# UNUSUAL STABILIZED FINITE ELEMENT METHODS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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## Abstract

Unusual stabilized finite element methods are introduced here for linear second order equations. The method consists in subtracting to the standard Galerkin method a mesh dependent term composed by the adjoint operator applied to the test function multiplied by the residual of the Euler-Lagrange equation. The method is numerically tested for advective dominated and zero order dominated regimes, when the equation presents singular behavior.

## 1. Introduction

We have pointed out in a recent communication [4] that for a certain model problem, bubble functions added to the usual finite element polynomials seem to subtract stability from the formulation. This finding contrasts our experience with other models such as the advective-diffusive equations and saddle-point problems [1-3, 12], where bubble functions are key to obtaining stable formulations.

In this work we show that the apparent contradiction raised in [4] is in fact an inspiration to develop *unusual* stabilized methods. The subtraction prescribed by the static condensation of the bubbles indicates that stability can be achieved in a nonstandard manner by stabilized methods that we present herein.

In order to clarify the terminology we recall that stabilized methods are constructed by adding a least-squares like term to the Galerkin method, preserving the high accuracy of the latter and getting more stability from the first. These are also known as Galerkin-least-squares methods (see [6-11] and references therein) and can be described roughly, for a linear differential equation,

$$Lu = f$$

as follows:

$$(Lu_k, v_k) + \sum_K (Lu_k, \tau L v_k)_K = (f, v_k) + \sum_K (f, \tau L v_k)_K$$

where $(\cdot, \cdot)$ denotes appropriate inner products and the trial $u_k$ and weighting $v_k$ functions are members of the usual finite element polynomials.

These stabilized finite element methods are:

• CONSISTENT METHODS: Since the residuals of the Euler-Lagrange equations are satisfied by the exact solutions, consistency is preserved in these methods.

• MORE STABLE THAN GALERKIN METHODS: The perturbation terms are designed to enhance stability of the original Galerkin formulation.

• CONVERGENT FOR USUAL ELEMENTS: Convergence results may be derived for a wide family of simple finite element interpolations.

From another viewpoint, if we insist in using the standard Galerkin method, i.e., for

$$Lu = f$$

we adopt

$$(Lu_h, v_h) = (f, v_h)$$

and for $u_h$ and $v_h$ we use the usual piecewise polynomials plus bubbles, then after static condensation the method resembles to

$$(Lu_k, v_k) - \sum_K (Lu_k, \tau L^* v_k)_K = (f, v_k) - \sum_K (f, \tau L^* v_k)_K$$

where $L^*$ is the adjoint operator associated with $L$, and the trial $u_k$ and weighting $v_k$ functions are members of the usual finite element polynomials.

These are Unusual Stabilized Finite Element Methods that are:

• CONSISTENT METHODS: Since the residuals of the Euler-Lagrange equations are satisfied by the exact solutions, consistency is preserved in these methods.

• POTENTIALLY MORE STABLE THAN GALERKIN METHODS: If $L$ is the first order operator of convection, then clearly this method adds stability to the Galerkin method in the same way as GLS. However for zero and second order operators, bubbles seem to suggest the *subtraction* of a square term from the Galerkin term.

• POTENTIALLY CONVERGENT FOR USUAL ELEMENTS: This needs to be verified for various equations. We start here with a second order linear scalar differential equation including first and zero order terms. This serves as model to advective-diffusive phenomena with sources that may be approximated in part by a zero order term (e.g., the transport equations of the turbulent quantities $k$ and $\epsilon$).

## 2. A model problem and bubbles' effects

We start with the model problem given by: find a scalar valued function $u(\mathbf{x})$ defined in $\Omega \subset I\!R^2$ such that

$$\sigma u + \mathbf{a} \cdot \nabla u - \kappa \Delta u = f \qquad \text{in } \Omega \tag{1}$$

$$u = 0 \qquad \text{on } \Gamma = \partial\Omega \tag{2}$$

where $\mathbf{a}$ is a given solenoidal velocity field (i.e. $\nabla \cdot \mathbf{a} = 0$), $\sigma$ and $\kappa$ are given positive constants and $f(\mathbf{x})$ is a given source function.

The variational formulation corresponding to (1)-(2) is: find $u \in H_0^1(\Omega)$ such that

$$(\sigma u, v) + (\mathbf{a} \cdot \nabla u, v) + (\kappa \nabla u, \nabla v) = (f, v) \qquad v \in H_0^1(\Omega) \tag{3}$$

where $H_0^1(\Omega)$ is the Hilbert space of functions with square-integrable value and derivative in $\Omega$ satisfying (2), and we use the notation $(f, g) = \int_\Omega fg \, d\Omega$ or $(\mathbf{f}, \mathbf{g}) = \int_\Omega \mathbf{f} \cdot \mathbf{g} \, d\Omega$ depending on whether the underlying inner product is performed between scalar or vector fields, respectively.

The standard Galerkin method is formulated on a subspace $V_h \subset H_0^1(\Omega)$ employing the variational formulation (3): find $u_h \in V_h$ such that

$$(\sigma u_h, v) + (\mathbf{a} \cdot \nabla u_h, v) + (\kappa \nabla u_h, \nabla v) = (f, v) \quad v \in V_h \tag{4}$$

We consider the following subspace $V_h = V_h^b$

$$V_h^b = \{ v \in H_0^1(\Omega) \,|\, v_{|K} \in P_1(K) \oplus B(K), \;\; K \in \mathcal{C}_h \} \,, \tag{5}$$

where $\mathcal{C}_h$ is a partition of $\Omega$ into regularly shaped triangles, $P_1(K)$ denotes the space of linear functions defined on the triangle $K$ and $B(K)$ denotes the space of bubble functions (e.g., spanned by a cubic function). The bubble basis function $\varphi \in B(K)$ satisfies

$$\begin{aligned} \varphi(\mathbf{x}) &> 0 \qquad \forall \mathbf{x} \in K \\ \varphi(\mathbf{x}) &= 0 \qquad \forall \mathbf{x} \in \partial K \,. \end{aligned} \tag{6}$$

We now proceed to eliminate the bubbles from (4) using the static condensation procedure. First we obtain the "bubble equation" by selecting in (4) $v = \varphi(\mathbf{x})$ for $\mathbf{x} \in K$ and $v = 0$ elsewhere in $\Omega$ to obtain

$$(\sigma u_h, \varphi)_K + (\mathbf{a} \cdot \nabla u_h, \varphi)_K + (\kappa \nabla u_h, \nabla \varphi)_K = (f, \varphi)_K \tag{7}$$

where the subscript $K$ indicates integration over $K$.

Denoting by $V_1$ the space of piecewise linears, i.e.,

$$V_1 = \{ v \in H_0^1(\Omega) \,|\, v_{|K} \in P_1(K), \;\; K \in \mathcal{C}_h \} \tag{8}$$

we may now decompose the unknown solution to (4) $u_h$ into its linear part $u_1 \in V_1$ and its part spanned by the bubble, i.e.,

$$u_h = u_1 + \sum_{K \in \mathcal{C}_h} u_b^K \varphi \tag{9}$$

where $u_b^K$ is the unknown bubble coefficient.

Substituting (9) into (7) we get

$$\begin{aligned} (\sigma u_1, \varphi)_K + & u_b^K (\sigma\varphi, \varphi)_K + (\mathbf{a} \cdot \nabla u_1, \varphi)_K + u_b^K (\mathbf{a} \cdot \nabla\varphi, \varphi)_K \\ & + (\kappa \nabla u_1, \nabla\varphi)_K + u_b^K (\kappa \nabla\varphi, \nabla\varphi)_K = (f, \varphi)_K \,. \end{aligned} \tag{10}$$

However, $\forall w_1 \in V_1$ we have by integration by parts

$$(\nabla w_1, \nabla\varphi)_K = -(\Delta w_1, \varphi)_K + (\nabla w_1 \cdot \mathbf{n}, \varphi)_{\partial K} = 0 \qquad w_1 \in V_1 \tag{11}$$

and we also have

$$(\mathbf{a} \cdot \nabla\varphi, \varphi)_K = \frac{1}{2} \left[ (\mathbf{a} \cdot \mathbf{n}, \varphi^2)_{\partial K} - (\varphi^2, \nabla \cdot \mathbf{a})_K \right] = 0 \tag{12}$$

Therefore, using (11)-(12) for $w_1 = u_1$, we reduce (10) to

$$u_b^K [\sigma \|\varphi\|^2_{0,K} + \kappa \|\nabla\varphi\|^2_{0,K}] = (f - \sigma u_1 - \mathbf{a} \cdot \nabla u_1, \varphi)_K \,, \tag{13}$$

where we have used the notation $\|v\|^2_{0,K} = \int_K v^2 \, d\Omega$. Solving (13) for the bubble coefficient in each element leads to

$$u_b^K = \frac{-1}{\sigma \|\varphi\|^2_{0,K} + \kappa \|\nabla\varphi\|^2_{0,K}} (\sigma u_1 + \mathbf{a} \cdot \nabla u_1 - f, \varphi)_K \,. \tag{14}$$

The procedure can be repeated to each element $K \in \mathcal{C}_h$ and equation (14) gives the value of each unknown coefficient $u_b^K$ as a function of the chosen bubble basis function $\varphi$, the data ($\sigma, \kappa, \mathbf{a}$ and $f$) and the unknown linear part $u_1$ of the solution $u_h$. The next step of static condensation consists in selecting in (4) $v = v_1 \in V_1$, and by using (9) and (11) we get:

$$\begin{aligned} (\sigma u_1, v_1) + (\mathbf{a} \cdot \nabla u_1, v_1) + & \sum_{K \in \mathcal{C}_h} u_b^K (\sigma\varphi + \mathbf{a} \cdot \nabla\varphi, v_1)_K \\ & + (\kappa \nabla u_1, \nabla v_1) = (f, v_1) \quad v_1 \in V_1 \end{aligned} \tag{15}$$

Therefore the resulting variational equation (15) is equivalent to using the standard Galerkin method for piecewise linear functions (i.e., choosing $V_h = V_1$ in (4)) "plus" a term that in view of (14) can be written as

$$\sum_{K\in\mathcal{C}_h} u_b^K(\sigma\varphi + \mathbf{a}\cdot\nabla\varphi, v_1)_K =$$
$$\sum_{K\in\mathcal{C}_h} \frac{-1}{\sigma\|\varphi\|_{0,K}^2 + \kappa\|\nabla\varphi\|_{0,K}^2}(\sigma u_1 + \mathbf{a}.\nabla u_1 - f, \varphi)_K(\varphi, \sigma v_1 - \mathbf{a}.\nabla v_1)_K \tag{16}$$

## 3. The unusual stabilized finite element method

Eq. (16) suggests the usage of the following stabilized finite element method: find $u_h \in V_1$ such that

$$B(u_h, v) = F(v) \qquad v \in V_1 \tag{17}$$

where $V_1$ is given by (8) and

$$\begin{aligned} B(u,v) &= (\sigma u, v) + (\mathbf{a}\cdot\nabla u, v) + (\kappa\nabla u, \nabla v) \\ &\quad - \sum_{K\in\mathcal{C}_h} (\sigma u + \mathbf{a}\cdot\nabla u - \kappa\Delta u, \tau_K(\sigma v - \mathbf{a}\cdot\nabla v - \kappa\Delta v))_K \end{aligned} \tag{18}$$

$$F(v) = (f, v) - \sum_{K\in\mathcal{C}_h} (f, \tau_K(\sigma v - \mathbf{a}\cdot\nabla v - \kappa\Delta v))_K \tag{19}$$

with the stability parameter $\tau_K$ given by

$$\tau_K = \begin{cases} \dfrac{h_K}{2|\mathbf{a}|_p} & , \mathrm{Pe}_K \geq 1 \\ \dfrac{h_K^2}{\sigma h_K^2 + \beta_K} & , \mathrm{Pe}_K < 1 \end{cases} \tag{20}$$

$$\mathrm{Pe}_K = \frac{2|\mathbf{a}|_p h_K}{\sigma h_K^2 + \kappa} \tag{21}$$

$$\beta_K = \begin{cases} |\mathbf{a}|_p h_K & , |\mathbf{a}|_p h_K \geq \kappa \\ \kappa & , |\mathbf{a}|_p h_K < \kappa \end{cases} \tag{22}$$

$$|\mathbf{a}|_p = \begin{cases} \left( \sum_{i=1}^{N} |a_i(\mathbf{x})|^p \right)^{1/p} & , \quad 1 \leq p < \infty \\ \max\limits_{i=1,N} |a_i(\mathbf{x})| & , \quad p = \infty \end{cases} \tag{23}$$

We present here two sets of data for the problem of advection skew to the mesh (see Figure 1 for the problem statement). On all simulations we used $\kappa = 7 \times 10^{-4}$, $\mathbf{a} = (0.15, 0.1)$ and 800 uniform triangles for the mesh over a unit square. In Figure 2 we compare the Galerkin method with the stabilized method for $\sigma = 0.3$ while Figure 3 shows the same comparison for $\sigma = 7$.

Figure 1. Problem statement: advection skew to the mesh

*Remarks:*

1. The case in which $\sigma = 0$, the method above reduces to the one introduced in [7].
2. The case in which $\mathbf{a} = \mathbf{0}$, the method above reduces to the method studied in [5].

Figure 2. Results for $\sigma = 0.3$.

Figure 3. Results for $\sigma = 7$.

3. This method deals with the more general possibility of having the differential operators of order 2, 1 and zero present in the same equation. This is of interest in applying these methods to the transport equations of turbulent quantities (such as the ones found in the $k-\epsilon$ model), or to applications involving chemical reactions, etc.
4. In closing, we would like to reiterate that bubbles prompt unusual stabilized methods in that instead of adding a least-squares form of the Euler-Lagrange equations to the Galerkin method, we subtract a term of the type

$$\sum_{K\in\mathcal{C}_h}(Lu-f,L^*v)_K$$

where $L$ is the differential operator associated with the scalar PDE and $L^*$ its adjoint. See [4] for an abstract theory. It should be noted from this work that such unusual methods keep the desired additional stability characteristics of Galerkin-least-squares methods and do have a nontrivial counterpart within the framework of the Galerkin method using 'virtual' bubbles.

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