The p-Competition Graphs of Symmetric Digraphs and p-Neighborhood Graphs

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Abstract. The p-competition graph G of a digraph D is a graph on the same vertex set as D, with $[x, y] \in E(G)$ if and only if $|Out(x) \cap Out(y)| \geq p$ in D. In this paper we focus on the case in which D is a *symmetric* digraph ((a, b) is an arc in D if and only if (b, a) is an arc in D). We relate the problem to 2-step graphs, squares, and a generalization of the neighborhood graph called the p-neighborhood graph. We also identify some familiar classes of graphs as 2-competition graphs of loopless symmetric digraphs.

I. Introduction. The p-competition graph was introduced in 1989 by Kim, McKee, McMorris, and Roberts [9] as a generalization of the competition graph first presented by Cohen [5] in 1968. The p-competition graph G of a digraph D has an edge between vertices x and y if and only if x and y have at least p common outneighbors in D (i.e. x and y have outarcs to at least p common vertices). We write $G = C_p(D)$. See Figure 1 for a digraph and its 2-competition graph.

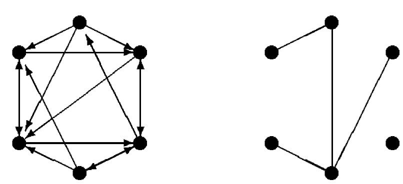

Figure 1: *A digraph and its 2-competition graph.*

The research in this area has thus far focused on determining whether or not a given graph is the p-competition graph of some digraph. The case where the digraph is acyclic is discussed by Kim, McKee, McMorris, and Roberts in [8], while p-competition graphs of loopless Hamiltonian and loopless strongly connected digraphs are explored by Langley, Lundgren, McKenna, Merz, and Rasmussen in [10]. In this paper we concentrate on the case where the digraph is symmetric. We relate the problem to 2-step graphs, squares, and a generalization of the neighborhood graph called the p-neighborhood graph in Section II. In Section III we identify some familiar graphs as 2-competition graphs of loopless symmetric digraphs. Throughout the paper we let $E(G), V(G), e(G), Z_4(G), \Delta(G), d(x,y), P_n,$ and$O(x)$ denote the edge

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set of G, the vertex set of G, the number of edges of G, the number of 4-cycles in G, the highest vertex degree in G, the distance between vertices x and y, the path on n vertices, and the outset of a vertex x in a digraph.

Given a graph G we construct the open neighborhood graph of G, $N(G)$, introduced by Acharya and Vartak [1], as follows. There is an edge between two vertices x and y in $N(G)$ if and only if $N(x) \cap N(y) \neq \emptyset$ in G (where $N(v)$ is the open neighborhood of vertex v, i.e., those vertices which are adjacent to v, not including v). Similarly, $[x, y]$ is an edge in the closed neighborhood graph of G, $N[G]$, if and only if $N[x] \cap N[y] \neq \emptyset$ in G, where $N[v] = N(v) \cup \{v\}$. For more on neighborhood graphs, see Boland, Brigham, and Dutton [2] and [3] and Brigham and Dutton [4]. These notions are generalized to the p-open neighborhood graph (or (p)-neighborhood graph) of G, denoted $N_p(G)$, and the p-closed neighborhood graph (or $[p]$-neighborhood graph) of G, denoted $N_p[G]$, as follows. In $N_p(G)$ we have edge $[x, y]$ if and only if $|N(x) \cap N(y)| \geq p$ in G. $N_p[G]$ is constructed in the expected way.

The 2-step graph of a graph G, $S_2(G)$, has an edge between vertices x and y if and only if there is a path of length 2 between x and y in G, and the square of G, G^2, has an edge between x and y if and only if there is a path of length 2 or less between x and y in G. Note that $S_2(G) = N(G)$ and $G^2 = N[G]$. For results on 2-step graphs, see Lundgren, Maybee, and Rasmussen [15], Lundgren, Merz, Maybee, and Rasmussen [12], and Lundgren and Rasmussen [14]. For work on the square of a graph, see Harary and McKee [6], Lundgren, Merz, and Rasmussen [13], Raychaudhuri [18], and Raychaudhuri and Roberts [19].

See Figure 2 for a graph G and its 2-step graph, square,(3)-neighborhood graph, and [3]-neighborhood graph.

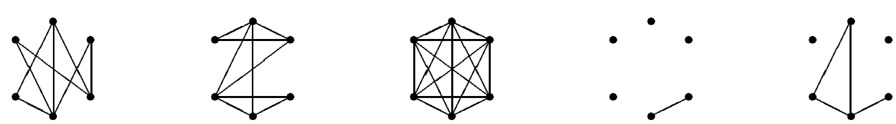

Figure 2: *From left to right, a graph G, its 2-step graph, square, (3)-neighborhood graph, and [3]-neighborhood graph.*

Next we discuss the relationship between these graphs and the p-competition graphs of symmetric digraphs.

II. p-Neighborhood Graphs as p-Competition Graphs of Symmetric Digraphs. We now consider only symmetric digraphs, and establish the relationship between p-competition graphs and p-neighborhood graphs. This generalizes the work in [11] on 2-step graphs and in [19] on the square of a graph.

Proposition 1 *If symmetric digraph D is loopless, then $C_p(D) = N_p(G)$, where G is the graph underlying D.*

Proof: Edge $[x, y] \in E(C_p(D)) \Leftrightarrow |O(x) \cap O(y)| \geq p$ in $D \Leftrightarrow |N(x) \cap N(y)| \geq p$ in $G \Leftrightarrow [x, y] \in E(N_p(G))$. $\square$

Proposition 2 *If symmetric digraph D has a loop at every vertex, then $C_p(D) = N_p[G]$, where G is the loopless underlying graph of D.*

Proof: Edge $[x, y] \in E(C_p(D)) \Leftrightarrow |O(x) \cap O(y)| \geq p$ in $D \Leftrightarrow |N[x] \cap N[y]| \geq p$ in $G \Leftrightarrow [x, y] \in E(N_p[G])$. $\square$

Using these propositions and a characterization of p-competition graphs of [9], we characterize those graphs which are p-neighborhood graphs as follows. The result in Theorem 2 is mentioned by Majors and McMorris in [17], and Theorems 1 and 2 are proved by Majors in [16]. The proofs in [16] do not make use of the competition graph relationship, and as a result are considerably more lengthy than those presented here.

Theorem 1 [16] *A graph G is a (p)-neighborhood graph if and only if there is a family $\mathbf{F} = \{S_1, \ldots, S_n\}$ of subsets of $V(G)$ with $n = |V(G)|$ satisfying the following three conditions:*

1. *For every set $i_1, \ldots, i_p$ of p distinct indices, $\mathbf{T} = S_{i_1} \cap \ldots \cap S_{i_p}$ is either empty or induces a clique of G and the collection of sets of the form $\mathbf{T}$ covers all edges of G.*
2. *Vertex $i \in S_j$ if and only if vertex $j \in S_i$.*
3. *Vertex $i \notin S_i$ for all $i \in \{1, \ldots, n\}$.*

Proof: $(\Rightarrow)$ If G is a (p)-neighborhood graph, then there exists a graph H such that $N_p(H) = G$. Let $S_i = N_H(i)$ for every vertex i. Then the family of sets S_i satisfies conditions (1)-(3).

$(\Leftarrow)$ Condition (1) convinces us that G is the p-competition graph of some digraph D. We can construct D by letting $S_i = In(v_i)$ for all vertices v_i in D. Condition (2) forces D to be a symmetric digraph, and condition (3) forces D to be loopless. We conclude that G is the p-competition graph of a loopless symmetric digraph D, and by Proposition 1 G is the (p)-neighborhood graph of the underlying graph of D. $\square$

Theorem 2 [16], [17] *A graph G is a $[p]$-neighborhood graph if and only if there is a family $\mathbf{F} = \{S_1, \ldots, S_n\}$ of subsets of $V(G)$ with $n = |V(G)|$ satisfying the following three conditions:*

1. *For every set $i_1, \ldots, i_p$ of p distinct subscripts, $\mathbf{T} = S_{i_1} \cap \ldots \cap S_{i_p}$ is either empty or induces a clique of G and the collection of sets of the form $\mathbf{T}$ covers all edges of G.*
2. *Vertex $i \in S_j$ if and only if vertex $j \in S_i$.*
3. *Vertex $i \in S_i$ for all $i \in \{1, \ldots, n\}$.*

Proof: The proof is analogous to that of Theorem 1, with condition (3) convincing us that D has a loop at every vertex. The result follows from Proposition 2. □

We now consider the relationship between p-neighborhood graphs and 2-step graphs and squares. The case $p = 1$, where D is loopless and symmetric, is studied in [11], and we have $S_2(G) = C(D) = N(G)$. Similarly, if symmetric D has a loop at every vertex, we find in [19] that $G^2 = N[G]$. We next give a necessary condition for the 2-step graph of a graph G to be the (p)-neighborhood graph of G (for $p > 1$), and a necessary condition for the square of a graph G to be the $[p]$-neighborhood graph of G (for $p > 1$).

Theorem 3 *Given a graph G, if $N_p(G) = S_2(G)$ and $p > 1$, then every P_3 of G lies in a $K_{2,p}$.*

Proof: Suppose $N_p(G) = S_2(G)$, and assume there exists a P_3 in G which does not lie in a $K_{2,p}$. Let the consecutive vertices of the P_3 be x, y, and z. Vertices x and z have fewer than p common neighbors, so $[x,z] \notin E(N_p(G))$. However, since x and z have a path of length 2 between them through y, $[x,z] \in E(S_2(G))$, a contradiction. Hence every P_3 of G lies in a $K_{2,p}$. □

For $[p]$-neighborhood graphs and squares, we present a similar necessary condition.

Theorem 4 *Given a graph G, $N_p[G] = G^2$ and $p > 1$ imply every induced P_3 of G lies in a $K_{2,p}$.*

Proof: Assume $N_p[G] = G^2$ and there exists an induced P_3, xyz, that does not lie in a $K_{2,p}$. Since xyz is an *induced* P_3, $[x,z]$ is not an edge of G, so $|N[x] \cap N[z]| \leq p-1$, and $[x,z] \notin E(N_p[G])$. However, $[x,z] \in E(G^2)$, a contradiction. □

Note that for general p, every induced P_3 of G lying in a $K_{2,p}$ is not a sufficient condition for $N_p[G] = G^2$. This is because $[x,y] \in E(G^2)$ does not imply $[x,y] \in E(N_p[G])$ (let $G = K_{3,3}$, for example). However, when $p = 2$ it is a sufficient condition, as we note next.

Theorem 5 *Given a graph G,*
(a) $N_2(G) = S_2(G)$ if and only if every P_3 of G lies in a 4-cycle, and
(b) $N_2[G] = G^2$ if and only if every induced P_3 of G lies in a 4-cycle.

Proof: For both (a) and (b), necessity follows as a corollary of Theorem 4, since $Z_4 = K_{2,2}$. To prove sufficiency in part (a), suppose every P_3 of G lies in a Z_4. Then $[x,y] \in E(N_2(G)) \Leftrightarrow x$ and y have at least 2 common neighbors in $G \Leftrightarrow [x,y] \in E(S_2(G))$. The last reverse implication is valid since $[x,y] \in E(S_2(G))$ implies x and y have at least one common neighbor, and by our assumption, one common neighbor implies the existence of another. For sufficiency in part (b), suppose every induced P_3 of G lies in a Z_4. Then $[x,y] \in E(G^2) \Leftrightarrow x$ and y have a path of length 2 between them in G or $[x,y] \in E(G)$ (perhaps both) $\Leftrightarrow x$ and y have an *induced* path of length 2 between them in G or $[x,y] \in E(G)$ (not both) $\Leftrightarrow x$ and y have

2 induced paths of length 2 between them or $[x,y] \in E(G)$ (not both) $\Leftrightarrow$ x and y are diagonally opposed on an induced 4-cycle of G or $[x,y] \in E(G)$ (not both) $\Leftrightarrow$ $|N[x] \cap N[y]| \geq 2 \Leftrightarrow [x,y] \in E(N_2[G])$. Hence $N_2[G] = G^2$. □

In the next section we identify some familiar graphs as (p)-neighborhood graphs for the case $p = 2$.

III. Which Graphs are (2)-Neighborhood Graphs? It is a relatively simple matter to construct the (p)-neighborhood graph of a graph G. However, identifying those graphs which are (p)-neighborhood graphs is considerably more complicated. Given a graph G, we must find a graph H such that $N_p(H) = G$. The case $p = 1$ is fully explored in [3], where the authors prove that the only nontrivial connected triangle-free open neighborhood graphs are the odd cycles on 5 or more vertices. For the case $p = 2$ we have identified some familiar graphs as (p)-neighborhood graphs, and continue to seek a full classification of such graphs. The following result regarding triangle-free graphs is a useful tool in proving the theorems of this section.

Lemma 1 *If a triangle-free graph G is the (2)-neighborhood graph of a graph H, then* $2(Z_4(H)) = e(G)$.

Proof: Suppose $G = N_2(H)$ and G is triangle-free. To see that $2(Z_4(H)) \geq e(G)$, we simply note that the $Z_4(H)$ 4-cycles in H produce at most $2(Z_4(H))$ edges in G. In fact, this inequality holds whether or not G is triangle-free. Now assume $2(Z_4(H)) > e(G)$. Then some pair of vertices, say x and y, is diagonally opposed on more than one 4-cycle of H, as shown in Figure 3. However, if the graph shown in Figure 3 is a subgraph of H, then triangle abc is a subgraph of $N_2(H) = G$, a contradiction. Hence $2(Z_4(H)) = e(G)$. □

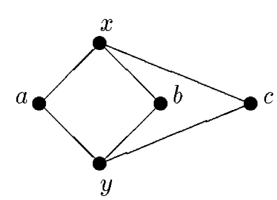

Figure 3: *If the graph above is a subgraph of H, then triangle abc is a subgraph of $N_2(H)$.*

Theorem 6 *The cycle Z_n is a (2)-neighborhood graph if and only if n is even and $n \geq 6$.*

Proof: By the lemma, odd cycles are not (2)-neighborhood graphs, so we consider the cycle Z_{2k}. If $k = 2$, we know from [7] that Z_4 is not the 2-competition graph of *any* digraph; in particular it is not the 2-competition graph of a loopless symmetric digraph, i.e. a (2)-neighborhood graph. The Möbius ladder (e.g. graph H in Figure 4 is a Möbius ladder of order 8) has Z_{2k} as its (2)-neighborhood graph when k is even and $k \geq 4$, and the ladder (e.g. graph H' in Figure 4 is a ladder of order 6) has Z_{2k} as its (2)-neighborhood graph when k is odd. (See Figure 4 for examples.) □

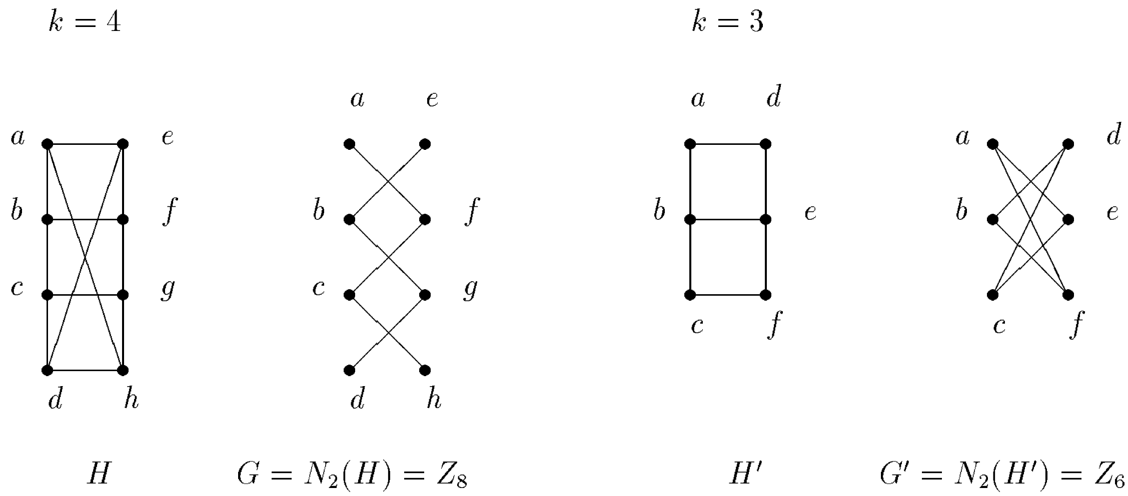

Figure 4: *Z_8 is the (2)-neighborhood graph of the Möbius ladder on 8 vertices H, while Z_6 is the (2)-neighborhood graph of the ladder on 6 vertices H'.*

In the triangle-free case, the challenge in determining whether or not a graph G is a (2)-neighborhood graph is in finding a special pairing of the edges of G. This pairing must determine the 4-cycles of H, so that $N_2(H) = G$. With C_{2k}, we see that if an edge is paired with the edge directly opposite it on the cycle and these pairings are used to determine the 4-cycles of H, we have a method of creating H such that $N_2(H) = C_{2k}$. A similar idea works for P_n.

Theorem 7 *P_n is a (2)-neighborhood graph if and only if n is odd and $n \geq 5$.*

Proof: Suppose P_n is a (2)-neighborhood graph. By Lemma 1, $e(P_n)$ must be even, so n is odd. The graphs on 3 vertices are K_3, P_3, $K_2 \cup I_1$, and I_3; none of these has P_3 as its (2)-neighborhood graph, so P_3 is not a (2)-neighborhood graph. Thus $n \geq 5$. Conversely, let the vertices of P_n be labelled consecutively as $v_1, v_2, \ldots, v_n$, with $n \geq 5$, $n = 2k+1$. We must construct H such that $N_2(H) = P_n$. Label the edges of P_n consecutively as $e_1, e_2, \ldots, e_k, e'_1, e'_2, \ldots, e'_k$. If we pair edge e_i with edge e'_i, then the four endpoints of those two edges are distinct for all i (since $n \geq 5$). With those four vertices we create the 4-cycle $v_i v_{i+k} v_{i+1} v_{i+k+1}$ in H. The diagonals of this 4-cycle are $\{v_i, v_{i+1}\}$ and $\{v_{i+k}, v_{i+k+1}\}$, giving edges $[v_i, v_{i+1}] = e_i$ and $[v_{i+k}, v_{i+k+1}] = e'_i$ in $N_2(H)$. Let H be the graph consisting only of these 4-cycles. Then $N_2(H) = P_n$. □

See Figure 5 for an example of the method described in the proof of Theorem 7 for P_7. The problem of determining which trees are (2)-neighborhood graphs remains unsolved. However, we do know exactly which caterpillars G are (2)-neighborhood graphs, as presented next. The proof uses a pairing of the edges of G that is similar to ideas used in the last two proofs.

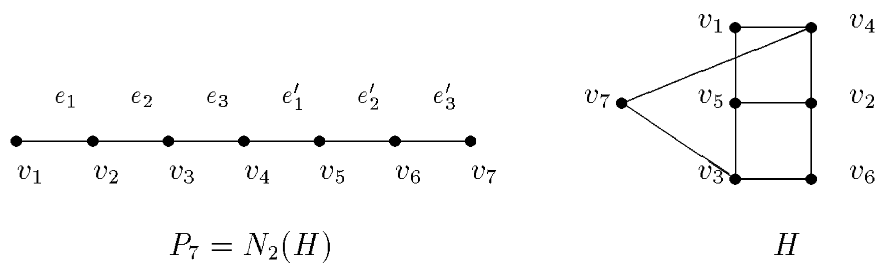

Figure 5: *P_7 is the (2)-neighborhood graph of H.*

Theorem 8 *A caterpillar G is a (2)-neighborhood graph if and only if $e(G) = 2k \geq 2\Delta(G)$.*

Proof: For necessity, suppose $G = N_2(H)$. By the lemma, $e(G) = 2k$. In order to see that $e(G) \geq 2\Delta(G)$, note that a vertex of degree $\Delta(G)$ in G must be on $\Delta(G)$ 4-cycles in H. Thus $Z_4(H) \geq \Delta(G)$, and by the lemma, $e(G) = 2(Z_4(H))$, so $\frac{e(G)}{2} \geq \Delta(G)$ and $e(G) \geq 2(\Delta(G))$.

The proof of sufficiency is algorithmic. We present the algorithm next, and give an example of its use in Figure 7.

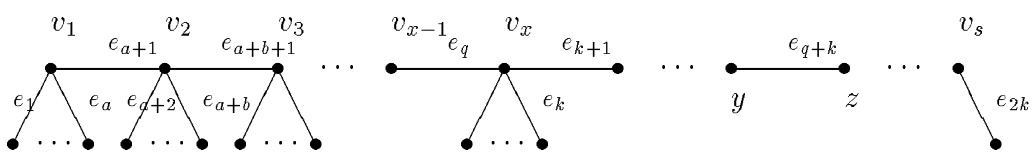

Figure 6: *A caterpillar labelled as in the proof of Theorem 8. Note that it is possible (although not depicted here) that both e_k and e_{k+1} are pendant edges at v_x, or that e_k is nonpendant (hence $e_k = e_q$) and e_{k+1} is pendant, or that both edges are nonpendant. Also note that e_{q+k} may or may not be a pendant edge. In either case, $y = v_i$ is a nonpendant vertex.*

Let G be a caterpillar with $e(G) = 2k \geq 2\Delta(G)$. Label G as in Figure 6. That is, choose as a starting point one of the nonpendant vertices which is adjacent to exactly one other nonpendant vertex and call it v_1. (There are two such vertices from which to choose.) Label the pendant edges incident at vertex v_1 with $e_1, \ldots, e_a$. Assign the label e_{a+1} to the edge connecting v_1 with the next nonpendant vertex, v_2. Label the pendant edges incident at v_2 as $e_{a+2}, \ldots, e_{a+b}$. Assign the label e_{a+b+1} to the edge connecting v_2 with its one unlabelled (and nonpendant) neighbor v_3. Continue in this manner until t he edges of G have been labelled $e_1, e_2, \ldots, e_{2k}$. Note that for $i = 1$ to k, e_i and e_{i+k} have four distinct vertices as endpoints. This is due to the method used in labelling the edges and to the fact that $k \geq \Delta(G)$.

Consider the edges of G to be in two halves, the first one being $e_1, \ldots, e_k$ (call this the *red* half) and the second one being $e_{k+1}, \ldots, e_{2k}$ (call this the *blue* half). Because of the method used to label the edges of G, exactly one vertex appears in both halves, call it vertex v_x. Vertex v_x is an endpoint of edge e_{k+1} and is nonpendant. It is also an

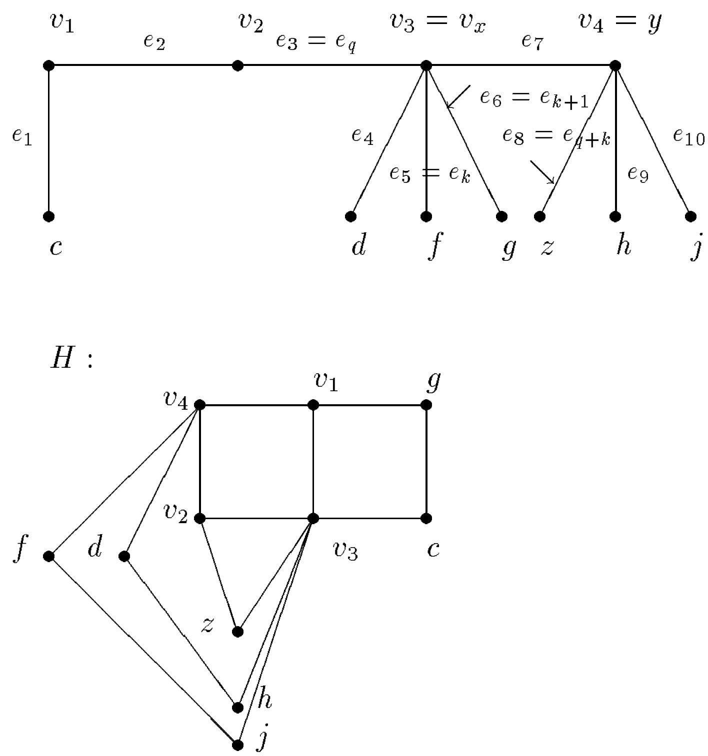

Figure 7: *The caterpillar above is the (2)-neighborhood graph of H.*

endpoint of edge e_k and possibly some edge(s) preceding e_k. Let e_q be the "first" edge in $e_1, \ldots, e_k$ which has vertex v_x as an endpoint. Then e_q connects two nonpendant vertices.

We use these labels and facts to construct a graph H such that $N_2(H) = G$. First, pair edge e_1 with edge e_{k+1}. By construction, the endpoints of these edges are distinct. With these four vertices, form a 4-cycle, C_1, on which the endpoints of e_1 are diagonally opposed and the endpoints of e_{k+1} are diagonally opposed. Create a 4-cycle, C_2, using the four vertices of e_2 and e_{k+2} in the same manner. Note, however, that C_2 shares an edge with C_1, since e_1 and e_2 are adjacent in G and e_{k+1} and e_{k+2} are adjacent in G. In fact, continuing in this manner, building one 4-cycle after another, each new 4-cycle will share exactly one edge with some 4-cycle(s) which preceded it, until we reach edge e_q. This edge will be paired up with edge e_{q+k}, but we cannot build the necessary 4-cycle by simply adding two new vertices as we have up to this point. This is because v_x was one of the vertices of C_1. The endpoints of e_q are v_{x-1} and v_x. Let the endpoints of e_{q+k} be y and z. Let $y = v_i$ for some i (so y is a nonpendant vertex), and if z is a nonpendant vertex, then $z = v_{i+1}$.

Up to this point, the graph (call it H') consists of 4-cycles connected in such a way that any given 4-cycle shares exactly one of its edges with other 4-cycles, so H' is bipartite. In fact, we can color the vertices from the red list red, and those from the blue list blue. Note that vertex v_x is colored blue, since it has not yet been used to form an edge from the red half of the list. Vertex v_{x-1} is red and vertex y is blue (vertex z is not yet part of th e construction).

The graph H' does contain edge $[v_{x-1}, y]$ (by construction). In order to create the 4-cycle $v_{x-1}yv_xz$, which we need in order to get edges e_q and e_{q+k} in $N_2(H)$, we must add edge $[y, v_x]$ to H' without creating any unwanted 4-cycles. So, we must check to see if there are any paths of length 3 between y and v_x in H'. Any path of length 3 in H' would have vertices of different colors as endpoints. Since y and v_x are both blue, no such path exists between them, so we can add edge $[y, v_x]$ to H' without creating any unwanted 4-cycles. By a similar argument, there is no path of length 2 between v_x and v_{x-1} (other than the new one through y), so we can add vertex z with edges $[z, v_{x-1}]$ and $[z, v_x]$, creating only the 4-cycle desired, $v_{x-1}yv_xz$.

¿From this point on, continue as before, building the necessary new 4-cycles from already existing edges by adding two new vertices each time. Continuing in this manner until all k 4-cycles have been completed yields a graph H such that $N_2(H) = G$. □

Although $e(G) = 2k \geq 2\Delta(G)$ is a necessary condition for any tree to be a (2)-neighborhood graph, it is not a sufficient condition, as the tree G in Figure 8 illustrates. That graph is the smallest tree that is not a caterpillar, and we refer to it as NC_7.

Theorem 9 *The tree NC_7 is not a (2)-neighborhood graph.*

Proof: Let NC_7 be labelled as in Figure 8. Suppose there exists a graph H such that $N_2(H) = NC_7$. Then vertex e must be on three distinct 4-cycles in H, no two of which share an edge. This is because if e is on two 4-cycles that share two edges in H, $N_2(H)$ contains a triangle (a contradiction), and if e is on two 4-cycles that share exactly one edge in H, then $N_2(H)$ contains a path of length 3 that does not include vertex e (also a contradiction). However, if in H vertex e is on 3 distinct 4-cycles, no two of which share an edge, H must consist of at least 10 vertices. We conclude that no graph H exists such that $N_2(H) = NC_7$. □

NC_7 :

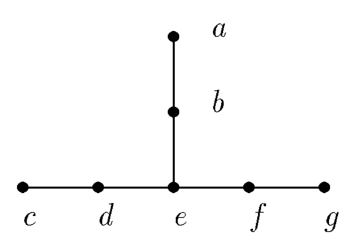

Figure 8: *Although tree NC_7 satisfies $e(NC_7) = 2k \geq 2\Delta(NC_7)$, NC_7 is not a (2)-neighborhood graph.*

The argument in the proof of Theorem 9 leads to an infinite family of trees that are not (2)-neighborhood graphs. If we append at vertex e paths of length one and/or two to NC_7, then the resulting tree is not a (2)-neighborhood graph. This result follows from precisely the same type of argument as in the proof.

There are, however, some trees which are not caterpillars which are (2)-neighborhood graphs, as we discuss next. We conclude that no forbidden subgraph characterizations are available in classifying those graphs which are (2)-neighborhood graphs, since every tree that is not a caterpillar has NC_7 as an induced subgraph.

Given any caterpillar G satisfying $e(G) = 2k \geq 2\Delta(G)$, we can create larger trees which have G as a subgraph and are (2)-neighborhood graphs.

Lemma 2 *Let G be a caterpillar, and suppose $G = N_2(H)$ for some graph H. Then if $[x, y] \in E(H)$, we can create a tree G', consisting of G with two new pendant vertices, one adjacent to x and one adjacent to y, such that G' is a (2)-neighborhood graph. In fact $G' = N_2(H')$, where H' is H plus a new 4-cycle appended to edge $[x, y]$.*

Proof: Suppose $G = N_2(H)$ is a caterpillar, and $[x, y] \in E(H)$. We create graph H' as follows. Begin with graph H. Add vertices x' and y' to H, with edges $[x, y'], [y', x']$, and $[x', y]$. Since $[x, y] \in E(H)$, we have created the 4-cycle $xy'x'y$ without making any other changes to H. Then $G' = N_2(H')$ is simply G plus two new pendant vertices x' and y', adjacent to x and y respectively.□

We illustrate the use of Lemma 2 with an example in Figure 9, and give a sufficient condition for a tree to be a (2)-neighborhood graph in Lemma 3.

Lemma 3 *Let T be a tree. If there exists a pair of vertices x and y in T meeting the conditions below, then T is a (2)-neighborhood graph.*

1. *Vertices x and y are pendant in T with respective neighbors x' and y', $x' \neq y'$.*
2. *There exists a graph H' such that $N_2(H') = T'$ and $[x', y'] \in E(H')$, where $T' = T - \{x, y\}$.*

Proof: This follows from Lemma 2. □

The next class of trees we consider are the *balanced binary trees*, where all leaves are at the same level, and every internal vertex has two children. We use Lemma 3 to show that all balanced binary trees are (2)-neighborhood graphs.

Lemma 4 *The balanced binary tree on 31 vertices, B_{31}, is a (2)-neighborhood graph.*

Proof: We begin with the caterpillar shown in bold in Figure 10; call this caterpillar G. We have $e(G) = 2k \geq 2\Delta(G)$, so we know G is a (2)-neighborhood graph, and certainly G is a subgraph of B_{31} (as depicted in the figure). We use G and the lemmas to show that B_{31} is a (2)-neighborhood graph. We find H such that $N_2(H) = G$, with the added property that H contains edges [1,9], [10,15], [11,14], and [12,13] (see Figure 10). Using H and the lemmas, we can create a graph H' such that $N_2(H') = B_{31}$ and H' contains edges [1,9], [10,15], [16,23], [17,22], [24,31], [25,30], [26,29], and [27,28]. We accomplish this by appending 4-cycles to existing edges in H. The graph H' consists of H plus the following 4-cycles: $< 12, 13, 18, 21 >, < 12, 13, 19, 20 >, < 11, 14, 16, 23 >, < 11, 14, 17, 22 >, < 21, 18, 31, 24 >, < 21, 18, 30, 25 >, < 20, 19, 29, 26 >$, and $< 20, 19, 28, 27 >$. □

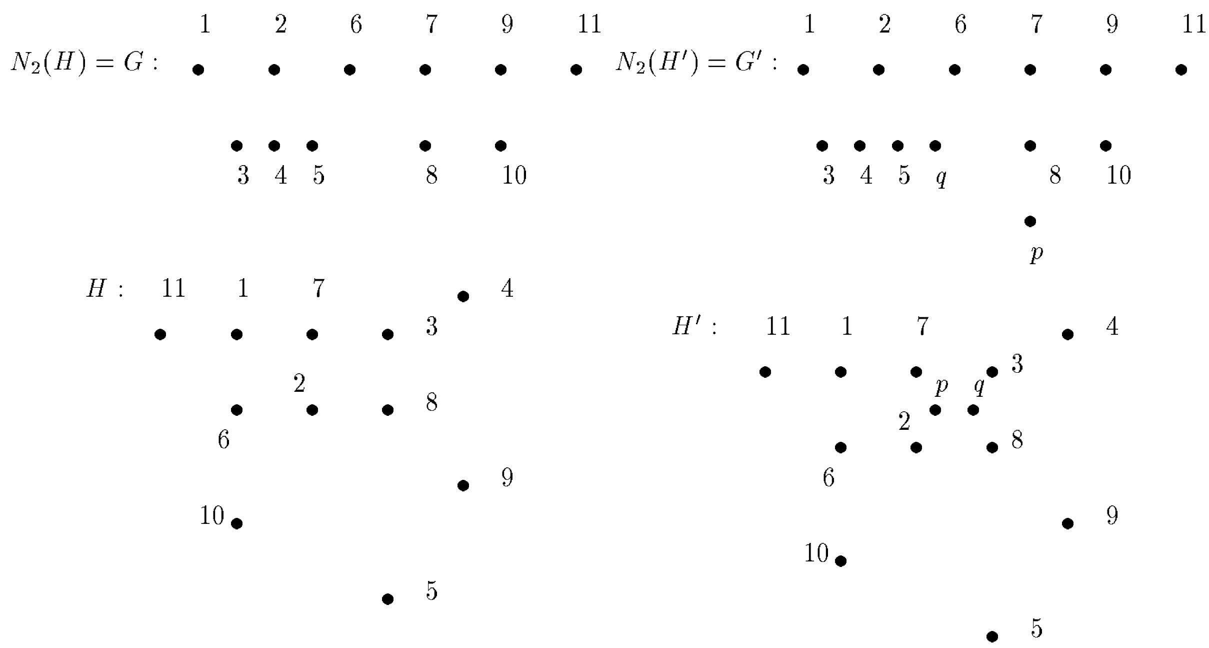

Figure 9: *We illustrate the use of Lemma 2 by choosing an edge in H (in this case, edge [2,8]), and add 4-cycle $2pq8$ to H to create H'. This gives tree $G' = N_2(H')$, which has caterpillar G as a generated subgraph.*

Theorem 10 *All balanced binary trees B_n on $n = 2^k - 1 \geq 7$ vertices are (2)-neighborhood graphs.*

Proof: We begin by considering the three smallest balanced binary trees. P_3 is the smallest, and as noted earlier is not a (2)-neighborhood graph. The next balanced binary tree B_7 is on 7 vertices and is pictured in Figure 11. B_7 is a caterpillar satisfying $e(B_7) = 2k \geq 2\Delta(B_7)$, so B_7 is a(2)-neighborhood graph by Theorem 8. The next B_n we must consider is B_{15}, which is shown in Figure 11 with a graph H such that $B_{15} = N_2(H)$, so B_{15} is a (2)-neighborhood graph.

For $k \geq 5$ we proceed using induction on k. We claim: Given a balanced binary tree B_{2^k-1} with $k \geq 5$, there exists a graph H such that $N_2(H) = B_{2^k-1}$, and there is an edge in H between every pair of "mirror image" leaves in B_{2^k-1}. (Note : we consider the graph to be drawn in the standard fashion as depicted in Figures 10 and 11 and can consequently pair each leaf with its reflection in the vertical line through the root vertex.)

See Lemma 4 for the base case, $k = 5$, B_{31}. Suppose for every $k < n$, $k > 5$, there exists a graph H satisfying the conditions just listed. Consider B_{2^n-1}. It has 2^{n-1} leaves. Label them and their neighbors as shown in Figure 12. Note that $(x \bowtie y)$ is the parent of leaves x and y.

Remove vertices $1, 2, \ldots, 2^{n-2} - 1, 2^{n-2}, 2^{n-2}*, 2^{n-2} - 1*, \ldots, 2*$, and $1*$. We now have $B_{2^{n-1}-1}$, a balanced binary tree whose leaves are labelled $(1 \bowtie 2), (3 \bowtie$

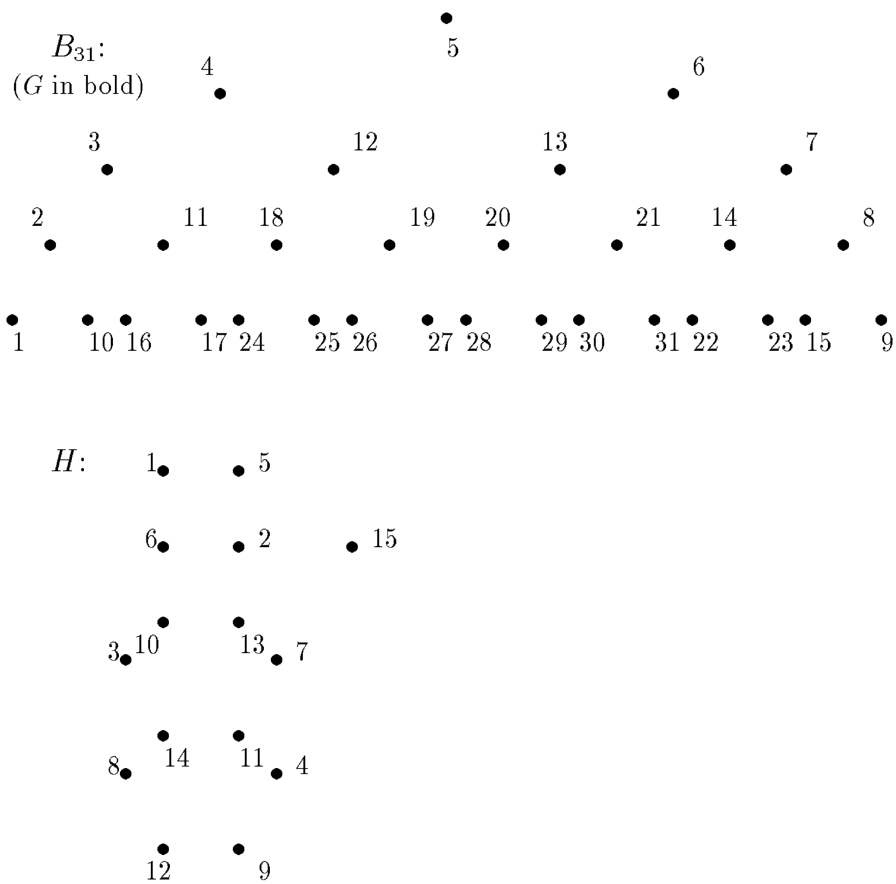

Figure 10: *Graph G (the caterpillar in bold) is a subgraph of B_{31}, and $N_2(H) = G$.*

$4), \ldots, (2^{n-2}-1 \bowtie 2^{n-2}), (2^{n-2}-1 \bowtie 2^{n-2})*, \ldots, (3 \bowtie 4)*$, and $(1 \bowtie 2)*$. By induction, we can create graph H such that $N_2(H) = B_{2^{n-1}-1}$ and H contains edges $[(1 \bowtie 2), (1 \bowtie 2)*], [(3 \bowtie 4), (3 \bowtie 4)*], \ldots,$ and $[(2^{n-2} - 1 \bowtie 2^{n-2}), (2^{n-2} - 1 \bowtie 2^{n-2})*]$. We now add vertices and 4-cycles to H, to create graph H' such that $N_2(H') = B_{2^n-1}$ and H' contains edges $[1, 1*], [2, 2*], \ldots, [2^{n-2}, 2^{n-2}*]$, as follows. To edge $[(1 \bowtie 2), (1 \bowtie 2)*]$ in H we append 4-cycles $< (1 \bowtie 2), (1 \bowtie 2)*, 1, 1* >$ and $< (1 \bowtie 2), (1 \bowtie 2)*, 2, 2* >$; to edge $[(3 \bowtie 4), (3 \bowtie 4)*]$ in H we append 4-cycles $< (3 \bowtie 4), (3 \bowtie 4)*, 3, 3* >$ and $< (3 \bowtie 4), (3 \bowtie 4)*, 4, 4* >$; $\ldots$; and to edge $[(2^{n-2} - 1 \bowtie 2^{n-2}), (2^{n-2} - 1 \bowtie 2^{n-2})*]$ in H we append 4-cycles $< (2^{n-2} - 1 \bowtie 2^{n-2}), (2^{n-2} - 1 \bowtie 2^{n-2})*, (2^{n-2} - 1), (2^{n-2} - 1)* >$ and $< (2^{n-2} - 1 \bowtie 2^{n-2}), (2^{n-2} - 1 \bowtie 2^{n-2})*, (2^{n-2}), (2^{n-2})* >$. Thus we have graph H' such that $N_2(H') = B_{2^n-1}$ and H' contains edges $[1, 1*], [2, 2*], \ldots, [2^{n-2}, 2^{n-2}*]$, as desired. □

This paper has just begun the effort of determining classes of graphs which are or are not (2)-neighborhood graphs. Future work should include attempts to generalize previous results on neighborhood graphs, 2-step graphs, and the square of a graph.

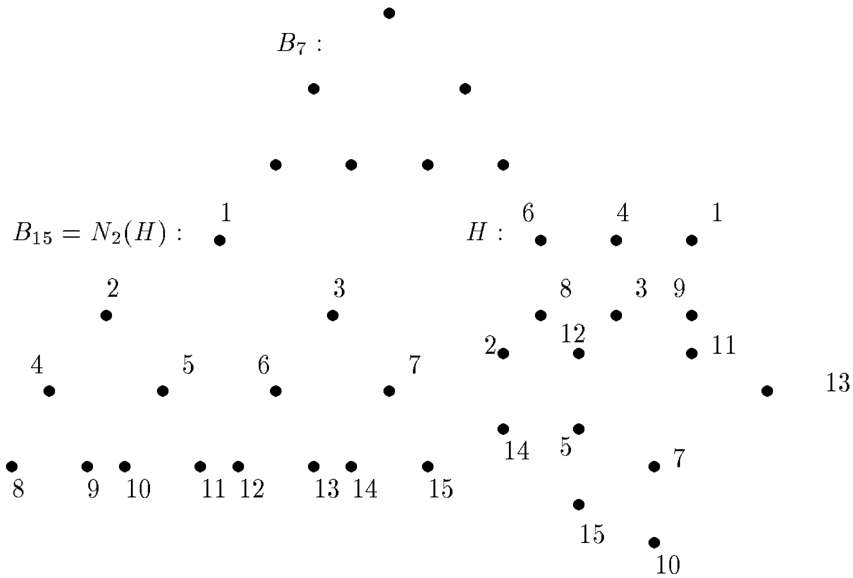

Figure 11: *B_7 is a (2)-neighborhood graph, since it satisfies Theorem 8, and B_{15} is a (2)-neighborhood graph, since $B_{15} = N_2(H)$.*

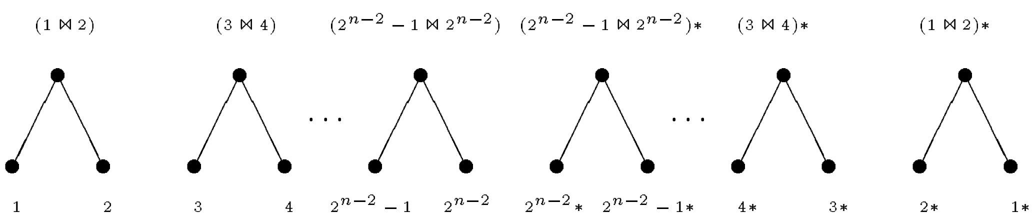

Figure 12: *Labels for leaves and their neighbors in B_{2^n-1}.*

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