Domination Graphs of Tournaments and Digraphs

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Abstract. The *domination graph* of a digraph has the same vertices as the digraph with an edge between two vertices if every other vertex loses to at least one of the two. Previously, the authors showed that the domination graph of a tournament is either an odd cycle with or without isolated and/or pendant vertices, or a forest of caterpillars. They also showed that any graph consisting of an odd cycle with or without isolated and/or pendant vertices is the domination graph of some tournament. This paper extends these results to oriented graphs. We also show that any caterpillar is the domination graph of some digraph, but a path on four or more vertices is not the domination graph of any tournament. Other results relate the domination graph of a tournament to its positive subtournament defined by Fisher and Ryan, and the possible and average number of edges in the domination graph of a tournament.

Let an n-*digraph* be an oriented graph on n vertices. Vertex x *beats* vertex y if there is an arc from x to y. For a vertex x, let $O(x)$ (the *out-neighbors* or *out-set* of x) denote the set of vertices which x beats, and let $I(x)$ (the *in-neighbors* or *in-set* of x) denote the set of vertices which beat x. Let $d^+(x) = |O(x)|$ and $d^-(x) = |I(x)|$ denote the *out-degree* and *in-degree* of x. An n-*tournament* is an oriented complete graph. See Moon [**7**] and Reid and Beineke [**8**] for more about tournaments.

Two vertices x and y *dominate* an n-digraph D if x and y beat all other vertices *i.e.*, $\{x, y\} \cup O(x) \cup O(y) = V(D)$ where $V(D)$ is the set of vertices of D. Two such vertices are called a *dominant pair*. The *domination graph* of D, denoted $dom(D)$, is a graph on the vertices of D with edges between vertices which are dominant pairs. Domination graphs were introduced in connection with competition graphs of tournaments. The domination graph of a tournament T is the complement of the competition graph of the reversal of T [**1**]. See Lundgren [**5**] for more about competition graphs and Lundgren, et al. [**6**] for more on the competition graphs of tournaments.

Figure 1 shows the domination graph of a tournament. For example, $\{1, 8\}$ is an edge in the domination graph because $O(1) = \{2, 3, 5, 7\}$ and $O(8) = \{1, 4, 5, 6\}$, but $\{4, 5\}$ is not an edge because $6 \notin O(4) \cup O(5)$.

This paper extends previous work of the authors [**1**] on domination graphs of tournaments. A brief summary of that work is included next. Recall that a *caterpillar* is a tree such that the removal of all pendant vertices results in a path (called the *spine*).

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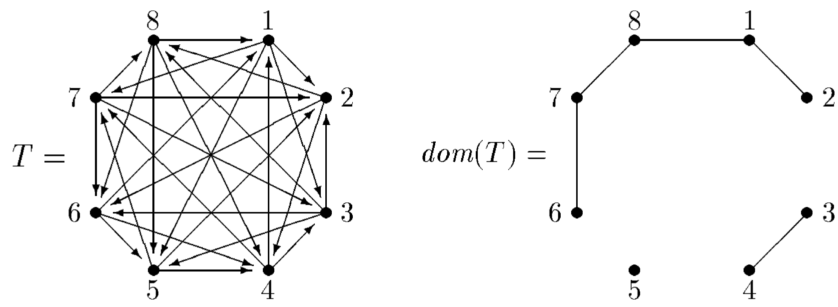

FIG. 1. **A tournament and its domination graph.**

PROPOSITION 0.1. *Let T be a tournament. Then dom(T) is either an odd cycle, with or without isolated and/or pendant vertices, or a forest of caterpillars. Further, any graph G consisting of an odd cycle with or without isolated and/or pendant vertices is the domination graph of some tournament.*

Let $k \geq 3$ be an odd integer. Let S be a $\left(\frac{k-1}{2}\right)$-set contained in $\mathcal{Z}_k$ (the integers mod k) where $0 \notin S$ and $s_1 + s_2 \not\equiv 0$ for all $s_1, s_2 \in S$. Form a regular tournament $T(S)$ called the *rotational tournament with symbol* S whose vertices are labeled by the elements of $\mathcal{Z}_k$ and with arcs (i, j) if $j - i \equiv s$ where $s \in S$. Let $U_k = T(\{1, 3, 5, \ldots, k-2\})$. Fisher, et al. [**1**] showed the following.

PROPOSITION 0.2. *If T is a tournament and dom(T) contains an odd k-cycle C, then the subtournament of T on the vertices of C is U_k.*

1. Tournament Games. Tournament Games are a generalization of the game "Scissors, Paper and Stone". Given a tournament T, two players simultaneously pick a vertex. A player wins if their vertex beats the opponent's vertex with a tie occurring if both players pick the same vertex.

If the winner pays the loser \$1 (with no money exchanged for ties), this is a two person zero-sum game. A *(mixed) strategy* assigns probabilities p_1, p_2, ..., p_n on the vertices 1, 2, ..., n so that vertex i is played with probability $p_i \geq 0$, where $p_1 + p_2 + \cdots + p_n = 1$. An *optimal* strategy is one in which the expected winnings (independent of how the opponent plays) is maximized. Since tournament games are symmetric, both players have the same optimal strategies and the expected winnings is 0. Thus an optimal strategy assures that no matter how one plays, the opponent is no more likely to win than lose. In other words for all vertices i, we have

$$\sum_{j \in O(i)} p_j \leq \sum_{j \in I(i)} p_j. \tag{1}$$

Figure 2 shows a strategy for a tournament game played on the tournament in Figure 1. Against this strategy, playing a vertex i wins with probability $\sum_{j \in O(i)} p_j$ and loses with probability $\sum_{j \in I(i)} p_j$. So, against this strategy, playing either a vertex marked $\frac{1}{3}$ wins with probability $\frac{1}{3}$, loses with probability $\frac{1}{3}$, and ties with probability $\frac{1}{3}$. Playing a vertex marked $\frac{1}{9}$ wins with probability $\frac{4}{9}$, loses with probability $\frac{4}{9}$ or $\frac{1}{3}$ (depending on which one is played), and loses with probability $\frac{5}{9}$ or $\frac{2}{3}$. So no matter where an opponent plays, the best they can do is make the expected winnings zero. Thus this strategy is optimal.

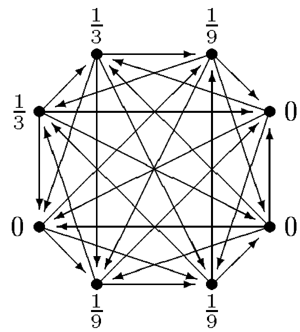

FIG. 2. **An optimal strategy for a tournament game.**

If vertex i is in an optimal strategy (*i.e.*, $p_i > 0$), it must win as often as it loses. Otherwise, it would not be used against an opponent playing the same optimal strategy. So (1) is an equality when $p_i > 0$, and hence

$$p_i > 0 \implies \sum_{j \in O(i)} p_j = \sum_{j \in I(i)} p_j = \frac{1}{2}(1 - p_i). \tag{2}$$

Fisher and Ryan [**2,3**], and Laslier, Laffond and Le Breton [**4**] (who examined this in a different context) independently showed the following.

PROPOSITION 1.1. *Let T be a tournament. Then the tournament game on T has a unique optimal strategy. Further, in this strategy, $p_i > 0$ for an odd number of vertices, and if $p_i = 0$, we have*

$$\sum_{j \in O(i)} p_j < \frac{1}{2} < \sum_{j \in I(i)} p_j. \tag{3}$$

Thus, Figure 2 shows the only optimal strategy for that tournament. Further, as predicted, the number of vertices with $p_i > 0$ is odd (in this case, five), and against this strategy, vertices with $p_i = 0$ are more likely to lose than win.

If the optimal strategy for a tournament game has $p_i > 0$ for all i, then the tournament is called *positive*. Since the number of vertices with $p_i > 0$ is odd, positive tournaments have an odd number of vertices. The 1-tournament is a positive tournament. The 3-cycle is the only positive 3-tournament (the optimal strategy has $p_i = 1/3$ for all i). The two positive 5-tournaments are shown in Figure 3, and the twelve positive 7-tournaments are shown in Figure 4.

2. Dominant Pairs in Positive Subtournaments. Given an optimal strategy of a tournament game, consider the subtournament on the vertices with $p_i > 0$. By definition, this subtournament is a positive tournament. For example, in Figure 2, this subtournament is the positive tournament of Figure 3(b). This subtournament will be called the *positive subtournament* of the tournament.

If $dom(T)$ has a k-cycle (k must be odd), then Proposition 0.2 shows that the subtournament of T on $V(C)$ is U_k. Further, since no vertex can beat adjacent

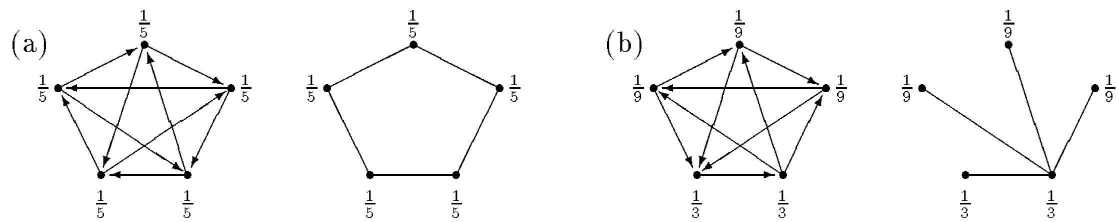

FIG. 3. **Positive tournaments on 5 vertices and their domination graphs.** *The labels show optimal strategies for tournament games on each tournament.*

vertices in C (or else the two vertices do not dominate T), a vertex of T can beat at most $(k-1)/2$ vertices in $V(C)$. Consider the strategy given by $p_i = 1/k$ if $i \in V(C)$ and $p_i = 0$ otherwise. As each vertex in C beats $(k-1)/2$ vertices in C and loses to $(k-1)/2$ vertices in C, against this strategy, an opponent wins and loses with probability $(k-1)/2k$ and ties with probability $1/k$. Each vertex not in C beats at most $(k-1)/2$ vertices in C and loses to at least $(k+1)/2$ vertices in C; so, against this strategy, an opponent wins with probability $(k-1)/2k$ or less and loses with probability $(k-1)/2k$ or more. So no matter what an opponent does, against this strategy, the best they can do is break even. Thus, this strategy is optimal. By Proposition 1.1, this is the unique optimal strategy for T. So the positive subtournament of T is the subtournament on the vertices of C.

Proposition 0.1 shows in part that if $dom(T)$ has a cycle, then at least one vertex in any dominant pair is in the cycle, and vertices not in the cycle can be in at most one dominant pair. Theorem 2.1 generalizes this to any tournament. It shows that at least one vertex in any dominant pair must be in the positive subtournament, and vertices not in the positive subtournament can be in at most one dominant pair. This can be seen in Figure 1 where vertices 2, 3, and 6 (those not in the positive subtournament) have degree at most one and form an independent set in $dom(T)$.

THEOREM 2.1. *Let S be the positive subtournament of a tournament T. Let $i \notin V(S)$. Then i has at most one neighbor in $dom(T)$ which must be in $V(S)$.*

Proof. Let $i, j \notin V(S)$. Then $p_i = p_j = 0$. Then by equation (3),

$$\sum_{k \in \{i,j\} \cup O(i) \cup O(j)} p_k \le p_i + p_j + \sum_{k \in O(i)} p_k + \sum_{k \in O(j)} p_k < 0 + 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Since the probability over all of the vertices is 1, there is a vertex that is not in $\{i,j\} \cup O(i) \cup O(j)$. Thus i and j do not dominate T.

Next suppose $i \notin V(S)$ and $\{i,j\}, \{i,k\} \in dom(T)$ for some $j \neq k$. Without loss of generality, assume j beats k in T. From above, $j \in V(S)$ and hence $p_j > 0$. Since i and k dominate T and k loses to j, we see that i beats j. Since i and j dominate T, we see that i also beats every vertex in $I(j)$ that beats j. Thus,

$$\sum_{\ell \in O(i)} p_\ell \ge p_j + \sum_{\ell \in I(j)} p_\ell = p_j + \frac{1}{2}(1 - p_j) = \frac{1 + p_j}{2} > \frac{1}{2}.$$

This contradicts (3) showing that i is in at most one dominant pair. $\square$

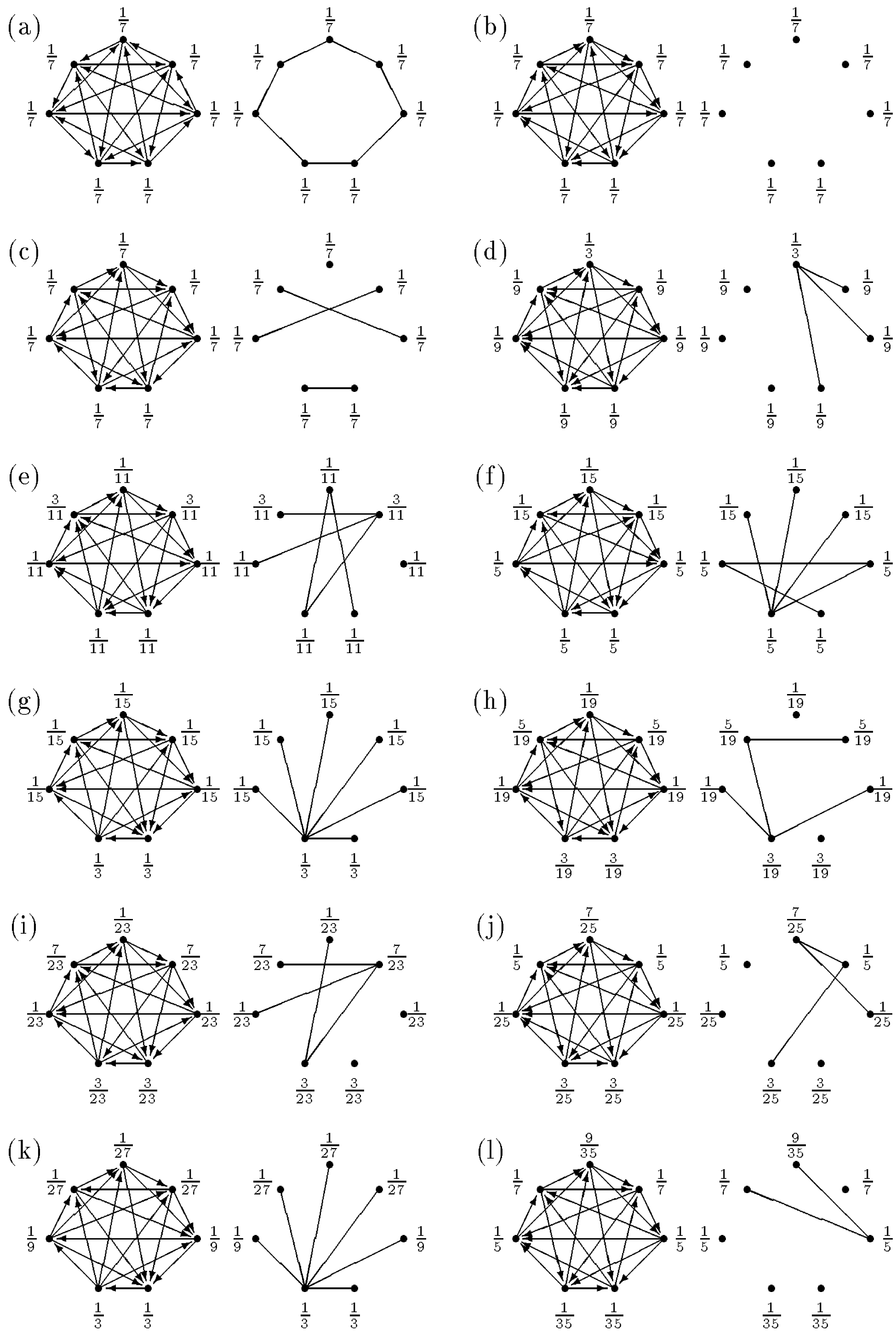

FIG. 4. **The positive tournaments on 7 vertices and their domination graphs.** *The labels show optimal strategies for tournament games on each tournament.*

3. The Size of Domination Graphs. *How many dominant pairs can an n-tournament have?* Fisher, et al. [**1**] showed that the domination graph of an n-tournament has at most n edges. However, for 3-tournaments, domination graphs can only have two (if it is transitive) or three (if it is a cycle) edges. So, some combinations of n vertices and e edges are not possible for domination graphs even when $e \leq n$. Theorem 3.1 shows which combinations are possible.

THEOREM 3.1. *Given nonnegative integers $e \leq n$, there is a tournament whose domination graph has n vertices and e edges if and only if (n, e) is not one of the following:* $(1,1)$, $(2,0)$, $(2,2)$, $(3,0)$, $(3,1)$, $(4,0)$, $(4,1)$, $(4,2)$, $(5,0)$, $(5,1)$, $(5,2)$, $(6,0)$, $(6,1)$, $(6,2)$, *or* $(7,1)$.

Proof. Let T be a tournament on at most 7 vertices and let S be its positive subtournament. Then S has 1, 3, 5, or 7 vertices.

If $|V(S)| = 1$, its vertex always wins. So, this vertex forms a dominant pair with all other vertices and no other dominant pairs exist. Thus, $dom(T)$ has $n-1$ edges.

If $|V(S)| = 3$, then S is a 3-cycle where at least two of its vertices beat each vertex of $T - S$. So any two vertices of S dominate T. Thus, $dom(T)$ has at least 3 edges.

If $|V(S)| = 5$, there are two cases. If S is the regular 5-tournament (Figure 3(a)), then the vertices in $T-S$ beat at most two vertices of S. So each vertex in $T-S$ can eliminate at most one edge from $dom(T)$. Thus $dom(T)$ has at least $10-n$ edges. If S is the tournament in Figure 3(b), then vertices in $T-S$ beat vertices of S whose probabilities sum to less than 1/2. For a vertex of $T-S$ to eliminate more than one edge from $dom(T)$, it would have to beat a vertex with $p_i = 1/3$ and at least two other vertices of S. Since the probabilities on these vertices sum to at least 5/9, this cannot happen. So again each vertex in $T-S$ eliminates at most one edge from $dom(T)$. Thus, $dom(T)$ has at least $9-n$ edges.

If $|V(S)| = 7$, then $T = S$ (because T has at most seven vertices). So T is one of the tournaments in Figure 4. None of these tournaments have exactly one dominant pair. Thus, $dom(T)$ cannot have exactly one edge.

Putting these results together shows that the forbidden (n, e) pairs in the statement of the theorem cannot occur. Now we must show that if (n, e) is allowed, we can find an n-tournament whose domination graph has e edges. This is trivial for $n < 3$. If $3 \leq e \leq n$, construct a tournament whose domination graph has a 3-cycle with $e-3$ pendant vertices and $n-e$ isolated vertices (Proposition 0.1 shows this is possible). If $e = 2$ and $n \geq 7$, add any $(n-7)$-tournament $\hat{(T)}$ to $\bar{(T)}$ of Figure 4(l) so that any vertex in $\hat{(T)}$ is beaten by all vertices of $\bar{(T)}$. If $e = 1$ and $n \geq 8$, let $\bar{(T)}$ be the tournament obtained from the tournament of Figure 4(l) by adding a vertex that beats a dominant pair and loses to all other vertices. Then add any $(n-8)$-tournament $\hat{(T)}$ to $\bar{(T)}$ so that any vertex in $\hat{(T)}$ is beaten by all vertices of $\bar{(T)}$. If $e = 0$ and $n \geq 7$, add any $(n-7)$-tournament $\hat{(T)}$ to $\bar{(T)}$ of Figure 4(b) so that any vertex in $\hat{(T)}$ is beaten by every vertex in $\bar{(T)}$. □

The proof of Theorem 3.1 suggests it is difficult to find tournaments where the number of dominant pairs is small compared to the number of vertices. This is not true. Theorem 3.2 shows that most tournaments have no dominant pairs.

THEOREM 3.2. *The average number of edges in domination graphs of n-tournaments is* $\binom{n}{2}(3/4)^{n-2}$.

Proof. Let T be a randomly selected n-tournament. Vertices i and j dominate T if no vertex beats both i and j. The probability a vertex beats both i and j is $1/4$. Since $n-2$ vertices could beat both i and j, the probability they dominate is $(3/4)^{n-2}$. Since there are $\binom{n}{2}$ pairs of vertices, the average number of dominant pairs is $\binom{n}{2}(3/4)^{n-2}$. □

Figure 5 shows both the possible number (the dots) and the average number (the curve) of edges in the domination graph of an n-tournament. The average is largest when $n = 7$ or 8 (a tie) with an average of $5103/1024 = 4.9833984375$ edges. The average approaches zero as the number of vertices grow.

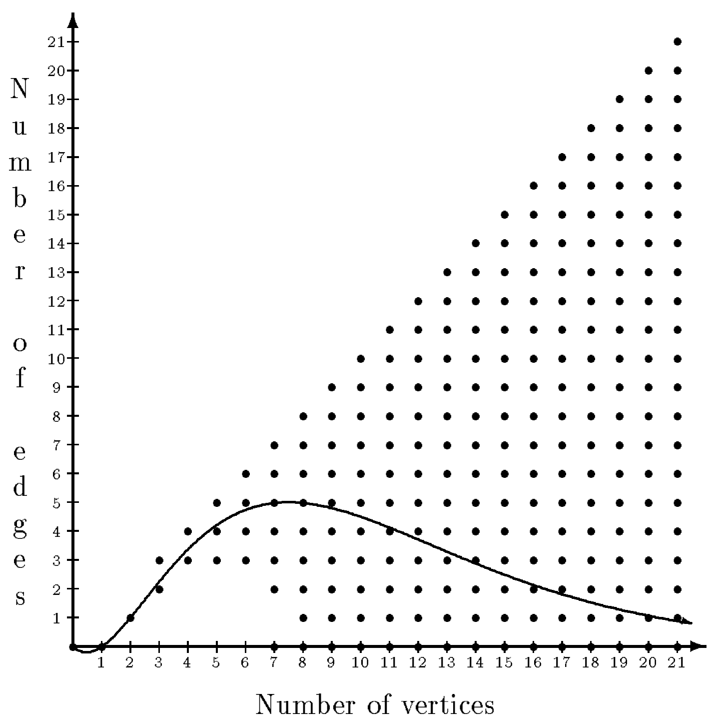

FIG. 5. **The number of edges in the domination graph.**

4. Orientations on Domination Graphs. For a tournament T, let the *domination digraph* of T, denoted $\mathcal{D}(T)$, be on the same vertices of T with an arc from vertex x to vertex y if x and y dominate T and x beats y in T. Thus $\mathcal{D}(T)$ is the orientation of $dom(T)$ induced by T. Figure 6 shows the domination digraph of the tournament in Figure 1.

Theorem 4.1 shows that vertices in $\mathcal{D}(T)$ can be beaten at most once. Further, it shows that a vertex can beat only one other vertex that has beaten other vertices. A crucial argument in the proof of Proposition 0.1 was that the graph in Figure 7 cannot be the domination graph of any tournament. Theorem 4.1 makes this easy.

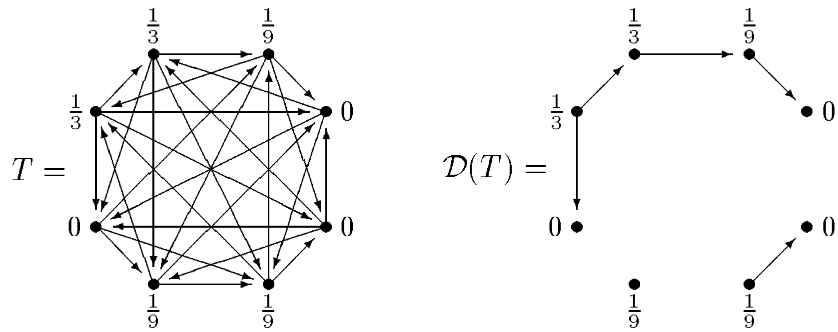

FIG. 6. **A tournament and its domination digraph.** *The labels are the optimal strategy for the tournament game on the tournament.*

Any orientation with maximum in-degree one would have the center vertex beating 2 or more neighbors which in turn would beat their pendant neighbor. But then two out-neighbors of the center vertex have out-neighbors violating Theorem 4.1. So the graph in Figure 7 cannot be the domination graph of any tournament.

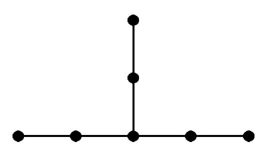

FIG. 7. **Any tree that is not a caterpillar has this as a subgraph.**

THEOREM 4.1. *Let T be a tournament. In $\mathcal{D}(T)$, a vertex can have at most one in-neighbor and at most one of its out-neighbors can have out-neighbors.*

Proof. First suppose vertices x and z both beat y in $\mathcal{D}(T)$. Without loss of generality, assume x beats z in T. So x beats both y and z in T, and y and z do not dominate T, a contradiction. Thus a vertex has at most one in-neighbor in $\mathcal{D}(T)$.

Now suppose in $\mathcal{D}(T)$ that vertex x beats both w and y where w beats v and y beats z. Without loss of generality, assume w beats y in T. Since w and x both beat y in T and since z and y dominate T, z must beat both w and x. But then w and x do not dominate T, a contradiction. So, a vertex has at most one out-neighbor in $\mathcal{D}(T)$ that has an out-neighbor. □

Theorem 4.2 relates the domination digraph of a tournament T to the optimal strategy for the tournament game on T. If S is the positive subtournament of T, Theorem 2.1 shows that the edges of $dom(T)$ are either in S or pendant to S. Theorem 4.2 show that arcs of $\mathcal{D}(T)$ pendant to S are directed away from S, and that the arcs of $\mathcal{D}(T)$ within S are oriented away from the vertex with higher probability (it does not give an orientation for vertices with equal probability).

THEOREM 4.2. *Let T be a n-tournament with $p_1, p_2, \ldots, p_n$ being the probabilities of vertices $1, 2, \ldots, n$ in an optimal strategy for the tournament game on T. If i beats j in $\mathcal{D}(T)$, then $p_i > 0$ and $p_i \geq p_j$.*

Proof. Assume that i and j dominate T and that i beats j in T. Then

$$V(T) = \{i\} \cup O(i) \cup O(j) = (V(T) - I(i)) \cup O(j)$$

$$\implies I(i) \subseteq O(j) \implies \sum_{k \in I(i)} p_k \leq \sum_{k \in O(j)} p_k.$$

If $p_i = 0$, then (3) gives that $\sum_{k \in I(i)} p_k > \frac{1}{2}$. Since $\sum_{k \in O(j)} p_k \leq \frac{1}{2}$ for any vertex j, this is a contradiction. If $p_j = 0$, the result is immediate. Finally if $p_i > 0$ and $p_j > 0$, we have

$$\frac{1}{2}(1 - p_j) = \sum_{k \in O(j)} p_k \geq \sum_{k \in I(i)} p_k = \frac{1}{2}(1 - p_i) \implies p_i \geq p_j. \square$$

Figure 6 illustrates this result. Theorem 4.2 prescribes the orientation of every arc of $\mathcal{D}(T)$ except for the arc between the vertices marked 1/3.

5. Domination Graphs of Digraphs. Tournaments record the result of contests where each team plays every other team exactly once and there are no ties. A natural generalization is to digraphs without cycles of length 1 or 2, which we will call *oriented (simple) graphs.* For these, the lack of an arc between two vertices indicates that the teams either tied or did not play. We begin by extending Proposition 0.1 to oriented graphs.

THEOREM 5.1. *If D is an oriented graph, then $dom(D)$ is either an odd cycle with or without isolated and/or pendant vertices, or a forest of caterpillars.*

Proof. Arbitrarily add arcs to D so as to form a tournament T. Since all dominant pairs in D are dominant pairs in T, we have that $dom(D)$ is a subgraph of $dom(T)$. The result then follows from Proposition 0.1. $\square$

Proposition 0.2 showed that a graph consisting of an odd cycle, with or without isolated and/or pendant vertices is the domination graph of some tournament. Since tournaments are oriented graphs, any graph consisting of an odd cycle, with or without isolated and/or pendant vertices is the domination graph of some oriented graph. However with oriented graphs, we can go further.

THEOREM 5.2. *If G is a caterpillar, then G is the domination graph of an oriented graph.*

Proof. Let 1, 2, ..., k be the consecutively labeled vertices of a longest path in G. Observe that both 1 and k have degree one. Let $j = k$ if k is odd and $j = k - 1$ if k is even. Let G' be the graph formed by adding edge $\{1, j\}$ to G. Since G' has an odd cycle with pendant edges, Proposition 0.2 shows there is a tournament T with $dom(T) = G'$, and that the subtournament of T on the vertices 1, 2, ..., j is U_j. Let D be the digraph formed by removing arc $(1, 2)$ from T. Then 1 and k do not dominate D because neither beats 2. However, 1 and 2 dominate D. All other dominant pairs of T are dominant pairs of D, because no other dominant pairs of T includes 1. So $dom(D) = G$. $\square$

A similar construction shows that a graph consisting of a caterpillar and an isolated vertex is the domination graph of some tournament (here edge $\{1, j\}$ is removed by adding a vertex that beats only vertices 1 and j). *Are all caterpillars the domination graph of a tournament?* The following shows the answer is no.

THEOREM 5.3. *A path on four or more vertices is not the domination graph of a tournament.*

Proof. For some n-tournament T with $n \geq 4$, suppose $dom(T)$ is a path. Up to symmetry, Figure 8 shows the only orientations of $\mathcal{D}(T)$ allowed by Theorem 4.1.

First assume orientation (a). Then for $1 \leq i \leq n-1$, vertex i beats vertex $i+1$ in T, and vertices i and $i+1$ dominate T. So for $1 \leq i \leq n-2$, since i beats $i+1$ and since $i+1$ and $i+2$ dominate T, $i+2$ must beat i. Next for $1 \leq i \leq n-3$, since $i+3$ beats $i+1$ and since i and $i+1$ dominate T, i must beat $i+3$. Then for $1 \leq i \leq n-4$, since i beats $i+3$ and since $i+3$ and $i+4$ dominate T, $i+4$ must beat i. Continuing this way shows that when $1 \leq i < j \leq n$, vertex i beats vertex j if $j-i$ is odd, and vertex j beats vertex i if $j-i$ is even. So the out-neighbors of vertex 1 are the even numbered vertices, and the out-neighbors of either vertex n (if n is odd), or vertex $n-1$ (if n is even) are the odd numbered vertices except n or $n-1$. So either $\{1,n\} \in dom(T)$ or $\{1,n-1\} \in dom(T)$. This is a contradiction when $n \geq 3$.

Now assume orientation (b). Similar arguments show that when $2 \leq i < j \leq n$, vertex i beats vertex j if $j-i$ is odd, and vertex j beats vertex i if $j-i$ is even. So the out-neighbors of vertex 2 are the odd numbered vertices (including 1 because of the orientation), and the out-neighbors of either vertex n (if n is even), or vertex $n-1$ (if n is odd) are the even numbered vertices except n or $n-1$. Thus either $\{2,n\} \in dom(T)$ or $\{2,n-1\} \in dom(T)$. This is a contradiction when $n \geq 4$. □

FIG. 8. **Orientation of a path.** *Up to symmetry, these are the only orientations of a path that are a domination digraph of a tournament.*

In a subsequent paper, we address the following questions. Which forests of caterpillars are the domination graph of a tournament? Which forests of caterpillars are the domination graph of an oriented graph? Which graphs are the domination graph of an arbitrary digraph?

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