The Domination and Competition Graphs of a Tournament

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Abstract. Vertices x and y *dominate* a tournament T if for all vertices $z \neq x, y$, either x beats z or y beats z. Let $\mathrm{dom}(T)$ be the graph on the vertices of T with edges between pairs of vertices that dominate T. We show $\mathrm{dom}(T)$ is either an odd cycle with possible pendant vertices or a forest of caterpillars. Since $\mathrm{dom}(T)$ is the complement of the competition graph of the tournament formed by reversing the arcs of T, complementary results are obtained for the competition graph of a tournament.

1. Introduction. Suppose n tennis players compete in a "round robin" tournament where each players plays every other players exactly once. If players are roughly even in ability and n is fairly large, it is unlikely that one player will beat every other player. It is more likely there are two players so that every other player is beaten by at least one of the two (such a pair might form a good "doubles" team). We show that regardless of the results, there are at most n such pairs.

The results can be modeled by a "tournament". A *digraph* D is a set $V(D)$ of vertices and a set $A(D)$ of ordered pair of vertices called arcs. We will denote an arc from x to y by $(x, y) \in A(D)$ and say x *beats* y. For all vertices x, let $O_D(x)$ or $O(x)$ (the *out-set* of x) be the set of the vertices that x beats. Similarly, let $I_D(x)$ or $I(x)$ (the *in-set*) be the set of vertices that beat x. Let $d^+(x) = |O(x)|$ be the *out-degree* of x. Let $\Delta^+(D)$ be the maximum of $d^+(x)$ over all vertices x in $V(D)$. A *tournament* T is a digraph without loops (i.e., arcs of the form (x, x)) in which for all x and y (distinct) in $V(T)$, either $(x, y) \in A(T)$, or $(y, x) \in A(T)$, but not both. An *n-tournament* is a tournament with n vertices. A regular tournament is one in which $d^+(x)$ is constant for all vertices x. See Moon [8] and Reid and Beineke [10] for more about tournaments.

Given a digraph D, vertices x and y *dominate* D if $O(x) \cup O(y) \cup \{x, y\} = V(D)$. Let the *domination graph* of D, denoted $\mathrm{dom}(D)$, be the graph with vertices $V(D)$ and edges between the pairs of vertices that dominate D (see Figure 1).

The domination graph is closely related to the "competition graph". Given a digraph D, the *competition graph* of D, denoted $C(D)$, is the graph with the same vertices as D and an edge between vertices x and y if and only if $O(x) \cap O(y) \neq \emptyset$. The domination graph of a tournament T is the complement of the competition graph of the tournament formed by reversing the arcs of T (this is not necessarily true for all digraphs). So for tournaments, results on domination graphs correspond to results on competition graphs. However, since the domination graph of a tournament generally

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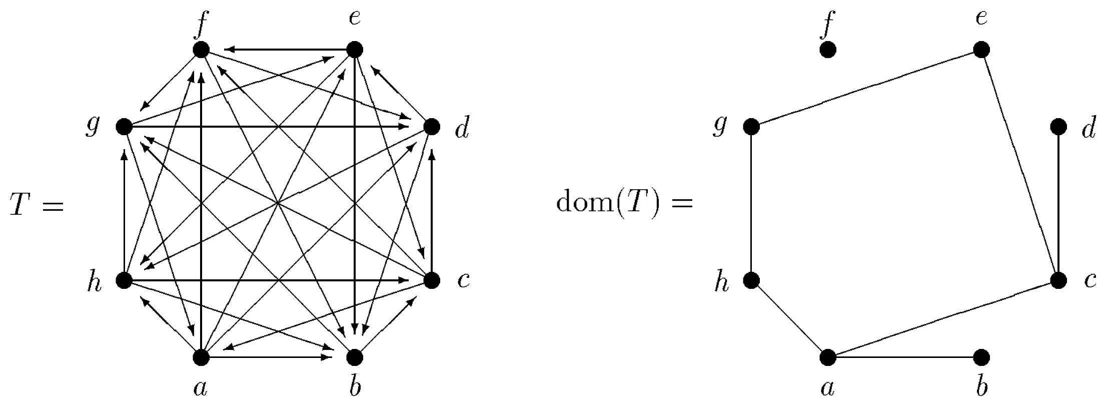

FIG. 1. *A tournament and its domination graph. The edges of the domination graph dom(T) are all pairs of vertices that dominate the tournament T. For example, vertices c and e are adjacent in dom(T) because in T vertex c beats a, g, f and d, and vertex e beats b, c, f and h.*

has fewer edges than the competition graph, it is more convenient to state and prove results on domination graphs.

Competition graphs were introduced by Cohen [2, 3] in the study of food webs. A food web can be modeled by a digraph whose vertices represent various species, with an arc from vertex x to vertex y if the species represented by x preys upon the species represented by y. If two vertices have arcs to a common vertex, this represents two species competing for common prey, hence the name "competition graph".

Competition graphs and their generalizations have been extensively studied, for example, by Brigham and Dutton [1], Kim, McKee, McMorris, and Roberts [5], and Roberts and Raychaudhuri [9]. Competition graphs of tournaments were first considered by Lundgren, Merz, and Rasmussen [7]. Comprehensive surveys on competition graphs are provided by Kim [4] and Lundgren [6].

2. Domination Graphs of Tournaments. *Which graphs can be the domination graph of a tournament?* We partially answer this by finding graphs that are not subgraphs of dom(T) for all tournaments T. A subset of the vertices of a graph G is *independent* if there are no edges between the vertices in the subset.

PROPOSITION 2.1. *Let T be a tournament with $z \in V(T)$. Then $O(z)$ is an independent set of dom(T).*

Proof. Let $x, y \in O(z)$. Then $z \notin O(x) \cup O(y) \cup \{x, y\}$. So x and y do not dominate T and hence are not adjacent in dom(T). □

Let $\alpha(G)$ (the *independence number* of G) denote the maximum cardinality of an independent set of G.

COROLLARY 2.2. *For any tournament T, we have $\Delta^+(T) \leq \alpha(dom(T))$.*

Let C_k denote the undirected cycle on k vertices.

LEMMA 2.3. *Let $k \geq 4$ be an even number and T be any k-tournament. Then C_k is not a subgraph of dom(T).*

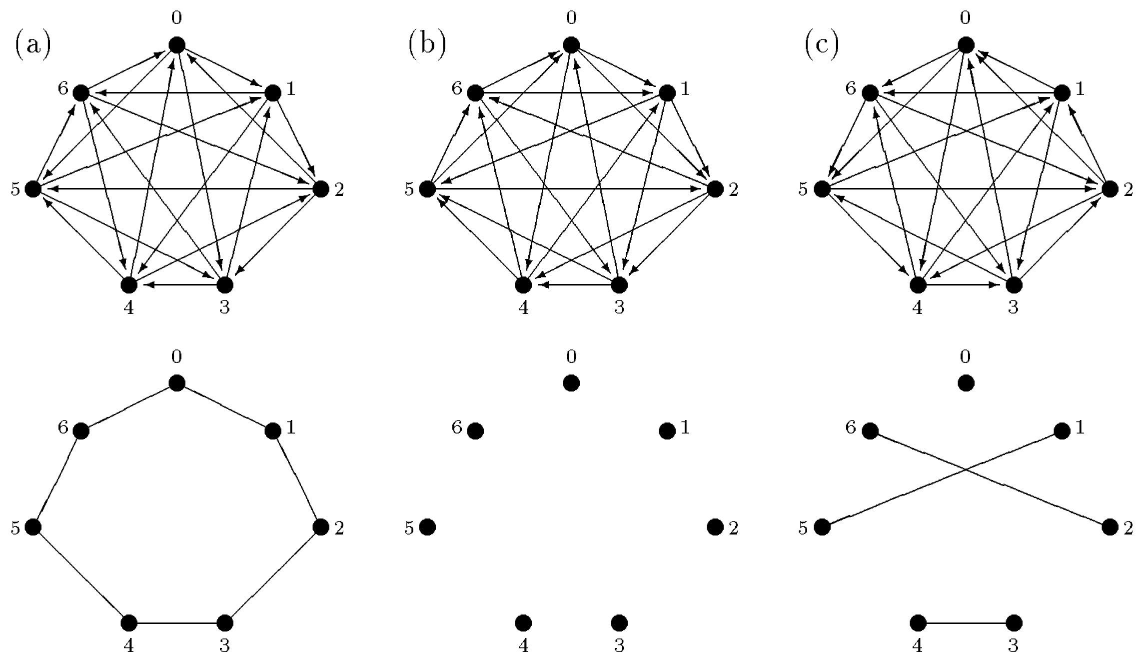

FIG. 2. *Regular tournaments on 7 vertices and their domination graphs.*

Proof. Suppose C_k is a subgraph of dom(T). Let $1, 2, \ldots, k$ be the consecutively labeled vertices of C_k. As k is even, $\Delta^+(T) \geq \frac{k}{2}$. Corollary 2.2 shows $\Delta^+(T) \leq \alpha(\text{dom}(T)) \leq \alpha(C_k) = \frac{k}{2}$. So $\Delta^+(T) = \frac{k}{2}$. The only two possible independent sets of order $\frac{k}{2}$ in dom(T) are the two maximal independent sets of C_k given by $A = \{1, 3, \ldots, k-1\}$ and $B = \{2, 4, \ldots, k\}$. Say $d^+(1) = \frac{k}{2}$. Then by Proposition 2.1, $O(1) = B$. Then for any $i \in B$, we have $O(i) \neq A$ because 1 beats i, and $O(i) \neq B$ because $i \in B$. For any $i \in A - \{1\}$, we have $O(i) \neq A$ because $i \in A$, and $O(i) \neq B$ because i beats 1. Thus $d^+(i) < \frac{k}{2}$ for all $i \neq 1$. So by symmetry, at most one vertex of T can have out-degree $\frac{k}{2}$. Thus the sum of the out-degrees of the vertices of T is at most $\frac{k}{2} + (k-1)\left(\frac{k}{2} - 1\right)$. For $k \geq 3$, this is less than the $\frac{k(k-1)}{2}$ arcs in T, a contradiction. □

Let $k \geq 3$ be an odd integer. Let S be a $\left(\frac{k-1}{2}\right)$-set contained in $\mathcal{Z}_k$ (the integers mod k) where $0 \notin S$ and $s_1 + s_2 \not\equiv 0$ for all $s_1, s_2 \in S$. For such sets, we can form a regular tournament $T(S)$ called the *rotational tournament with symbol* S whose vertices are labeled by the elements of $\mathcal{Z}_k$ and with arcs (i, j) if $j - i \equiv s$ where $s \in S$. Let $U_k = T(\{1, 3, \ldots, k-2\})$. Figure 2(a) shows U_7. Figure 2(b) shows $T(\{1, 2, 4\})$. The regular 7-tournament in Figure 2(c) is not rotational as $O(0)$ and $O(4)$ induce distinct 3-tournaments.

LEMMA 2.4. *Let k be an odd number and T a k-tournament. Then C_k is a subgraph of dom(T) if and only if T is isomorphic to U_k.*

Proof. ($\Leftarrow$) The only dominating pairs in U_k are i and j where $j - i \equiv 1$ or $k-1$. So dom(U_k) is the k-cycle with vertices labelled consecutively by $0, 1, 2, \ldots, k-1$.

($\Rightarrow$) Assume C_k is a subgraph of dom(T). Consecutively label the vertices of C_k

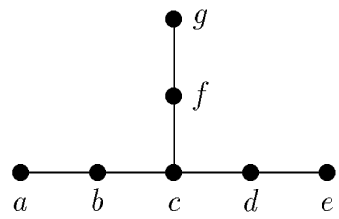

FIG. 3. *This graph is not the induced subgraph of the domination graph of a tournament.*

by $0, 1, 2, \ldots, k-1$. Corollary 2.2 shows $\Delta^+(T) \leq \alpha(\text{dom}(T)) \leq \alpha(C_k) = \frac{k-1}{2}$. Since the average out-degree in a k-tournament is $\frac{k-1}{2}$, we have that $d^+(i) = \frac{k-1}{2}$ for all i. Without loss of generality, assume 0 beats 1. The only independent set of C_k of order $\frac{k-1}{2}$ without either 0 or 1 is $\{2, 4, \ldots, k-1\}$, so $O(1)$ is this set. In particular, 1 beats 2. The only independent set of C_k of order $\frac{k-1}{2}$ without either 1 or 2 is $\{3, 5, \ldots, k\}$, so $O(2)$ is this set. Continuing in this way along the vertices of C_k shows that T is isomorphic to U_k. □

A *tree* is a connected acyclic graph. A *caterpillar* is a tree such that the removal of all pendant (degree one) vertices yields a path. Figure 3 shows the smallest tree that is not a caterpillar. This tree will be called NC7 (non-caterpillar on 7 vertices). It is well known that a tree which is not a caterpillar must contain a copy of NC7.

LEMMA 2.5. *Let T be a 7-tournament. Then NC7 is not a subgraph of dom(T).*

Proof. Suppose NC7 is a subgraph of dom(T) with the vertices of T labeled as in Figure 3. Corollary 2.2 shows $\Delta^+(T) \leq \alpha(\text{dom}(T)) \leq \alpha(\text{NC7}) = 4$. Since the average out-degree in a tournament with 7 vertices is 3, either T is regular or T has a vertex with out-degree 4. Figure 2 shows the domination graphs of the only three nonisomorphic regular 7-tournaments. None of these graphs has NC7 as a subgraph. Thus, T has a vertex with out-degree 4. Since $\{a, c, e, g\}$ is the only 4 vertex independent set in NC7, by Proposition 2.1 only b, d, or f can have out-degree 4 in T. By symmetry, assume without loss of generality that b has out-degree 4. Then b beats a, c, e and g, and loses to d and f. Since b and c dominate T, we have c beats d and f. Now further assume without loss of generality that d beats f. Since f and g dominate T and c and d both beat f, we see that g beats c and d. But then c and d do not dominate T because neither beats g, a contradiction. Thus NC7 is not a subgraph of the domination graph of a 7-tournament. □

LEMMA 2.6. *Let S be an induced subdigraph of a digraph D. Then the induced subgraph of dom(D) on the vertices of S is a subgraph of dom(S).*

Proof. Let $x, y \in S$ where $\{x, y\}$ is an edge in dom(D). Then $O_D(x) \cup O_D(y) \cup \{x, y\} = V(D)$. As $V(S) \subseteq V(D)$, we have $O_S(x) \cup O_S(y) \cup \{x, y\} = V(S)$. Thus $\{x, y\}$ is an edge in dom(S). □

A *spiked cycle* is a connected graph such that the removal of all pendant vertices yields a cycle.

THEOREM 2.7. *Let T be an n-tournament. Then dom(T) is either a spiked odd cycle, with or without isolated vertices, or a forest of caterpillars.*

Proof. Lemmas 2.3 and 2.6 show that dom(T) has no even cycles. First assume dom(T) has a k-cycle C where k is odd. Lemmas 2.4 and 2.6 show that the subtournament of T on the vertices of C is U_k. By Lemma 2.6, the induced subgraph of dom(T) on C is a subgraph of $\text{dom}(U_k) = C_k$. So $C = C_k$ is an induced subgraph of dom(T). By Proposition 2.1, if x is not on C, then $O(x) \cap V(C)$ is an independent set. Since the independent sets in a k-cycle have at most $\frac{k-1}{2}$ vertices, two vertices not in C cannot beat all k vertices in C. So the subgraph induced on the vertices that are not on C has no edges. If some vertex x that is not on C is adjacent in dom(T) to at least two vertices on C, say y and z, then edges $\{x, y\}$ and $\{x, z\}$ together with one of the two path connecting y and z form an even cycle in dom(T), a contradiction. Thus dom(T) is a spiked odd cycle possibly with isolated vertices.

Otherwise, assume dom(T) is cycle-free. Then by Lemmas 2.5 and 2.6, each component must be a caterpillar. So, dom(T) is a forest of caterpillars. □

PROPOSITION 2.8. *All graphs G consisting of a spiked odd cycle C with possible isolated vertices are the domination graph of some tournament.*

Proof. Let G be such a graph on n vertices with a cycle, C, of length k. Consecutively label the vertices of C as $\{0, 1, \ldots, k-1\}$. For $i \in \{0, 1, \ldots, k-1\}$, let N_i be the set of vertices pendant to i. Let J be the set of isolated vertices. We then construct a tournament T with dom(T) $= G$ as follows. Create arcs between vertices in $V(C)$ so that the induced subgraph on $V(C)$ is U_k. Let i beat all vertices in N_i. Let all vertices in N_i beat all vertices in $V(C)$ which dominate i and let all vertices in $V(C)$ that are dominated by i beat all vertices in N_i. For all $i, j \in \{0, 1, \ldots, k-1\}$ with $i \neq j$, if i beats j, then let each vertex in N_i beat all vertices in N_j. Let each vertex not in J beat all vertices in J. Pairs of vertices in N_i and pairs of vertices in J are joined by arcs in an arbitrary manner. Then dom(T) $= G$. □

Not all forests of caterpillars are the domination graph of some tournament (*e.g.*, a path on 5 vertices is not the domination graph of any tournament). In a subsequent paper we will address the problem of which forests of caterpillars occur as domination graphs of tournaments.

3. Consequences of the Characterization. Since Theorem 2.7 gives such a sharp characterization, it is straightforward to deduce results about various graph parameters for domination and competition graphs of tournaments. Below are some examples. The bounds in Corollaries 3.1 to 3.6 are achieved for all allowed values of n.

COROLLARY 3.1. *For $n \geq 2$, then the maximum possible number of edges in the domination graph of an n-tournament is n.*

COROLLARY 3.2. *For $n \geq 2$, the minimum possible number of edges in the competition graph of an n-tournament is $\binom{n}{2} - n$.*

A subset of the vertices of G form a *clique* if there are edges between every pair of vertices in the subset. Let $\omega(G)$ (the *clique number* of G) be the maximum cardinality of a clique of G. Clearly, $\alpha(G) = \omega(\overline{G})$ where $\overline{G}$ is the complement of G. A *coloring* of G is a labeling of its vertices so that adjacent vertices do not have the same label. Let $\chi(G)$ (the *chromatic number* of G) be the minimum possible number of labels in a coloring of G.

COROLLARY 3.3. *For $n \geq 3$, let T be an n-tournament. The clique number and chromatic number of the domination graph of T are at most 3.*

A *clique cover* of G is a labeling of its vertices so that nonadjacent vertices do not have the same label. Let $\mathrm{cc}(G)$ (the *clique cover number* of G) be the minimum possible number of labels in a clique cover of G. Clearly, $\mathrm{cc}(G) = \chi(\overline{G})$.

COROLLARY 3.4. *For $n \geq 3$, let T be an n-tournament. The independence number and the clique cover number of the competition graph of T are at most 3.*

COROLLARY 3.5. *For $n \geq 3$, let T be an n-tournament. The independence number and the clique cover number of the domination graph of T are at least $\lfloor n/2 \rfloor$.*

COROLLARY 3.6. *For $n \geq 3$, let T be an n-tournament. The clique number and the chromatic number of the competition graph of T are at least $\lfloor n/2 \rfloor$.*

A digraph or graph is *vertex transitive* if for every pair of vertices i and j, there is an automorphism that maps i to j. Rotational tournaments are vertex transitive. Corollary 3.7 shows that the tournaments U_k are the only vertex transitive tournaments that have a dominating pair of vertices. The three regular 7-tournaments illustrate this. Figure 2(a) is U_7 (which is vertex-transitive) and its domination graph is a 7-cycle. Figure 2(b) is also vertex transitive and its domination graph is edgeless. The domination graph of Figure 2(c) is neither a cycle nor is it edgeless, but this regular tournament is not vertex transitive.

COROLLARY 3.7. *Let T be a vertex-transitive k-tournament. Then either dom(T) is C_k and T is U_k or dom(T) is edgeless.*

Proof. Since a vertex transitive tournament is regular and regular tournaments have an odd number of vertices, $|T|$ is odd. Further, since T is vertex transitive, $\mathrm{dom}(T)$ is also vertex transitive. So $\mathrm{dom}(T)$ is a regular graph on an odd number of vertices. Thus, the common degree of the vertices of $\mathrm{dom}(T)$ is even. Corollary 3.1 shows that $\mathrm{dom}(T)$ has at most n edges. Therefore this degree is either 2 or 0. If it is 2, then $\mathrm{dom}(T)$ is the disjoint union of cycles. However, Theorem 2.7 states that $\mathrm{dom}(T)$ can have only one cycle. Thus $\mathrm{dom}(T)$ is C_k. By Lemma 2.4, $T = U_k$. Otherwise, the degree is 0 and $\mathrm{dom}(T)$ is edgeless. □

REFERENCES

[1] R. C. Brigham and R. D. Dutton. A characterization of competition graphs. *Discr App M*, 6:315–317, 1983.

[2] J. E. Cohen. Food webs and the dimensionality of trophic niche space. *P NAS US*, 74:4533–4536, 1977.

[3] J.E. Cohen and Z.J Palka. A stochastic theory of community food webs. V.: intervality and triangulation in the trophic-niche overlap graph. *Am Natural*, 135(3):435–463, 1990.

[4] S. Kim. The competition number and its variants. In: J. Gimbel, J.W. Kennedy, and L.V. Quintas, eds., Quo Vadis, Graph Theory?, Ann Discr M 55 (1993) 313-326.

[5] S.R. Kim, T.A. McKee, F.R. McMorris, and F.S. Roberts. p-competition graphs. To appear in *Lin Alg App*, 1994.

[6] J. R. Lundgren. Food webs, competition graphs, competition-common enemy graphs, and niche graphs. In F.S. Roberts, editor, *Applications of Combinatorics and Graph Theory to the Biological and Social Sciences.* Springer-Verlag, 1989. In *IMH Volumes in Mathematics and Its Applications*, vol. 17.

[7] J.R. Lundgren, S.K. Merz, and C.W. Rasmussen. On direct computation of chromatic numbers of competition graphs. To appear in *Lin Alg App.*

[8] J. W. Moon. *Topics On Tournaments.* Holt, Rinehart and Winston, 1968.

[9] A. Raychaudhuri and F. S. Roberts. Generalized competition graphs and their applications. In P. Brucker and P. Pauly, editors, *Meth Op Res*, pages 295–311. Anton Haim, Konigstein, Germany, 1985.

[10] K.B. Reid and L.W Beineke. Tournaments. In L.W. Beineke and R.J. Wilson, editors, *Selected Topics in Graph Theory*, pages 169–204. Academic Press, New York, 1979.