A CHARACTERIZATION OF GRAPHS WITH INTERVAL SQUARES

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Abstract. The competition graph of a symmetric digraph D with a loop at each vertex is the square of the underlying graph of D with loops removed. In the interest of efficiently assigning radio frequencies in a communication network, we would like to determine which graphs have interval squares. Necessary and sufficient conditions on the graph G are given for G^2 to be interval for several large classes of graphs. We first discuss results related to the Fulkerson-Gross characterization of interval graphs. Second, we discuss necessary conditions involving forbidden subgraphs and consider why such an approach does not produce sufficient conditions. Open problems are also considered.

1. Introduction. The *square* of a graph $G = (V, E)$, denoted G^2, is a graph on the same vertex set V such that two vertices x and y are adjacent in the square if and only if there is a path of length one or two between x and y in G. Squares of graphs have been studied by Balakrishnan and Paulraja [1], [2], Laskar and Shier [6], Raychaudhuri [11], and Raychaudhuri and Roberts [13]. Squares of graphs are useful in the study of radio communication networks. Such a network can be modeled by a digraph $D = (V, A)$. Each vertex in V is representative of a station in the network and there is an arc from x to y if the station y can receive a signal transmitted by station x. Note that such communication networks are usually symmetric; if $(x, y) \in A$ then $(y, x) \in A$. Thus we will restrict our study to the case of symmetric digraphs. Observe that given a symmetric digraph $D = (V, A)$ we can construct the underlying graph $G = (V, E)$ on the same vertex set with $(x, y) \in E$ if and only if $(x, y) \in A$ and $(y, x) \in A$. Further observe that in such a communication network we can think of each station receiving its own signal. This results in a loop at each vertex. We can then model the transmission interferences between stations with the *competition* or *conflict* graph of D, denoted $C(D)$. The competition graph of D is a graph on the same vertex set as D with an edge between x and y if there exists a vertex z such that (x, z) and (y, z) are both arcs in D. That is, x and y compete for z. The relationship between $C(D)$ and G^2 is given in the following result of Raychaudhuri and Roberts [13].

PROPOSITION 1.1. *Let D be a symmetric digraph with a loop at each vertex. Let G be the underlying graph of D with the loops removed. Then $C(D) = G^2$.*

Thus the square of a graph modeling a radio communication network is useful in assigning frequencies. Cozzens and Roberts [4] demonstrated that if G^2 is unit interval we can optimally assign frequencies in $O(v^2)$ time. Observe that if G^2 is interval there are fast algorithms to compute the chromatic number. Recall that a graph G is *interval* if and only if G is the intersection graph of a set of intervals on the real line. Thus in the interest of optimally assigning frequencies, we would like to know which graphs have interval squares. We will present ideas and results pertaining to the determination of whether or not the square of a given graph G is interval using the properties of G.

2. Preliminaries. These results are related to results obtained in characterizing graphs with interval two-step graphs [9]. The two-step graph of a graph G, denoted $S_2(G)$, is a graph on the same set of vertices with two vertices x and y adjacent in $S_2(G)$ if and only if they are joined by a path of length two in G. Given a loopless symmetric digraph D with underlying graph G it is easily observed that $C(D) = S_2(G)$ [10]. There are two characterizations of interval graphs that will be the primary focus of our result. We recall that if a graph G contains an induced cycle of length $n \geq 4$ then G is not interval. We will show a similar necessary condition on the length of an induced cycle in G in order that G^2 be interval. But first, we will use the following characterization of interval graphs from Fulkerson and Gross [5] to characterize graphs with interval squares.

PROPOSITION 2.1. *A graph G is an interval graph if and only if the family of maximal cliques (complete subgraphs) of G has a consecutive ranking.*

Suppose we can define a family of sets $MC(G^2)$ found in G that correspond to the maximal cliques in the square. Then we will be able to say G^2 is interval if and only if $MC(G^2)$ has a consecutive ranking. We begin by noting the following result of Raychaudhuri[12].

PROPOSITION 2.2. *For all positive integers $k \geq 2$, if G^{k-1} is an interval graph, then G^k is an interval graph.*

Thus it has been previously established that interval graphs have interval squares. Nonetheless, the proof given in the work cited cannot be easily generalized to graphs that are not interval. We also observe that there are graphs which are not interval that have interval squares. Consider the forbidden subgraphs for interval graphs provided by Lekkerkerker and Boland [7]. These are depicted in Figure 1. Note that some of these forbidden subgraphs for an interval graph have interval squares.

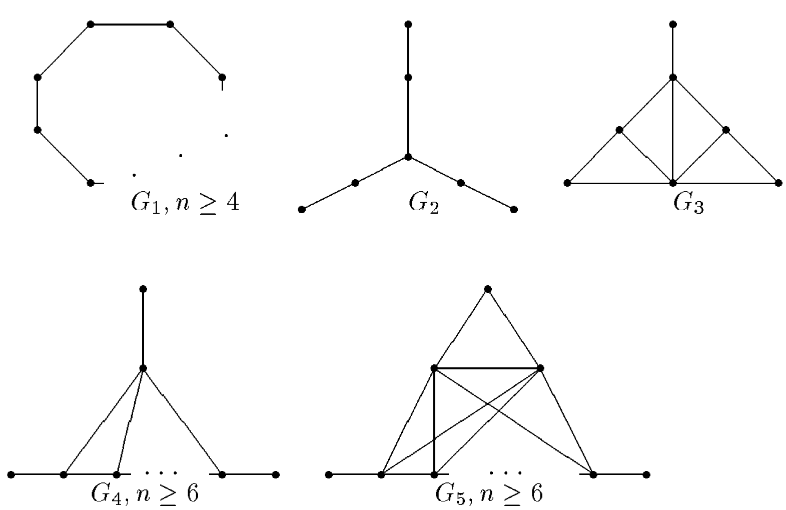

FIG. 1. *A graph is interval if and only if it contains no subgraph isomorphic to a G_1, G_2, G_3, G_4, G_5. Note the squares of $G_1, (n = 4, 5), G_3, G_4$ and G_5 are interval.*

We also observe that except for $G_5(n = 6)$ and $G_1(n = 4, 5)$, the maximal cliques in the squares of these forbidden subgraphs correspond to the closed neighborhoods of the nonsimplicial vertices in the original graph (a vertex is simplicial if its neighborhood is a clique). We say a closed neighborhood of a nonsimplicial vertex is *maximal* if it is not properly contained in the closed neighborhood of any other nonsimplicial vertex in the graph. We then ask the following question.

> *For what classes of graphs G is $MC(G^2)$ precisely the set of maximal closed neighborhoods of the nonsimplicial vertices in G?*

We define classes of graphs having this property as having *the closed neighborhood property.*

3. Classes of Graphs with the Closed Neighborhood Property. A good starting point is the class of interval graphs. We now prove that if G is an interval graph, $MC(G^2)$ is the family of maximal closed neighborhoods of the nonsimplicial vertices in G.

THEOREM 3.1. *Let G be a connected noncomplete interval graph. Then the maximal cliques in G^2 correspond to the maximal closed neighborhoods of the nonsimplicial vertices in G.*

Proof. (neighborhood $\Rightarrow$ maximal clique) Let z be a nonsimplicial vertex in G such that $N[z]$ is maximal. Since G is interval the cliques of G have a consecutive ranking $\{C_1, C_2, \ldots, C_t\}$. We claim that the consecutively ranked cliques $C_i, \ldots, C_k$ containing z form a maximal clique in the square. Suppose not. Then there exists a vertex w such that w is adjacent to every vertex in $C_i, \ldots, C_k$ in G^2, but w is not in $C_i, \ldots, C_k$ in G. WLOG, assume $w \in C_j$ such that $j > k$. Let x denote an arbitrary vertex in C_i not equal to z such that x is not in C_{i+1}. First note that w is not adjacent to x in G since $x \notin C_m$ for $m > i$ and $w \notin C_m$ for $m \leq k$ and $i < k$. Hence, w and x are joined by a path of length two so there exists a vertex t such that $w, x \in N(t)$. Thus $t \in C_l$ for some $l \leq i$ and $t \in C_m$ for some $m > k$. This ranking of the maximal cliques in G is consecutive therefore t is an element of each clique in the set $\{C_i, \ldots, C_k\}$. This implies $N[z]$ is properly contained in $N[t]$, a contradiction. Thus no such w can exist, i.e., $N[z]$ is a maximal clique in G^2.

(maximal clique $\Rightarrow$ neighborhood) Let C be a maximal clique in G^2. We will show there exists a nonsimplicial vertex z such that $C = N[z]$ in G. Since G is interval the maximal cliques of G have a consecutive ranking $\{C_1, \ldots C_l\}$. Note that $C_i \cap C_{i+1} \neq \emptyset$ so $C_i \cup C_{i+1}$ is a clique in G^2. Hence $C \nsubseteq C_i$ for any i. Suppose there does not exist a vertex z such that $C \subseteq N[z]$. Let i be the smallest integer such that C_i contains vertices in C but there exists a vertex x that is an element of C_i and C, but $x \notin C_{i+1}$. This must occur since otherwise $C \subseteq C_l$. Let j be the largest integer such that C_j contains vertices in C, but there exists a vertex y that is an element of both C_j and C , but $y \notin C_{j-1}$. This must occur since otherwise $C \subseteq C_1$. Note i must be less than j, for if not then $C \subseteq C_k$ for all $j \leq k \leq i$. Since this is a consecutive ranking of the cliques, x and y are not adjacent. Therefore x and y must be joined by a path of length two, so there exists a vertex z such that x and z are contained in a maximal clique in G and y and z are contained in a maximal clique in G. Since this ranking is consecutive and $x \notin C_{i+1}$, z must be in a clique C_k such that $k \leq i$.

Since $y \notin C_{j-1}$, z must be in a clique C_m such that $m \geq j$. Therefore $z \in C_p$ for all $p, i \leq p \leq j$. Note that every vertex of C is contained in a clique C_p such that $i \leq p \leq j$. Thus $C \subset N[z]$, a contradiction. Therefore there must exist a vertex z such that $C \subseteq N[z]$.
Since C is a maximal clique, it follows that $C = N[z]$. If z is simplicial, then C is a clique in G. Since G is connected and not complete, there exists a vertex $x \notin C$ such that x is adjacent to a vertex $y \in C$. But then $C' = \{x\} \cup C$ would be a clique in G^2, a contradiction. So z is nonsimplicial, completing the proof. □

As a result of this theorem, we use the consecutive ranking of the maximal cliques in an interval graph to conclude the following.

COROLLARY 3.2. *Let G be a connected interval graph. Then G^2 is interval.*

Proof. If G is complete, the statement is clearly true so assume G is not complete. We claim that since the maximal cliques of G have a consecutive ranking then the maximal nonsimplicial closed neighborhoods of G have a consecutive ranking. Let $\{C_1, C_2, \ldots, C_k\}$ be the consecutively ranked maximal cliques of G. Since G is connected, $C_i \cap C_{i+1} \neq \emptyset$, and if $v \in C_i \cap C_{i+1}$, then v is nonsimplicial. On the other hand, any nonsimplicial vertex v must be in the intersection of consecutive cliques. So $N[v]$ is the union of the maximal cliques containing v. Thus the maximal nonsimplicial closed neighborhoods of G have a consecutive ranking. □

Though this result has been previously established, this approach to the problem may be more easily generalized to other classes of graphs, for example trees. We now show that trees have the closed neighborhood property.

THEOREM 3.3. *Let T be a tree with $n \geq 3$ vertices. Then the maximal nonsimplicial closed neighborhoods in T are precisely the maximal cliques in T^2.*

Proof. (neighborhood $\Rightarrow$ *maximal* clique) Let v be a nonsimplicial vertex in T such that $N[v]$ is maximal. Clearly $N[v]$ is a clique in T^2. Suppose it is not maximal. Then there exists $w \notin N[v]$ such that w is joined to all $x \in N[v]$ by a path of length one or two. Since w and v are not adjacent there exists x such that $w, v \in N(x)$. Since $N[v]$ is not properly contained in $N[x]$ there exists $y \in N[v]$ such that y and x are not adjacent. If w and y are adjacent we are done since $wxvyw$ is a 4-cycle so w and y are joined by a path of length two, i.e., there exists a $(a \neq x, a \neq v)$ such that $w, y \in N(a)$. Then $wxvyaw$ is a 5-cycle, a contradiction. Thus no such w can exist, i.e., $N[v]$ is a maximal clique in T^2.

(maximal clique $\Rightarrow$ neighborhood) Let C be a maximal clique in T^2. We claim there exists z such that $C \subseteq N[z]$ in T. Let R be a subset of C. The proof is by induction on $|R|$. Let $R = \{x, y\}$. Then either x and y are adjacent in T so $R \subseteq N[x]$ or $x, y \in N(z)$ for some z, so $R \subseteq N[z]$.

Let $R = \{x, y, z\}$. Assume the claim is false. First suppose x and y are adjacent. Then z is not adjacent to x or y so there exist a, b (possibly $a = b$) such that $x, z \in N(a)$ and $y, z \in N(b)$. If $a = b$ we are done since xya is a 3-cycle. If $a \neq b$ we are done since $xyazbx$ is a 5-cycle. So assume no two vertices in R are adjacent. Then there exists a such that $x, y \in N(a)$, $b(b \neq a)$ such that $y, z \in N(b)$ and $c(c \neq a, c \neq b)$ such that $x, z \in N(c)$. Then $xaybzcx$ is a 6-cycle, a contradiction.

Assume the claim is true for all R such that $|R| < |C|$. Let $R = C$. We assume $|C| \geq 4$. Let w be an arbitrary vertex in C. Consider $R' = R - \{w\}$. By induction hypothesis there exists z_0 such that $R' \subseteq N[z_0]$. Suppose the claim is false for R. Then w and z_0 are not adjacent. Since $|R'| \geq 3$ there exists $x, y \in R'$ such that $x \neq z_0$ and $y \neq z_0$. If w is not adjacent to x or y we are done since there exist a, b (possibly $a = b$) such that $w, x \in N(a)$ and $w, y \in N(b)$. Then if $a \neq b$ $waxz_0ybw$ is a 6-cycle and if $a = b$ then axz_0ya is a 4-cycle. So w is adjacent to some $x \in R'$. Since $R \not\subseteq N[x]$ there exists $y \in N[z_0]$ such that y and x are not adjacent. If w and y are adjacent we are done since wxz_0yw is a 4-cycle therefore there exists $v (v \neq z_0, v \neq x)$ such that $w, y \in N(v)$. Then wxz_0yvw is a 5-cycle, a contradiction. Therefore there exists z such that $R \subseteq N[z]$. Since C is a maximal clique, $C = N[z]$. If z is simplicial then C is a clique in T. T is connected and not complete therefore there exists $x \notin C$ such that x is adjacent to some $y \in C$. But then $C' = \{x\} \cup C$ is a clique in T^2, a contradiction. So z is nonsimplicial, completing the proof. □

COROLLARY 3.4. *Let T be a tree with $n \geq 3$ vertices. Then T^2 is interval if and only if the maximal nonsimplicial closed neighborhoods of T have a consecutive ranking.*

Next we consider the class of graphs which have girth at least 7. Recall that the *girth* of a graph is the length of the shortest cycle in the graph and that girth is not defined in the event the graph has no cycles. Careful examination of the proof of Theorem 3.3 shows that we obtained contradictions by producing cycles of length at most six. So the same proof will work for graphs with girth at least seven, proving the following result.

THEOREM 3.5. *Let G be a connected graph with girth at least 7. Then the maximal cliques in G^2 are precisely the maximal nonsimplicial closed neighborhoods.*

COROLLARY 3.6. *Let G be a connected graph with girth at least 7. Then G^2 is interval if and only if the maximal nonsimplicial closed neighborhoods have a consecutive ranking.*

In the next section we will show that for the two previously examined classes of graphs, trees and graphs with girth at least 7, the square is interval if and only if the original graph is interval. We now consider a class of graphs containing noninterval examples of graphs with interval squares. This class of graphs is the graphs such that every edge is contained in a triangle and there is no 6-cycle. Figure 2 shows such a graph G such that G is not interval but G^2 is interval.

THEOREM 3.7. *Let G be a connected noncomplete graph such that every edge is contained in a triangle and G contains no 6-cycle. Then the maximal cliques in G^2 correspond to the maximal closed neighborhoods of the nonsimplicial vertices.*

Proof. (neighborhood $\Rightarrow$ maximal clique) Let v be a nonsimplicial vertex in G such that the closed neighborhood of v is maximal. Clearly $N[v]$ is a clique in G^2. Suppose it is not maximal. Then there exists a vertex $w \notin N[v]$ such that for all $x \in N[v]$, w and x are joined by a path of length at most two. Note that if for some $x \in N[v]$, w and x are adjacent then every edge contained in a triangle implies w

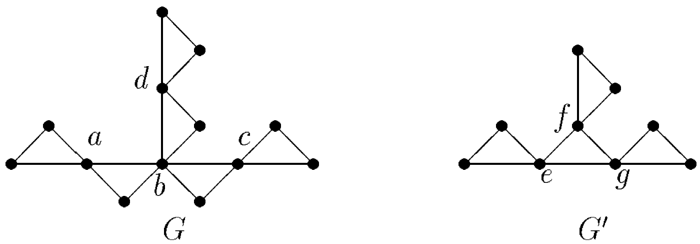

FIG. 2. *G is not interval. G^2 is not interval. We observe this by noting the maximal cliques in G^2, $N[a], N[b], N[c], N[d]$ do not have a consecutive ranking. G' is not an interval graph, but G'^2 is an interval graph. We observe this by noting the maximal cliques in G'^2 have a consecutive ranking, namely $N[e], N[f], N[g]$.*

and x are joined by a path of length two. Since w and v are joined by a path of length two there exists x such that $w, v \in N(x)$. Observe w and v are not adjacent. Since $N[v]$ is not properly contained in $N[x]$ there exists $y \in N[v]$ such that x and y are not adjacent. Since w and y are joined by a path of length two there exists a vertex u $(u \neq x, u \neq v)$ such that $w, y \in N(u)$. If u and x are not adjacent we are done since (w, x) must be contained in a triangle implies there exists a vertex b $(b \neq y, b \neq v)$ such that $w, x \in N(b)$ and then $wbxvyuw$ is a 6-cycle. So assume u and x are adjacent. Similarly if v and u are not adjacent we are done since (x, v) must be contained in a triangle implies that there exists $c(c \neq y, c \neq w)$ such that $x, v \in N(c)$ and $xcvyuwx$ forms a 6-cycle. So assume v and u are adjacent.

Since $N[v]$ is not properly contained in $N[u]$, there exists $z \in N[v]$ such that u and z are not adjacent. If x and z are adjacent we are done since $wxzvyuw$ is a 6-cycle so assume x and z are not adjacent. If w and z are adjacent we are done since $zwxuyvz$ is a 6-cycle so assume w and z are not adjacent. If $w, z \in N(y)$ we are done since $zywxuvz$ is a 6-cycle so assume w and z are not both in $N(y)$. Then w and z are joined by a path of length two implies there exists a vertex $d(d \neq y, d \neq u, d \neq x, d \neq v)$ such that $w, z \in N(d)$. Then $wdzvyuw$ is a 6-cycle, a contradiction. Therefore no such w can exist, i.e., $N[v]$ is a maximal clique in G^2.

(maximal clique $\Rightarrow$ neighborhood) Let C be a maximal clique in G^2. We claim there exists a vertex z such that $C \subseteq N[z]$. Let R be a subset of C. The proof is by induction on $|R|$. Let $R = \{x, y\}$. Then x and y are joined by a path of length one or two. Clearly in either case $\{x, y\} \subset N[z]$ for some z. Assume the statement is true for all R such that $|R| < |C|$. Let $R = C$. Let w be an arbitrary vertex in R. Consider $R' = R - \{w\}$. By the inductive hypothesis there exists z_0 such that $R' \subseteq N[z_0]$. Suppose there does not exist z such that $R \subseteq N[z]$.

Case 1: Assume w is adjacent to all $x \in N(z_0)$. Then $z_0 \in R$. Let $x \in N(z_0)$. Then x and w are adjacent. Since $R \not\subseteq N[x]$ there exists $y \in z_0$ such that x and y are not adjacent. Since w and y are assumed to be adjacent they are joined by a path of length two so there exists $v(v \neq x, v \neq z_0)$ such that $w, y \in N(v)$. If x and z_0 are not adjacent to v then we are done since there exists $a(a \neq v, a \neq w, a \neq y)$ such that $x, z_0 \in N(a)$ and then xaz_0yvwx is a 6-cycle. So assume x and z_0 are adjacent to v. Then $R \not\subseteq N[v]$ implies there exists $t \in R'$ such that v and t are not adjacent. Then

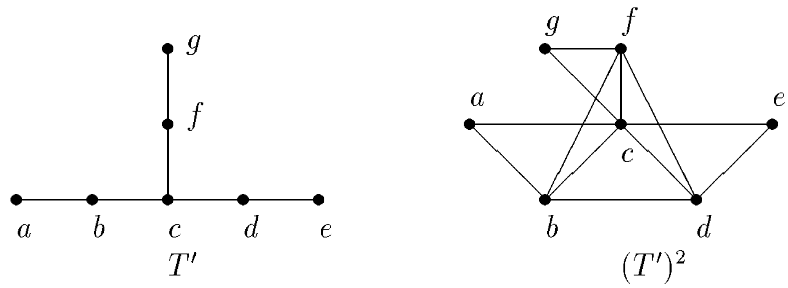

FIG. 3. *T is a noninterval tree. T^2 is noninterval.*

w and t are adjacent and wtz_0yvxw is a 6-cycle, a contradiction.

Case 2: There is a vertex $y \in N(z_0)$ such that y and w are not adjacent. Since w and y are joined by a path of length two there exists a vertex $v(v \neq z_0)$ such that $w, y \in N(v)$. Since $R \not\subseteq N[v]$ there exists a vertex $x \in R'$ such that x and v are not adjacent. If w and x are joined by a path of length two we are done since there exists a vertex $a(a \neq y, a \neq z_0, a \neq v)$ such that $w, x \in N(a)$ and $wvyz_0xaw$ is a 6-cycle. So w and x are not joined by a path of length two. Then w and x must be adjacent and (w, x) is contained in a triangle implies w and x are joined by a path of length two, a contradiction. Therefore there exists z such that $C \subseteq N[z]$. Since C is a maximal clique in G^2 it follows that $C = N[z]$. If z is simplicial then C is a clique in G. Since G is connected and not complete there exists a vertex $x \notin C$ such that x is adjacent to a vertex $y \in N$. But then $C' = \{x\} \cup C$ is a clique in G^2, a contradiction. So z is simplicial, completing the proof. □

COROLLARY 3.8. *Let G be a connected noncomplete graph such that every edge is contained in a triangle and G contains no 6-cycle. Then G^2 is interval if and only if the maximal closed nonsimplicial neighborhoods have a consecutive ranking.*

4. Necessary Conditions Using Forbidden Subgraphs. By Corollary 3.2, we know the square of an interval tree is interval, but can the square of a non-interval tree be interval? Consider the non-interval tree T' in Figure 3. Recall that a tree is interval if and only if it does not contain an induced subgraph isomorphic to T'. Observe that $(T')^2$ in Figure 3 is not interval.

We then ask: if T is a tree and contains an induced subgraph isomorphic to T', does T^2 contain an induced subgraph isomorphic to $(T')^2$? In general, if H is an induced subgraph of G, H^2 is not an induced subgraph of G^2. This is illustrated in Figure 4. For this reason, the forbidden subgraph approach to characterizing graphs with interval square does not generally work. We now show this approach will work for trees.

LEMMA 4.1. *Let T be a tree and H a connected subgraph of T. Then H^2 is an induced subgraph of T^2.*

Proof. Suppose not. Then there are vertices $x, y \in H$ such that (x, y) is an edge in T^2 but not in H^2. Since (x, y) is not an edge in H^2, x and y are not adjacent in H therefore there exists $z \in T$ but $z \notin H$ such that $x, y \in N(z)$. Since H is connected

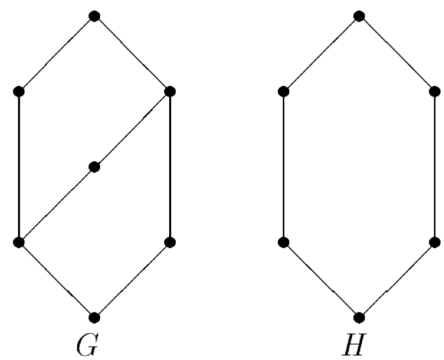

FIG. 4. *H is an induced subgraph of G. H^2 is not an induced subgraph of G^2.*

there is a path $x, v_1, \ldots, v_k, y$ joining x and y. Then $yzxv_1 \cdots v_k y$ is a cycle in T, a contradiction. Therefore H^2 is an induced subgraph of T^2. □

THEOREM 4.2. *Let T be a tree. Then T^2 is interval if and only if T is interval.*

Proof. ($\Leftarrow$) This follows from the previous result that interval graphs have interval squares.
($\Rightarrow$) Assume T is not interval. Suppose T^2 is interval. Then T contains an induced subgraph isomorphic to the graph T' in Figure 3. By the lemma $(T')^2$ is an induced subgraph of T^2, but $(T')^2$ is not interval, a contradiction. Thus T^2 is not interval. □

Note that the noninterval tree is G_2 in Figure 1 and that G_2 and $G_1 (n \geq 6)$ were the only forbidden subgraphs of an interval graph with noninterval squares. We now consider graphs containing subgraphs isomorphic to $G_1 (n \geq 6)$. Such graphs have noninterval squares if the graphs have large enough girth.

LEMMA 4.3. *Let G be a graph with girth $p \geq 6$. Then G contains an induced p-cycle C, and C^2 is an induced subgraph of G^2.*

Proof. Suppose not. Then there exist vertices $x, y \in C$ such that (x, y) is an edge in G^2 but not in C^2. Since (x, y) is not an edge in C^2, x and y are not adjacent in C so there exists $z \in G$ such that $x, y \in N(z)$. Since x and y are on a p-cycle with $p \geq 6$, the length of the shortest path joining them is $q \leq p - 3$. Let $x, v_1, \ldots, v_{q-1}, y$ denote this path. Then $yzxv_1 \cdots v_{q-1} y$ is a cycle of length $q + 2 < p$, a contradiction. Hence C^2 is an induced subgraph of G^2. □

LEMMA 4.4. *Let G be a graph with girth $p \geq 6$. Then G^2 contains an induced cycle of length l, $l \geq 4$.*

Proof. <u>Case 1:</u> $p = 6$. Then G contains an induced 6-cycle, $C = v_1 v_2 \cdots v_6 v_1$, so C^2 contains an induced four cycle, namely $v_1 v_2 v_4 v_5 v_1$. By Lemma 4.3 C^2 is an induced subgraph of G^2. Hence G^2 contains an induced four cycle.

<u>Case 2:</u> $p > 6$, even. G contains an induced p-cycle, $C = v_1 v_2 \cdots v_p v_1$, so C^2 contains an induced $\frac{p}{2}$-cycle, namely $v_1 v_3 v_5 \cdots v_{p-1} v_1$. By Lemma 4.3 C^2 is an induced subgraph of G^2. Hence G^2 contains an induced $\frac{p}{2}$-cycle, ($\frac{p}{2} \geq 4$).

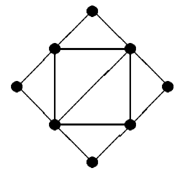

FIG. 5. *A chordal graph whose square is not chordal.*

Case 3: $p > 6$, odd. G contains an induced p-cycle, $C = v_1v_2 \cdots v_pv_1$, so C^2 contains an induced $\frac{p+1}{2}$-cycle, namely $v_1v_3v_5 \cdots v_pv_1$. By Lemma 4.3 C^2 is an induced subgraph of G^2. Hence G^2 contains an induced $\frac{p+1}{2}$-cycle,($\frac{p+1}{2} \geq 4$), completing the proof. □

Using Lemma 4.4 we obtain the following result.

THEOREM 4.5. *Let G be a connected graph with girth greater than five. Then G^2 is not interval.*

5. Conclusions and Open Problems. We have characterized two large classes of graphs which have interval squares, namely interval graphs and graphs without 6-cycles such that every edge is contained in a triangle and the maximal closed nonsimplicial neighborhoods have a consecutive ranking. We have also characterized three large classes of graphs which do not have interval square, namely noninterval trees, graphs with girth at least six, and graphs without 6-cycles such that every edge contained is in a triangle and the maximal closed nonsimplicial neighborhoods do not have a consecutive ranking. In the interest of fully characterizing graphs with interval squares the following questions should be considered.

1. We know a chordal graph may have a non-chordal square (see Figure 5). Can we characterize chordal graphs with interval squares?
2. We know the maximal cliques in the square do not correspond to the maximal closed nonsimplicial neighborhoods for all chordal graphs (see Figure 6). Given a chordal graph (or graph such that every edge is contained in a triangle), can we define a set easily found in the graph that corresponds to the maximal cliques in the square?

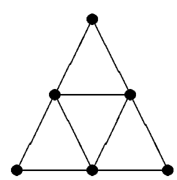

FIG. 6. *A chordal graph in which the maximal cliques in the square do not correspond to the maximal closed nonsimplicial neighborhoods.*

3. For graphs with the closed neighborhood property, the following observation can be made. If C is a maximal clique in the square then there exists a set of maximal cliques in the original graph containing C which have a vertex in common, namely the nonsimplicial vertex. There are examples of graphs in which the maximal cliques combining to form a maximal clique in the square

do not all share a common vertex (though pairwise they must). See Figure 7. Can we characterize graphs with such maximal cliques in the square?

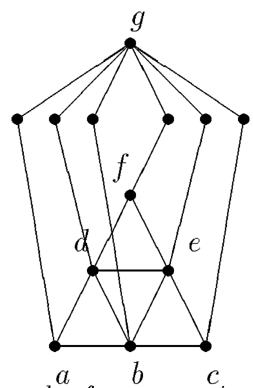

FIG. 7. *Observe that a, b, c, d, e, f, and g form a maximal clique in the square but there is not a set of maximal cliques in this graph containing these vertices such that there is a vertex common to all elements of the set.*

4. Chang and Nemhauser [3] and Lasker and Shier [6] proved that the chordal graphs with chordal squares are precisely the strongly chordal graphs. Lubiw [8] proved that the square of a strongly chordal graph is strongly chordal. Do strongly chordal graphs have the closed neighborhood property?
5. We have characterized graphs with girth at least 6 whose squares are interval. Similarly we have characterized trees and graphs with every edge contained in a triangle and no six cycle whose squares are interval. Can similar approaches be used to characterize graphs with girth 4 and 5 whose squares are interval?

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