Interval p-Neighborhood Graphs

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Abstract. The (p)-neighborhood graph of a graph G, denoted $N_p(G)$, is defined on the same vertex set as G, with $[x, y] \in E(N_2(G))$ if and only if $|N(x) \cap N(y)| \geq p$ in G, where $N(v)$ is the open neighborhood of vertex v. The $[p]$-neighborhood graph of G, $N_p[G]$, is defined similarly, using closed neighborhoods rather than open ones. If G is the underlying graph of a symmetric digraph D, then the p-neighborhood graph of G is the p-competition graph of D. The case $p = 1$ has been studied by several authors. We consider the general case, asking, "which graphs have interval p-neighborhood graphs", and give specific results for the case $p = 2$.

I. Introduction. The *p-competition graph H* of a digraph D has the same vertex set as D, with edge $[x, y]$ in $E(H)$ if and only if x and y have at least p common outneighbors in D. The paper by Kim, McKee, McMorris, and Roberts [6] in which the concept was introduced prompted several others. The case where D is acyclic is studied by Kim, McKee, McMorris, and Roberts in [5], and the instance where D is strongly connected or Hamiltonian is explored in [7] by Langley, Lundgren, McKenna, Merz, and Rasmussen. In order to study the case where D is symmetric, we consider the *(p)-neighborhood graph* (read "p-open-neighborhood graph") of a graph G, denoted $N_p(G)$, where $[x, y] \in E(N_p(G))$ if and only if $|N(x) \cap N(y)| \geq p$ in G. The *$[p]$-neighborhood graph* (read "p-closed neighborhood graph") of G, denoted $N_p[G]$, is defined in the expected fashion. The graph $N_p(G)$ is the p-competition graph of a loopless symmetric digraph D, where G is the graph underlying D. Similarly, $N_p[G]$ is the p-competition graph of a symmetric digraph D' that has a loop at every vertex, where G is the graph underlying D'. These graphs are generalizations of the "neighborhood graphs" studied by several authors (see, for example, Acharya and Vartak [1], Brigham and Dutton [2], Raychaudhuri [12], Harary and McKee [4]). In these papers, the case $p = 1$ was explored, with the open neighborhood graph sometimes referred to as the "2-step graph" and the closed neighborhood graph called the "square" of a graph. Identification of p-neighborhood graphs in the case $p = 2$ is the main subject of [9] by Lundgren, McKenna, Merz, and Rasmussen.

In this paper we consider the following question: given a graph G, what conditions are necessary and sufficient for the p-neighborhood graph of G to be an interval graph? This question generalizes one posed by Raychaudhuri and Roberts [13] and explored by Lundgren, Rasmussen, and Maybee [11], where the case $p = 1$ was considered. We focus on the case $p = 2$. We consider the 2-open instance in Section II, giving a sufficient condition for a chordal graph to to have an interval (2)-neighborhood graph. In Section III, the 2-closed instance is examined, and we prove that the [2]-neighborhood graph of an interval graph is interval. The definition that follows is of

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use in both of those sections. A family of sets $\mathcal{S} = S_1, S_2, \ldots, S_r$ of vertices of G is called a *(p)-neighborhood cover of* G if the following condition is satisfied: Vertices $i, j \in S_m$ for some m if and only if $|N(i) \cap N(j)| \geq p$. The definition is similar for the $[p]$-*neighborhood cover of* G. Essentially, a (p)-neighborhood cover of G is an edge-clique cover of $N_p(G)$, as the example that follows illustrates.

Example 1 For graph G depicted in Figure 1, the following sets form a (2)-neighborhood cover: $\{v, x\}$, $\{a, w\}$, $\{b, w\}$, $\{a, x\}$, $\{a, b\}$, and $\{x, y, z\}$. We see from the figure that this family of sets also forms an edge-clique cover of $N_2(G)$.

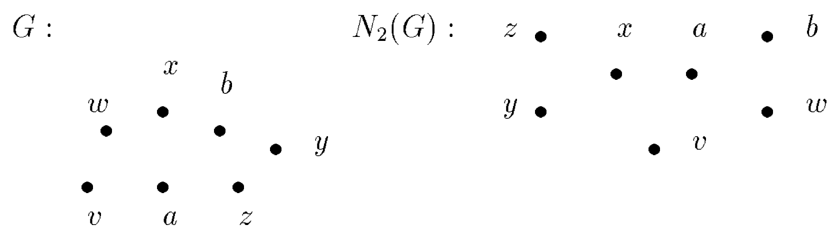

Figure 1: *A graph G and its (2)-neighborhood graph.*

II. Which graphs have interval (2)-neighborhood graphs? We begin by presenting three more examples. Consider the graphs in Figure 2; we have G_1, a graph which is not interval, whose (2)-neighborhood graph is interval; graph G_2 is not interval and $N_2(G_2)$ is not interval; lastly we have interval graph G_3, whose (2)-neighborhood graph is not interval. We see from these examples that whether or not $N_2(G)$ is interval is not dependent upon G being interval. For this reason we seek to identify which interval graphs have interval (2)-neighborhood graphs.

By Fulkerson and Gross [3] we have $N_2(G)$ is an interval graph if and only if its maximal cliques can be consecutively ranked. That is, there exists an ordering of the maximal cliques C_i of $N_2(G)$ such that if vertex x is in cliques C_k and C_j then x is in every clique between C_k and C_j in the ranking.

Proposition 1 *$N_p(G)$ is interval if and only if there exists a (p)-neighborhood cover of G with a consecutive ranking.*

Proof: If $N_p(G)$ is an interval graph, then its maximal cliques form a (p)-neighborhood cover of G with a consecutive ranking. Conversely, let $S_1, \ldots, S_r$ be a consecutively ranked (p)-neighborhood cover of G. Suppose vertex $x \in S_i, \ldots, S_j$ and $x \notin S_k$, for $k < i$, $k > j$. Let I_x be the interval $[i, j]$ on the real line. We show that $N_p(G)$ is the intersection graph of these intervals I_x. For two vertices x and y in $N_p(G)$, $I_x \cap I_y$ is nonempty if and only if there exists a set S_q such that $x, y \in S_q$ if and only if $[x, y] \in E(N_p(G))$. Hence $N_p(G)$ is an interval graph. $\square$

The difficulty lies in actually *finding* a (p)-neighborhood cover of a graph G with a consecutive ranking. If we can find one that corresponds to the maximal cliques of

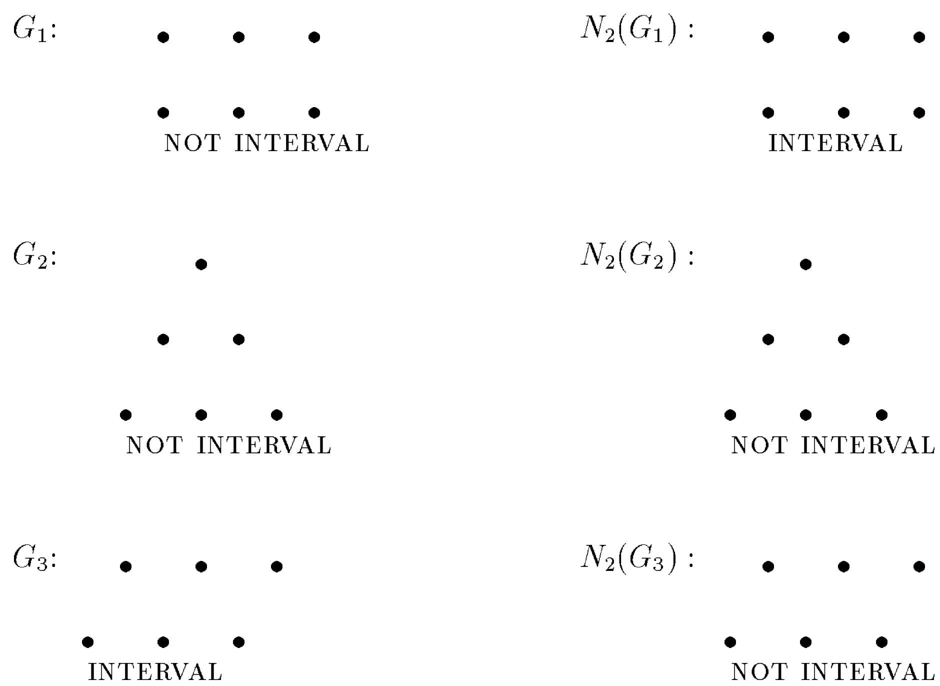

Figure 2: *Three graphs and their (2)-neighborhood graphs.*

$N_p(G)$, that is an added bonus. In studying the case $p = 1$, the authors of [8] identified a family of sets that not only formed a (1)-neighborhood cover of a given interval graph G, but actually corresponded to the maximal cliques of $N_1(G)$. They concluded that for interval G, $N_1(G)$ is interval if and only if the family has a consecutive ranking. In that case, the *nonsimplicial vertices* of G, i.e. those vertices whose neighborhoods are not cliques, played a crucial role in identifying the collection of sets. Similarly in the case $p = 2$ we seek a (2)-neighborhood cover of G that has a consecutive ranking. In fact, we hope to find one that consists of the maximal cliques of $N_2(G)$. If such a cover has a consecutive ranking we can conclude that $N_2(G)$ is an interval graph. In this case we believe that the *nonsimplicial edges* of G, i.e. those edges in G that are contained in more than one maximal clique of G, will be very useful in finding such a cover of G. This notion is the natural generalization of work done in the case $p = 1$ [8]. We can partition the nonsimplicial edges of any graph G into three types:

Type 1: nonsimplicial edge e_i is contained only in cliques of size ≥ 4,

Type 2: nonsimplicial edge e_i is contained only in cliques of size 3, and

Type 3: nonsimplicial edge e_i is contained in both cliques of size 3 and cliques of size ≥ 4.

We may use the nonsimplicial edges of a chordal graph G to construct a (2)-neighborhood cover $\mathcal{S}$ of G. Let $e_1, e_2, \ldots, e_k$ be the nonsimplicial edges of G. For each nonsimplicial e_i, construct the set(s) as follows:

- If e_i is Type 1, then $S_i = N[e_i]$ (call this a Type 1 set),
- If e_i is Type 2, then $S_{i_1} = N(e_i)$ (call this a Type 2a set) and $S_{i_2} = \{e_i\}$ (a Type 2b set), and

- If e_i is Type 3, then $S_{i_1} = \cup\{C_j \mid e_i \in C_j, |C_j| \geq 4\}$ (a Type 3a set), and $S_{i_2} = N(e_i)$ (a Type 3b set).

We must add one last type of set, one that is not related to nonsimplicial edges. For every maximal clique C_j in G where $|C_j| \geq 4$ and C_j has no nonsimplicial edges, let $C_j \in \mathcal{S}$; call C_j a Type 4 set.

Theorem 1 Let G be a connected chordal graph. The family of sets $\mathcal{S}$ just described forms a (2)-neighborhood cover of G.

Proof: We must show: $p, q \in S_k$ for some k if and only if p and q have at least 2 common neighbors in G.

($\Rightarrow$) Let $p, q \in S_k$ for some k. If S_k is a clique of size at least four with no nonsimplicial edges, then p and q have at least two common neighbors in G in that clique. Otherwise, S_k is associated with some nonsimplicial edge, say e_k.
<u>Case i:</u> Edge e_k is Type 1. Then e_k is contained only in cliques of size at least 4, $S_k = N[e_k]$, and p and q have at least two common neighbors in G.
<u>Case ii:</u> Edge e_k is Type 2. Then e_k is only contained in cliques of size 3. It is possible that $[p, q] = e_k$. If this is the case, then $[p, q]$ is contained in more than one maximal clique (since e_k is nonsimplicial), which implies p and q have at least 2 common neighbors. Otherwise $p, q \in S_k = N(e_k)$, so two common neighbors of p and q are the endpoints of edge e_k.
<u>Case iii:</u> Edge e_k is Type 3. Then e_k is contained both in cliques of size at least 4 and in cliques of size 3. Then either

(1) $S_k = \cup\{C_j \mid e_k \in C_j, |C_j| \geq 4\}$, or

(2) $S_k = N(e_k)$.

If (1) is true, then p or q (perhaps both) could be an endpoint of e_k; whether or not this happens, p and q have at least two common neighbors. If (2) is true, then the endpoints of edge e_k are two common neighbors of p and q. We conclude that $p, q \in S_k$ for some k implies p and q have at least two common neighbors in G.

($\Leftarrow$) Assume p and q have at least two common neighbors, say a and b, in G. Then $paqb$ is a 4-cycle in G. G chordal implies $[a, b] \in E(G)$ or $[p, q] \in E(G)$, perhaps both.

(1) $[a, b] \in E(G)$ and $[a, b]$ nonsimplicial implies $p, q \in S_k$ for S_k associated with edge $[a, b]$, no matter what type of nonsimplicial edge $[a, b]$ is.

(2) $[a, b] \in E(G)$ and $[a, b]$ simplicial implies $[a, b]$ is in exactly one maximal clique in G, which implies $[p, q] \in E(G)$. Then $abpq$ is contained in some maximal clique in G (just one, since $[a, b]$ is simplicial). The maximal clique containing $abpq$ either has a nonsimplicial edge or it doesn't. If it doesn't, then p and q appear together in a Type 4 set, S_k. Assume the maximal clique containing $abpq$ has a nonsimplicial edge e_k. Edge $[a, b]$ simplicial implies $[a, b] \neq e_k$. Edge e_k is of either Type 1 or Type 3, and in either case p and q are both in a set S_k associated with edge e_k.

(3) Suppose $[a, b] \notin E(G)$. Then certainly $[p, q] \in E(G)$. In fact, $[p, q]$ is nonsimplicial, since $[a, b] \notin E(G)$. Then p and q appear together in some set S_k associated with edge $[p, q]$: S_k is of Type 1 if $[p, q]$ is Type 1, S_k is of Type 2b if $[p, q]$ is Type 2, and S_k is of Type 3a if $[p, q]$ is Type 3.

We conclude that $p, q \in S_k$ for some k if and only if p and q have at least two common neighbors in G. $\square$

The graph G of Figure 3 convinces us that chordality is required in the theorem. We have G, a non-chordal graph with no nonsimplicial edges and no cliques of size at least 4, hence $\mathcal{S} = \emptyset$. However, $N_2(G)$ is not an empty graph, thus $\mathcal{S}$ is not a (2)-neighborhood cover of G.

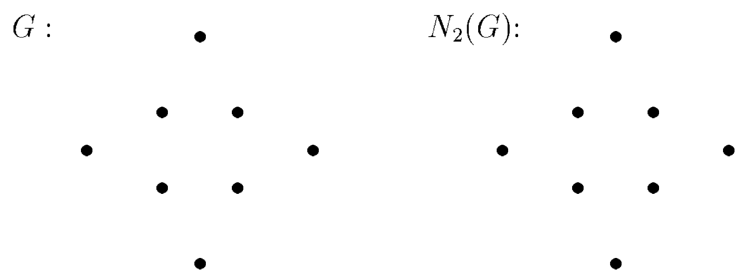

Figure 3: *G is not chordal, and $\mathcal{S} = \emptyset$ does not form a (2)-neighborhood cover of G.*

Corollary 1 *If G is chordal and $\mathcal{S}$ has a consecutive ranking, then $N_2(G)$ is an interval graph.*

Graph G of Figure 1 serves as a counterexample to the converse of Corollary 1. We see this by examining Example 1; the (2)-neighborhood cover of chordal graph G listed is precisely the family of sets $\mathcal{S}$ which come from the nonsimplicial edges of G. We see that these sets have no consecutive ranking (check $\{a, b\}$, $\{b, w\}$, $\{a, w\}$), however $N_2(G)$ is an interval graph. The problem is that the sets $\mathcal{S}$ do not correspond in this case to the *maximal* cliques of G. In the case $p = 1$, the analogous collection of sets (based on nonsimplicial vertices) worked for both necessity and sufficiency when G was an interval graph. However, for the case $p = 2$, Example 1 (where G is interval) convinces us that our collection $\mathcal{S}$ needs to be altered. We suspect that it will be useful to restrict G to being an interval graph, so that we may make use of the consecutive ranking of G's maximal cliques. The exact altered family of sets will appear in a subsequent paper.

III. Interval graphs have interval [2]-neighborhood graphs. We now consider the closed case. The instance $p = 1$ was explored in [12], where the author proved that the square (or closed neighborhood graph) of an interval graph is interval. In this section, we show that the [2]-neighborhood graph of an interval graph is interval by proving that every interval graph has a [2]-neighborhood cover with a consecutive ranking. If G is an interval graph with consecutively ranked maximal cliques $C_1, C_2, \ldots, C_k$, define S_{ij}, where $i \leq j$, to be the set $C_i \cup C_{i+1} \cup \cdots \cup C_{j-1} \cup C_j$.

Lemma 1 *Suppose G is an interval graph with consecutively ranked maximal cliques $C_1, C_2, \ldots, C_k$, then S_{ij} is a clique in $N_2[G]$ if and only if $C_i \cap C_{i+1} \cap \cdots \cap C_j \geq 2$.*

Proof: Clearly $C_i \cap C_{i+1} \cap \cdots \cap C_j \geq 2$ implies S_{ij} is a clique in $N_2[G]$, so assume S_{ij} is a clique in $N_2[G]$.

Let $x \in C_i - C_{i+1}$ and let $z \in C_j - C_{j-1}$. Observe $x \notin C_j$ and $z \notin C_i$ since the cliques C_i are consecutively ranked. Since S_{ij} is a clique in $N_2[G]$, there exist a and b such that $x, y \in N[a] \cap N[b]$. Then $x \notin C_{i+1}$ implies $a \in C_p$ for some $p \leq i$ and $z \notin C_{j-1}$ implies $a \in C_q$ for some $q \geq j$. Then since $C_1, C_2, \ldots, C_k$ is a consecutive ranking, we have $a \in C_l$ for all $p \leq l \leq q$. Similarly, we have $b \in C_m$ for all $u \leq m \leq v$, for some integers u and v, where $u \leq i$, $v \geq j$. Hence $a, b \in C_i \cap C_{i+1} \cap \cdots \cap C_j$, and $|C_i \cap C_{i+1} \cap \cdots \cap C_j| \geq 2$. □

If we take all sets S_{ij} that are cliques in $N_2[G]$, we have a [2]-neighborhood cover of G, as the next lemma shows.

Lemma 2 *Let $\mathcal{S} = \{S_{ij} \ni |C_i \cap \ldots \cap C_j| \geq 2\}$. The collection $\mathcal{S}$ is a [2]-neighborhood cover of G.*

Proof: We must show x and y are in some set S_{ij} of $\mathcal{S}$ if and only if $|N[x] \cap N[y]| \geq 2$ in G.

($\Rightarrow$) Let $x, y \in S_{ij}$. Since S_{ij} is a clique of $N_2[G]$, $[x, y] \in E(N_2[G])$ (by Lemma 1), and $|N[x] \cap N[y]| \geq 2$.

($\Leftarrow$) Suppose $|N[x] \cap N[y]| \geq 2$ in G. If $[x, y] \in E(G)$ then $x, y \in C_j$ for some maximal clique C_j of G and $x, y \in S_{jj}$. If $[x, y] \notin E(G)$ then x and y have (at least) two common neighbors in G, say a and b. Since $[x, y]$ is not an edge of G and G is interval, we have $[a, b] \in E(G)$. Thus there exist distinct integers p and q such that $x, a, b \in C_p$ and $y, a, b \in C_q$. Assume $p < q$. Then $a, b \in C_r$ for all $p \leq r \leq q$, so $|C_p \cap \ldots \cap C_q| \geq 2$, and $x, y \in S_{pq}$.

□

Let $\mathcal{S}'$ contain only those sets of $\mathcal{S}$ which are contained in no other set of $\mathcal{S}$. Then certainly $\mathcal{S}'$ is also a [2]-neighborhood cover of G. With the next theorem we see that if G is an interval graph then these sets have a consecutive ranking.

Theorem 2 *If G is an interval graph, then the relation on $\mathcal{S}'$ determined by $S_{ac} < S_{gi}$ if and only if $a < g$ and $c < i$ is a consecutive ranking of $\mathcal{S}'$.*

Proof: Since no set in the family is properly contained in any other, $a < g$ if and only if $c < i$, so the ranking orders all the sets in $\mathcal{S}'$.

Let $x \in S_{ac}$ and $x \in S_{gi}$ for $a < g$ and $c < i$. Consider $S_{df} \in \mathcal{S}'$ for $a < d < g$ and $c < f < i$. We claim $x \in S_{df}$ Observe that $x \in C_b$ for some $a \leq b \leq c$ and that $x \in C_h$ for some $g \leq h \leq i$. We must show that $x \in C_e$ for some $d \leq e \leq f$.

<u>Case i:</u> $b \leq h$. Since the cliques C_i are consecutively ranked we have

$$x \in C_b \cap C_{b+1} \cap \cdots \cap C_h.$$

Note that $d \leq h$ and $b \leq f$. These inequalities combined with $b \leq h$ and $d \leq f$ imply the existence of an integer e between b and h such that $C_e \subseteq S_{df}$. This implies $x \in S_{df}$ and we are done.

Case ii: $h < b$. Since the cliques C_i are consecutively ranked we have

$$x \in C_h \cap C_{h+1} \cap \cdots \cap C_b.$$

Since $d < h < b < f$ and $x \in C_e$ for all e between h and b, we have $x \in S_{df}$, completing the proof. □

Corollary 2 *If G is an interval graph, then $N_2[G]$ is interval.*

Note that graph G_1 of Figure 2 is not an interval graph. However, its [2]-neighborhood graph (two K_4's sharing exactly one edge - not depicted in the figure) is interval. An interesting problem would be to determine what other classes of graphs have interval [2]-neighborhood graphs, as Lundgren, Merz, and Rasmussen [10] did for the case $p = 1$.

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