The Competition Graphs of Interval Digraphs

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Abstract. Given a digraph D, we construct the competition graph of D, $C(D)$, on the same vertex set as D with x and y adjacent in $C(D)$ if and only if there exists a vertex z such that x and y both have arcs to z in D. A digraph is an interval digraph if and only if two intervals $S(x)$ and $T(x)$ on the real line can be assigned to vertex x such that $(x, y) \in A(D)$ if and only if $S(x) \cap T(y) \neq \emptyset$.

In this paper we show that the competition graph of an interval digraph is an interval graph and that every interval graph is in fact the competition graph of some digraph. These results are then related to previous work on competition graphs.

1. Introduction. Given a digraph D, we construct the *competition graph* of D, $C(D)$ on the same vertex set as D with x and y adjacent in $C(D)$ if and only if there exists a vertex z such that x and y both have arcs to z in D. Though introduced by Cohen [1] in the study of food webs, competition graphs have been applied to other models such as communication networks by Raychaudhuri and Roberts [8]. A digraph may be used to model a food web as follows. Let each vertex in the digraph represent a species in the food web. Allow $(x, y) \in A(D)$ if and only if the species corresponding to x preys upon the species corresponding to y. A graph G is an *interval graph* if and only if it is possible to associate an interval $I(x)$ on the real line with every vertex x of $V(G)$ and $\{x, y\} \in E(G)$ if and only if $I(x) \cap I(y) \neq \emptyset$. Cohen observed that most competition graph of digraphs constructed from empirical data on food webs are interval. This fact has created much interest in the study of competition graphs. Though many have worked on this problem [3], [5], [7], [10], an explanation for this phenomenon remains elusive. One possible approach is to identify classes of digraphs with interval competition graphs. In this paper, we prove interval digraphs to be such a class.

Several equivalent characterizations of interval graphs are required in the following section. A *consecutive ranking* of the maximal cliques of a graph is an ordering of the maximal cliques $C_1, \ldots, C_p$ such that if a vertex x is in C_i and C_k with $i < k$ then x is in C_j for all $j, i < j < k$. An *asteroidal triple* in a graph is a set of three vertices x, y and z such that there is a path between each pair of vertices that contains no neighbor of the third vertex.

PROPOSITION 1.1. *Given an undirected simple graph G, the following are equivalent.*

1. *G is an interval graph.*
2. *G has a consecutive ranking of the maximal cliques.*
3. *G has no Z_n, $n \geq 4$ and no asteroidal triples.*

Part 2 of the above proposition is due to Fulkerson and Gross [2]. Part 3 is due to Lekkerkerker and Boland [4].

A digraph is an *interval digraph* if two intervals $S(x)$ and $T(x)$ on the real line can be assigned to each vertex x such that $(z, y) \in A(D)$ if and only if $S(z) \cap T(y) \neq \emptyset$. A digraph is an *interval-point* digraph if $T(x)$ is a single point for each vertex x. Interval digraphs and interval-point digraphs were introduced by Sen, Das, Roy, and West [9].

The problem of identifying which classes of graphs $\mathcal{G}$ are the competition graphs of a particular class of digraphs $\mathcal{D}$ has been considered by two of the present authors [5]. They showed that if G is an interval graph then G is the competition graph of a Hamiltonian digraph. The latter construction has importance in the application of competition graphs to models of communication networks. In this paper we show if G is an interval graph, then G is the competition graph of an interval digraph.

2. Characterization. Not only is every interval graph the competition graph of an interval digraph, but the competition graph of every interval digraph is an interval graph. To prove the latter result, we need the following lemma.

LEMMA 2.1. *If D is an interval digraph and C is a maximal clique of $C(D)$ then there is some $x \in V(D)$ such that Inset(x) equals C.*

Proof. Consider the vertices of C. Let x be a vertex such that $\text{Inset}(x) \cap C$ is not properly contained in $\text{Inset}(v) \cap C$ for any other vertex v. Suppose $\text{Inset}(x) \cap C \neq C$. Then there exists a vertex $a \in C$ such that $a \notin \text{Inset}(x)$. Since $\text{Inset}(x)$ is not properly contained in the inset of any other vertex there must exist $b, c \in \text{Inset}(x) \cap C$ and y, z such that

$$(a, y), (c, y), (a, z), (b, z) \in A(D) \text{ but } (b, y), (c, z) \notin A(D).$$

Since D is an interval digraph we have intervals on the real line $S(a), S(b), S(c), T(x), T(y)$, and $T(z)$ such that the intersection graph G of these six intervals contains graph H in Figure 1 as a subgraph.

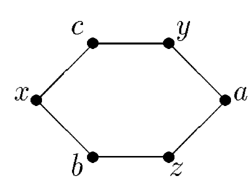

FIG. 1.

Observe that $G \neq H$ since H is not interval. Further observe that $\{x, a\}, \{y, b\}$, and $\{z, c\} \notin G$. Without loss of generality, assume $\{x, y\} \in G$. Then $xyazbx$ is a five cycle. Since G is interval this five cycle must have a chord. Suppose $\{b, a\} \in E(G)$. Then G has the chordless four cycle xyabx. So $\{b, a\} \notin E(G)$ and therefore $\{y, z\}$ or $\{x, z\}$ must be in G. If only one of these edges is in G, G has a chordless four cycle, so both $\{y, z\}$ and $\{x, z\} \in E(G)$. Suppose $\{c, b\} \in E(G)$. Then $cbzy$ is a chordless four cycle, so $\{c, b\} \notin E(G)$. Suppose $\{a, c\} \in E(G)$. Then $cazx$ is a chordless four cycle, so $\{a, c\} \notin E(G)$. Thus $E(G)$ is completely determined and G is shown in Figure 2. Since the vertices a, b and c form an asteroidal triple, we have a contradiction, i.e., no such z can exist, completing the proof. □

THEOREM 2.2. *If D is an interval digraph, then $C(D)$ is an interval graph.*

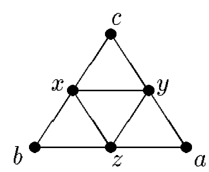

FIG. 2.

Proof. By Lemma 2.1, every maximal clique of $C(D)$ is the inset of some vertex x. Choose a unique such vertex for each maximal clique of $C(D)$. Consider the set of terminal intervals for these vertices. If $T(x) \subseteq T(y)$ then $\text{Inset}(x) \subseteq \text{Inset}(y)$. Thus no interval in this set may be properly contained in another interval and we may order the maximal cliques $C_1, C_2, \ldots, C_p$, where $C_i \leq C_j$ implies the left endpoint of the interval corresponding to C_i is strictly less that the left endpoint for C_j. We claim this ranking is consecutive.

Let x_i denote the vertex chosen such that $\text{Inset}(x_i) = C_i$. If $x \in C_i$ then $S(x) \cap T(x_i) \neq \emptyset$. Suppose $x \in C_i$, $x \in C_k$ and $i < j < k$. Then $S(x) \cap T(x_i) \neq \emptyset$ and $S(x) \cap T(x_k) \neq \emptyset$. If x is not in C_j then $S(x) \cap T(x_i)$ is to the left of $T(x_j)$. But $S(x) \cap T(x_k)$ is to the right of $T(x_j)$ and since $S(x)$ is an interval, this is impossible. Thus $C_1, C_2, \ldots, C_n$ is a consecutive ranking of the maximal cliques of $C(D)$, i.e., $C(D)$ is an interval graph. □

THEOREM 2.3. *If G is an interval graph, then G is the competition graph of an interval-point digraph D.*

Proof. Let $C_1, C_2, \ldots, C_p$ be a consecutive ranking of the maximal cliques of G. Construct D as follows. Assign to each vertex x the interval $S(x) = [i, j]$ where C_i is the first clique containing x and C_j is the last clique containing x. Observe this is possible since there are at least as many vertices as maximal cliques. For each C_i, choose $x_i \in C_i$ and assign $T(x_i) = i$. Assign $T(x) = p + 1$ for all other vertices.

Two vertices in G are adjacent if and only if there is some maximal clique that has both in common. If two vertices x and y appear in a maximal clique C_i, by the above construction their source intervals will contain the point i, so (x, x_i) and $(y, x_i) \in A(D)$ implies $\{x, y\} \in E(C(D))$. Conversely, x will only be directed toward vertices that correspond to cliques containing x, so if $\{x, y\} \notin G$, then x and y are not directed toward the same vertex in D, i.e., $\{x, y\} \notin E(C(D))$. Thus $C(D) = G$. □

COROLLARY 2.4. *If G is an interval graph, then G is the competition graph of an interval digraph D.*

Finding the chromatic number of the competition graph of a digraph was considered by two of the present authors and Rasmussen [6]. The size of a maximal inset in a directed graph is denoted $\Delta^-(D)$. We now see that the chromatic number of the competition graph of an interval digraph is easily computed.

COROLLARY 2.5. *If D is an interval digraph, then $\Delta^-(D) = \chi(C(D))$.*

Proof. Since $C(D)$ is interval it is perfect and so $\chi(C(D))$ is the size of the largest clique. By Lemma 2.1 we conclude that $\chi(C(D)) = \Delta^-(D)$. □

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