big)^{\frac{1}{2}} \\ &\le c\Big(\sum_{e\subset\Omega}\sum_{|\alpha|=2}[h_e^{-1}\|\partial_\alpha\theta - M_e(\partial_\alpha\theta)\|_{0,e}^2 + h_e|\partial_\alpha\theta - M_e(\partial_\alpha\theta)|_{1,e}^2]\Big)^{\frac{1}{2}} \\ &\quad \Big(\sum_{e\subset\Omega}\sum_{|\alpha|=1}[h_e^{-1}\|\partial_\alpha(w - L_e w)\|_{0,e}^2 + h_e|\partial_\alpha(w - L_e w)|_{1,e}^2\Big)^{\frac{1}{2}} \\ &\le c\Big(\sum_{e\subset\Omega}\sum_{|\alpha|=2}h_e|\partial_\alpha\theta|_{1,e}^2\Big)^{\frac{1}{2}}\Big(\sum_{e\subset\Omega}\sum_{|\alpha|=1}h_e|\partial_\alpha w|_{1,e}^2\Big)^{\frac{1}{2}} \\ &\le ch|\theta|_{3,\Omega}|w|_{h,\Omega}. \end{aligned}$$

Therefore, (2.3) is established.

We notice that for each edge F, if $F \subset \partial\Omega$, it is obvious that $\int_F \frac{\partial w}{\partial \nu} = 0$; if $F = \partial e_1 \cap \partial e_2$ for some elements e_1, e_2, then $\sum_{i=1}^{2} \int_{F\subset\partial e_i} \frac{\partial w}{\partial \nu} = 0$. In addition, $\int_F \left[\frac{\partial w}{\partial \nu} - M_F(\frac{\partial w}{\partial \nu})\right] = 0$ for each edge F. In the same manner as above, we have

$$\begin{aligned} E_2(\theta, w) &= \sum_{e\subset\Omega} \sum_{F\subset\partial e} \int_F \left[\triangle\theta - (1-\gamma)\frac{\partial^2\theta}{\partial s^2} - M_F\left(\triangle\theta - (1-\gamma)\frac{\partial^2\theta}{\partial s^2}\right)\right]\left[\frac{\partial w}{\partial \nu} - M_F(\frac{\partial w}{\partial \nu})\right] \\ &\leq ch|\theta|_{3,\Omega}|w|_{h,\Omega}. \end{aligned}$$

So we get (2.4). □

<u>**The proof of Corollary 2.5.**</u> Let $(\lambda, \mu) \in \Re^I \times \Re^J$. Take $v \in V_h^0$ s.t. $v = E_h^k(\lambda, \mu)$ on Ω_k, $k = 1, 2$. Then by the definition of T_h^1 and (2.7), we see that

$$\begin{aligned} A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu)) &= -A_1(T_h^1(\lambda,\mu), E_h^1(\lambda,\mu)) \\ &\leq \Big(A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu))\Big)^{\frac{1}{2}} \Big(A_1(E_h^1(\lambda,\mu), E_h^1(\lambda,\mu))\Big)^{\frac{1}{2}} \\ &\leq \Big(A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu))\Big)^{\frac{1}{2}} \Big(\sigma A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu))\Big)^{\frac{1}{2}}. \end{aligned}$$

So

$$A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu)) \geq \frac{1}{\sigma} A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu)).$$

Take $v \in V_h^0$ s.t. $v = \begin{cases} T_h^1(\lambda,\mu), & \text{on } \Omega_1 \\ E_h^2(r_0T_h^1(\lambda,\mu), r_1T_h^1(\lambda,\mu)), & \text{on } \Omega_2 \end{cases}$. Then by the definition of T_h^1 and (2.8), we get

$$\begin{aligned} A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu)) &= -A_2(E_h^2(\lambda,\mu), E_h^2(r_0T_h^1(\lambda,\mu), r_1T_h^1(\lambda,\mu))) \\ &\leq \Big(A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu))\Big)^{\frac{1}{2}} \\ &\quad \Big(A_2(E_h^2(r_0T_h^1(\lambda,\mu), r_1T_h^1(\lambda,\mu)), E_h^2(r_0T_h^1(\lambda,\mu), r_1T_h^1(\lambda,\mu)))\Big)^{\frac{1}{2}} \\ &\leq \Big(A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu))\Big)^{\frac{1}{2}} \\ &\quad \Big(\tau A_1(E_h^1(r_0T_h^1(\lambda,\mu), r_1T_h^1(\lambda,\mu)), E_h^1(r_0T_h^1(\lambda,\mu), r_1T_h^1(\lambda,\mu)))\Big)^{\frac{1}{2}} \\ &= \Big(A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu))\Big)^{\frac{1}{2}} \Big(\tau A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu))\Big)^{\frac{1}{2}}. \end{aligned}$$

So we have

$$A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu)) \leq \tau A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu)).$$

Combining the above inequalities yields

$$\frac{1}{\sigma} A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu)) \leq A_1(T_h^1(\lambda,\mu), T_h^1(\lambda,\mu)) \leq \tau A_2(E_h^2(\lambda,\mu), E_h^2(\lambda,\mu)).$$

In the same manner, we obtain

$$\frac{1}{\tau} A_1(E_h^1(\lambda,\mu), E_h^1(\lambda,\mu)) \leq A_2(T_h^2(\lambda,\mu), T_h^2(\lambda,\mu)) \leq \sigma A_1(E_h^1(\lambda,\mu), E_h^1(\lambda,\mu)).$$

□

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