A Characterization of Graphs with Interval Two-Step Graphs

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Dedicated by the other authors to Professor John Maybee on the occasion of his 65th birthday.

Abstract. One of the intriguing open problems on competition graphs is determining what digraphs have interval competition graphs. This problem originated in the work of Cohen [5, 6] on food webs. In this paper we consider this problem for the class of loopless symmetric digraphs. The competition graph of a symmetric digraph D is the two-step graph of the underlying graph H of D, denoted $S_2(H)$. The two-step graph is also known as the neighborhood graph, and has been studied recently by Brigham and Dutton [4] and Boland, Brigham and Dutton [1, 2]. This work was motivated by a paper of Raychaudhuri and Roberts [20] where they investigated symmetric digraphs with a loop at each vertex. Under these assumptions, the competition graph is the square of the underlying graph H without loops. Here we will first consider forbidden subgraph characterizations of graphs with interval two-step graphs. Second, we will characterize a large class of graphs with interval two-step graphs using the Gilmore-Hoffman characterization of interval graphs.

1. Introduction. Let $G = (V, E)$ be a graph. The *two-step graph*, denoted $S_2(G)$, is a graph on the same vertex set as G with an edge joining vertices x and y in V if and only if there exists a vertex z in V such that $x, y \in N(z)$, the open neighborhood of z. The two-step graph is closely related to the *competition graph* of a digraph. Let $D = (V, A)$ be a digraph. Then the competition graph of D, denoted $C(D)$, is a graph on the same set of vertices with an edge between two distinct vertices x and y in V if and only if there exists a vertex z in V such that there is an arc from x to z and from y to z in A. If D is a symmetric digraph with underlying graph H, it is easily seen that the two-step graph of H and the competition graph of D are identical (see [13]). The problem of which digraphs have interval competition graphs originated in the work of Cohen [5, 6], on food webs. This problem has been studied for several special cases (see [11, 12, 22, 23]), but remains unsolved in general. Raychaudhuri and Roberts [20] were able to answer the following question: given a symmetric digraph D with a loop at each vertex and underlying interval graph H, what conditions are necessary and sufficient for the competition graph of D to be interval? Lundgren, Maybee, and Rasmussen [13] were able to solve this problem for loopless symmetric digraphs with underlying interval graph H. We will use ideas from [16] to characterize a large class of graphs which have interval two-step graphs.

First we will consider necessary conditions involving forbidden subgraphs. This will lead to a characterization related to the Gilmore-Hoffman characterization of interval graphs: a graph G is interval if and only if the family of maximal cliques of G can be ordered $C_1, C_2, \ldots, C_r$ such that if a vertex $v \in C_i$ and $v \in C_k$, then $v \in C_j$ for all $i \leq j \leq k$. Such an ordering is called a *consecutive ranking* (for a comprehensive introduction to interval graphs see Golumbic [9]). We will restrict our discussion to connected noncomplete graphs since disconnected graphs can be examined by connected component and the two-step graph of the complete graph K_n is K_n.

2. The Forbidden Subgraph Approach. In earlier work, Lundgren and Rasmussen [17] take the forbidden subgraph approach to characterizing trees with an interval two-step graph. In general this approach does not work. For example, consider the forbidden subgraphs of an interval graph in Figure 1. Some of these graphs have an interval two-step graph, while others do not. Trees are one class of graphs for which a forbidden subgraph approach does work as illustrated by the following result of Lundgren and Rasmussen.

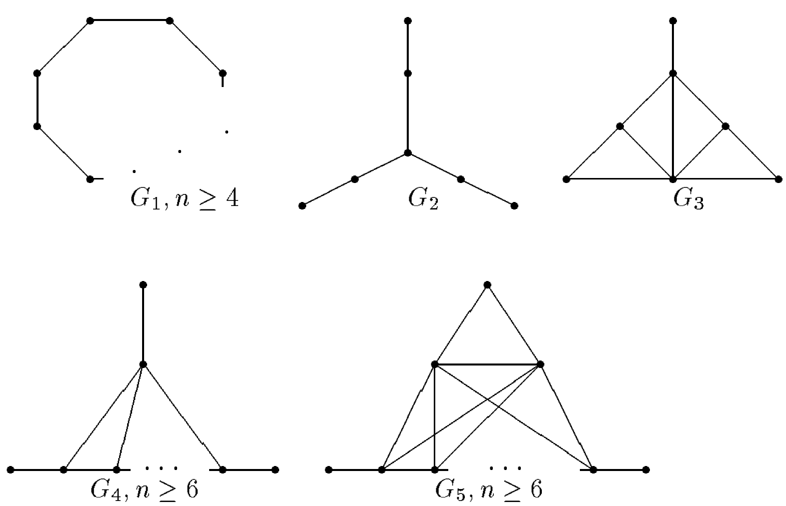


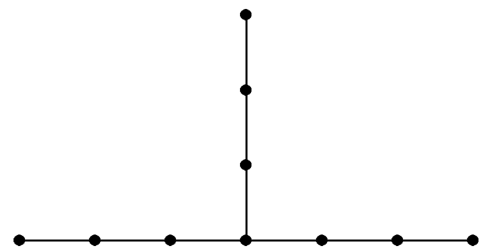
FIG. 1. *A graph is interval if and only if it contains no subgraph isomorphic to G_1, G_2, G_3, G_4, or G_5. Note the two-step graphs of $G_1 (n = 4, 6), G_2, G_3$ and G_5 are interval while the two-step graphs of the others are not.*

PROPOSITION 2.1. *[17] Let T be a tree. Then $S_2(T)$ is interval if and only if T does not contain an induced H, where H is the graph of Figure 2.*

We provide some necessary conditions using forbidden subgraphs which establish the two-step graph as noninterval. The basic idea behind the following two theorems is that if the minimum length cycle in a graph is large enough, the two-step graph contains an induced cycle of length greater than 3.

THEOREM 2.2. *Let G be a graph with girth 5. Then $S_2(G)$ is not interval.*

Proof. Let $C = x_1x_2x_3x_4x_5x_1$ be a cycle in G of length five. Since G has girth 5, C is an induced subgraph of G. We claim $S_2(C)$ is an induced subgraph of $S_2(G)$.



H

FIG. 2.

$S_2(T)$ is interval if and only if T contains no subgraph isomorphic to H

Suppose $S_2(C)$ is not an induced subgraph of $S_2(G)$. Then there are two vertices x_i and x_j in C that are adjacent in $S_2(G)$ but are not in the open neighborhood of a vertex in C. Therefore x_i and x_j are adjacent in C. Since x_i and x_j are joined by a path of length two in G but not in C, there exists a vertex z in G such that $x_i, x_j \in N(z)$. Then $x_i x_j z x_i$ is a cycle in G of length less than 5, a contradiction. Thus $S_2(C)$ is an induced subgraph of $S_2(G)$. It is easy to check that the two-step graph of a five cycle is also a five cycle; thus $S_2(G)$ contains an induced subgraph isomorphic to a cycle of length five which implies $S_2(G)$ is not chordal and therefore not interval, completing the proof. □

Observe that such an approach will not work for graphs with girth three, four, or six, as the two-step graph of each of these graphs is a triangle, two paths of length one, and two triangles respectively. We can eliminate graphs of all other girth.

THEOREM 2.3. *Let G be a graph with girth $p \geq 7$. Then $S_2(G)$ is not interval.*

Proof. Let $C = x_1 x_2 \ldots x_p x_1$ be a cycle in G of length p. Since G has girth p, C is an induced subgraph of G. Suppose $S_2(C)$ is not an induced subgraph of $S_2(G)$. Then there are two vertices x_i and x_j in C that are adjacent in $S_2(G)$ but are not in the open neighborhood of a vertex in C. Therefore x_i and x_j are more than distance two apart on the cycle or they are adjacent. Since x_i and x_j are joined by a path of length two in G but not in C, there exists a vertex z in G such that $x_i, x_j \in N(z)$. If x_i and x_j are adjacent then $x_i x_j z x_i$ is a cycle of length less than p, a contradiction. Otherwise, $x_1 x_2 x_3 \ldots x_i z x_j \ldots x_p, x_1$ is a cycle in G of length less than p, a contradiction. Thus $S_2(C)$ is an induced subgraph of $S_2(G)$. If p is odd, it is easy to check that $S_2(C)$ is a cycle of length p. Thus $S_2(G)$ contains an induced subgraph isomorphic to a cycle of length $p \geq 7$, i.e., $S_2(G)$ is not interval. If p is even, it is easy to check that $S_2(C)$ is a graph isomorphic to two cycles of length $p/2$. Thus $S_2(G)$ contains an induced subgraph isomorphic to a cycle of length $q = p/2 \geq 4$, i.e., $S_2(G)$ is not interval, completing the proof. □

In the sections that follow, we will draw an important connection between open and/or closed neighborhoods and the maximal cliques in the two-step graph. One consequence of this approach is a result involving open neighborhoods in graphs of girth at least 7.

3. Using Open and Closed Neighborhoods to Find Maximal Cliques. We begin with a relatively simple class of graphs: trees. Though a characterization of trees with an interval two-step graph has already been provided, we consider that searching a graph for a forbidden subgraph is not necessarily an easy task. If we can find the maximal cliques of the two-step graph in the original graph easily, we can then use known linear-time algorithms to test for a consecutive ranking. We will disregard maximal cliques in the two-step graph of magnitude 1, since these maximal cliques can be arbitrarily added at either the beginning or end of a consecutive ranking, should one exist. Recall a pendant vertex is a vertex with precisely one neighbor.

THEOREM 3.1. *Let T be a tree. Then the maximal cliques in $S_2(T)$ of magnitude at least 2 correspond to the open neighborhoods of the nonpendant vertices in T.*

Proof. Let $S = N(v)$, where v is a nonpendant vertex in T. Clearly $N(v)$ is a clique in $S_2(T)$. Suppose it is not maximal. Then there exists a vertex $w \notin S$ that is joined to every vertex in S by a path of length two. Since $|S| \geq 2$, there exist distinct vertices x and y in $N(v)$. Since T is a tree, x and y are not adjacent. Then there exist vertices t and u such that $x, w \in N(t)$ and $y, w \in N(u)$. If $t = u$, $vxuyv$ is a cycle in T, a contradiction. Therefore $t \neq u$. Then $vxtwuyv$ forms a cycle in T, a contradiction. Thus no such w can exist; therefore $N(v) = S$ is a maximal clique in $S_2(T)$. Furthermore, if $N(v) = N(z)$ for two vertices v and z, then there exist x and $y \in N(v) \cap N(z)$ and $vxzyv$ is a cycle, a contradiction.

Let S be a maximal clique in $S_2(T)$. Then $|S| \geq 2$, so there exist distinct x and y in S. Since S is a maximal clique in $S_2(T)$, there exists a vertex z such that $x, y \in N(z)$. Suppose $S \neq N(z)$. Then there exists a vertex $w \in S$ such that $w \notin N(z)$. Since T is a tree, x and y are not adjacent. Since S is a maximal clique in $S_2(T)$ there exist vertices t and u such that $w, x \in N(t)$ and $w, y \in N(u)$. If $t = u$, $txzyt$ is a cycle in T, a contradiction. Therefore $t \neq u$. Then $wtxzyuw$ is a cycle in T, a contradiction. Thus no such w can exist, i.e., $N(z) = S$, completing the proof. □

Using the Gilmore-Hoffman characterization of interval graphs we obtain the following corollary.

COROLLARY 3.2. *Let T be a tree. Then $S_2(T)$ is interval if and only if the maximal open neighborhoods of the nonpendant vertices in T have a consecutive ranking.*

We would like to take this characterization further to triangle-free graphs. Again the 6-cycle poses a problem. This is captured in the following lemma, the proof of which is easily observed.

LEMMA 3.3. *Let G be a graph and let x, y and z be vertices contained in a maximal clique in $S_2(G)$. If there does not exist v such that $x, y, z \in N[v]$ then there must exist distinct a, b and c such that $x, y \in N(a)$, $y, z \in N(b)$ and $x, z \in N(c)$, i.e., $xaybzcx$ is a 6-cycle.*

So in order to find classes of graphs in which the maximal cliques of the two-step graph correspond to open or closed neighborhoods in the original graph, we must exclude graphs containing 6-cycles.

THEOREM 3.4. *Let $G = (V, E)$ be a connected, noncomplete triangle- and 6-cycle-free graph. Then $C \subseteq V$ such that $|C| \geq 2$ is a maximal clique in $S_2(G)$ if and only if $C = N(z)$ for some z in G such that the open neighborhood of z is not properly contained in the open neighborhood of any other vertex.*

Proof. ($\Rightarrow$) Let C be a maximal clique in $S_2(G)$. If $|C| = 2$ the statement is clearly true so assume $|C| \geq 3$. Let $R \subseteq C$. We prove by induction on $|R|$ that there exists z such that $R \subseteq N[z]$ in G. By Lemma 3.3 if $|R| = 3$ the claim is true so assume $|R| \geq 4$. Assume the claim is true for all R such that $|R| < k \leq |C|$ and consider the case $|R| = k \leq |C|$. Pick arbitrary $x \in R$. Let $R' = R - \{x\}$. By induction hypothesis there exists z_1 such that $R' \subseteq N[z_1]$ in G. Pick arbitrary $y \neq x \in R$. Let $R'' = R - \{y\}$. By induction hypothesis there exists z_2 such that $R'' \subseteq N[z_2]$. Since x and y are in R there exists z such that $x, y \in N(z)$. If z is z_1 or z_2 we are done so assume not. Observe $z_1, z_2 \notin R$ since G is triangle-free (for example, if $z_1 \in R$ then y and z_1 are adjacent and joined by a path of length two). Since $|R| \geq 4$ there exists $w \in R$ ($w \neq z, w \neq x, w \neq y, w \neq z_1, w \neq z_2$) such that w is adjacent to z_1 and z_2. Then $xzyz_1wz_2x$ is a 6-cycle in G, a contradiction. Therefore without loss of generality we conclude $z = z_1$, i.e., $R \subseteq N[z_1]$ for all $R \subseteq C$. In particular $C \subseteq N[z_1]$ and so by maximality of C we conclude $C = N(z_1)$.

($\Leftarrow$) Let z be a vertex in G such that the open neighborhood of z is not properly contained in the open neighborhood of any other vertex in G. Clearly $N(z)$ is a clique in $S_2(G)$. Suppose it is not maximal. Then there is a vertex $w \notin N(z)$ such that w is joined by a path of length two to every vertex in $N(z)$ in G. Let $x \in N(z)$. Since w and x are joined by a path of length two in G there exists a such that $x, w \in N(a)$. But $N(z)$ is not properly contained in $N(a)$ so there exists $y \in N(z)$ such that a and y are not adjacent. Then w and y are joined by a path of length two implies there exists a distinct vertex b such that $y, w \in N(b)$. Since G is triangle-free, $b \neq x$. Then $zxawbyz$ is a 6-cycle in G, a contradiction. Thus $N(z)$ forms a maximal clique in $S_2(G)$, completing the proof. □

If an open neighborhood has the property that it is not properly contained in the open neighborhood of any other vertex, we say it is *maximal*. This result does not state that there is a one-to-one correspondence between the maximal cliques in $S_2(G)$ and the maximal closed neighborhoods of G. For example, consider the graph in Figure 3. In this graph, $N(v_1) = N(v_2)$. Since the existence of a consecutive ranking of a family of sets is not affected by allowing a set in the family to appear more than once, we use the Gilmore-Hoffman characterization of interval graphs to conclude the following.

COROLLARY 3.5. *Let G be a connected, noncomplete, triangle- and 6-cycle-free graph. Then $S_2(G)$ is interval if and only if the maximal open neighborhoods of G have a consecutive ranking.*

Theorem 2.3 and Corollary 3.5 then prove:

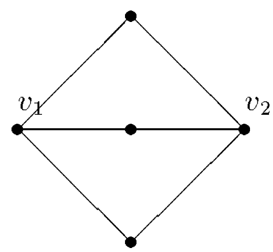


FIG. 3.

The maximal cliques in the two-step of this graph do not correspond one-to-one with the maximal open neighborhoods of the original graph.

COROLLARY 3.6. *Let G be a graph with girth $p \geq 7$. Then the maximal open neighborhoods of G do not have a consecutive ranking.*

Now consider 6-cycle-free graphs such that every edge is contained in a triangle.

THEOREM 3.7. *Let $G = (V, E)$ be a connected, noncomplete, 6-cycle-free graph such that every edge is contained in a triangle. Then $C \subseteq V$ such that $|C| \geq 2$ is a maximal clique in $S_2(G)$ if and only if $C = N[z]$ for some z in G such that the closed neighborhood of z is not properly contained in the closed neighborhood of any other vertex.*

Proof. ($\Rightarrow$) Let C be a maximal clique in $S_2(G)$. By an analogous argument to that in Theorem 3.4 we can show that there exists z such that $C \subseteq N[z]$. Since every edge is contained in a triangle and C is maximal we conclude $C = N[z]$.

($\Leftarrow$) Let z be a vertex in G such that $N[z]$ is not properly contained in another closed neighborhood in G. Since every edge of G is contained in a triangle, clearly $N[z]$ forms a clique in $S_2(G)$. Suppose it is not maximal. Then there exists w such that w is joined to every vertex in $N[z]$ by a path of length two but w and z are not adjacent. Since w and z are joined by a path of length two there exists a vertex v such that $w, z \in N(v)$. Since $N[z]$ is not properly contained in $N[v]$ there exists $y \in N[z]$ such that $y \notin N[v]$. Then w and y are joined by a path of length two so there exists u such that $w, y \in N(u), (u \neq v)$. Then $v, z \in N(u)$ since otherwise the edge (v, z) is contained in a triangle implies there exists a vertex t such that $v, z \in N(t)$ and $ztvwuyz$ is a 6-cycle. But then $N[z]$ is not properly contained in $N[u]$ so there exists $x \in N[z]$ such that $x \notin N[u]$. If $x \notin N(v)$ we are done since x and w are joined by a path of length two implies there exists s (possibly y) such that $w, x \in N(s)$ and then $wvuzxsw$ forms a 6-cycle in G. Thus x and v are adjacent. Then $wvxzyuw$ forms a 6-cycle in G. This contradiction proves no such w can exist, completing the proof. □

COROLLARY 3.8. *Let G be a connected, noncomplete, 6-cycle-free graph such that every edge is contained in a triangle. Then $S_2(G)$ is interval if and only if the maximal closed neighborhoods of G have a consecutive ranking.*

To generalize these results we need some definitions.

4. The Competition Cover Approach. We begin with the following definition from Lundgren, Maybee, and Rasmussen [16]. Let G be a graph. A family $S =$

$\{S_1, \ldots, S_r\}$ of sets of vertices of G is called a *competition cover* of G if the following conditions are satisfied:

1. $i, j \in S_m$ implies there exists a vertex k such that $i, j \in N(k)$.
2. if $i, j \in N(k)$ for some k, then $i, j \in S_m$ for some m.

This definition leads to the following result.

PROPOSITION 4.1. [16] *Let G be a graph. Then $S_2(G)$ is interval if and only if G has a competition cover S which has a consecutive ranking.*

The difficulty with this result is finding the right competition cover. Furthermore, it is very difficult to use this characterization to prove that the two-step graph of a given graph is not interval. This leads to the following question: can we define a specific family of sets in G that determines whether or not $S_2(G)$ is interval? We have already shown this family of sets is the open neighborhoods for trees and triangle- and 6-cycle-free graphs and the closed neighborhoods for 6-cycle-free graphs such that every edge is contained in a triangle. Using the competition cover approach, this problem was solved for interval graphs in [16]. The family of sets is found through categorizing the *nonsimplicial vertices* of G (recall a simplicial vertex is a vertex whose neighborhood is a clique). Let v_i be a nonsimplicial vertex in G. We say v_i is Type I if every maximal clique containing v_i contains three or more vertices. We say v_i is Type II if every maximal clique containing v_i contains exactly two vertices. Otherwise we say v_i is Type III.

Let G be a noncomplete connected graph with nonsimplicial vertices $\{v_1, \ldots, v_r\}$. Define $S(G) = \{S_1, \ldots, S_r\}$, where S_i is

1. $N[v_i]$, the closed neighborhood of v_i, if v_i is Type I.
2. $N(v_i)$, the open neighborhood of v_i, if v_i is Type II.
3. actually two sets S_{i_1} and S_{i_2} otherwise, where

$$S_{i_1} = \mathcal{C}_{v_i} = \bigcup\{C | C \in \mathcal{C}, v_i \in C, |C| \geq 3\} \quad \text{and} \quad S_{i_2} = N(v_i),$$

where $\mathcal{C}$ is the family of maximal cliques in G.

Define $S'(G)$ as the set of all sets in $S(G)$ such that no set is properly contained in any other. We note the following previous result.

PROPOSITION 4.2. [16] *Let G be a connected noncomplete interval graph. $S'(G)$ is a competition cover of G.*

For this reason $S'(G)$ is called the *maximal nonsimplicial competition cover* of G. $S'(G)$ is particularly useful in characterizing interval graphs with interval two-step graphs as was proved in the following result.

PROPOSITION 4.3. [16] *Let G be a connected noncomplete interval graph. $S_2(G)$ is interval if and only if $S'(G)$ has a consective ranking.*

Observe that Proposition 4.2 does not say anything about the maximal cliques in $S_2(G)$. A competition cover of a graph does not necessarily correspond precisely to the maximal cliques in the two-step graph. For example, the open neighborhoods of a

6-cycle form a competition cover, but the two-step graph of a 6-cycle is two triangles. Figure 4 gives another example in which this is not the case. We now ask the following question: when does the competition cover $S'(G)$ correspond to the maximal cliques in $S_2(G)$?

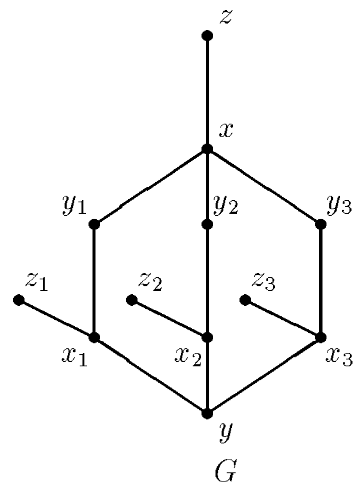


FIG. 4.

Observe that $\{x, x_1, x_2, x_3\}$ forms a maximal clique in $S_2(G)$ but this set is not a member of $S'(G)$.

Though Proposition 4.3 already characterizes interval graphs with interval two-step graphs, we consider whether or not for an interval graph G, $S'(G)$ corresponds to the maximal cliques in $S_2(G)$. The following result of Lundgren, Maybee, and Rasmussen proves the first half of the next theorem.

PROPOSITION 4.4. [16] *Let G be a connected, noncomplete, interval graph. Let $S'(G) = \{S_1, \ldots, S_m\}$ be the maximal nonsimplicial competition cover of G. Let $x \in V(G)$. If x is connected by a path of length two to every vertex in some $S_i \in S'(G)$, then $x \in S_i$.*

THEOREM 4.5. *Let $G = (V, E)$ be a connected, noncomplete, interval graph. Then $C \subseteq V$ is a maximal clique in $S_2(G)$ if and only if $C \in S'(G)$.*

Proof. ($\Leftarrow$) Let $C \in S'(G)$. Clearly C is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists a vertex $w \notin C$ such that w is joined to every vertex in C by a path of length two. But Proposition 4.4 implies w must be an element of C. This contradiction proves C must be a maximal clique in $S_2(G)$.

($\Rightarrow$) Let C be a maximal clique in $S_2(G)$. Since G is interval the maximal cliques of G have a consecutive ranking $\{C_1, \ldots, C_l\}$. We claim there exists a nonsimplicial vertex z such that $C \subseteq N[z]$. First we will show there exists a vertex z such that $C \subseteq N[z]$. Suppose not. Let i be the smallest integer such that there exists a vertex x that is an element of both C_i and C, but $x \notin C_{i+1}$. This must occur since $C \not\subseteq N[x]$. Let j be the largest integer such that there exists a vertex y that is an element of both C_j and C, but $y \notin C_{j-1}$. This must occur since $C \not\subseteq N[y]$. Note i must be less than j, for if not then $C \subseteq C_k$ for all $j \leq k \leq i$. Since x and y are joined by a path of length two, there exists a vertex z such that x and z are contained in a maximal clique and y and z are contained in a maximal clique. Since this ranking is consecutive and $x \notin C_{i+1}$, z must be in a clique C_k such that $k \leq i$. Since $y \notin C_{j-1}$, z must be in a clique C_m such that $m \geq j$. This ranking of cliques is consecutive, therefore $z \in C_p$ for all p, $i \leq p \leq j$. Note every vertex of C is contained in a clique C_p such that $i \leq p \leq j$. Thus $C \subseteq N[z]$, a contradiction. Thus there must exist a vertex z such that $C \subseteq N[z]$.

We now return to the proof of our claim. If z is simplicial, then C is a clique in G. Since G is connected and not complete, there exists a vertex $x \notin C$ such that x is adjacent to a vertex $y \in C$. If y is nonsimplicial we are done since $C \subseteq N[y]$, so assume y is simplicial. Then $\{x\} \cup C$ is a clique in $S_2(G)$ containing C, a contradiction. Therefore z is nonsimplicial, completing the proof of our claim.

If $z \in C$, since C is a maximal clique and z is joined to every vertex in C by a path of length two, it follows that $C = \mathcal{C}_z$. If $z \notin C$, since C is a maximal clique it follows that $C = N(z)$. In either case, $C \in S'(G)$. □

Proposition 4.3 is then an immediate corollary.

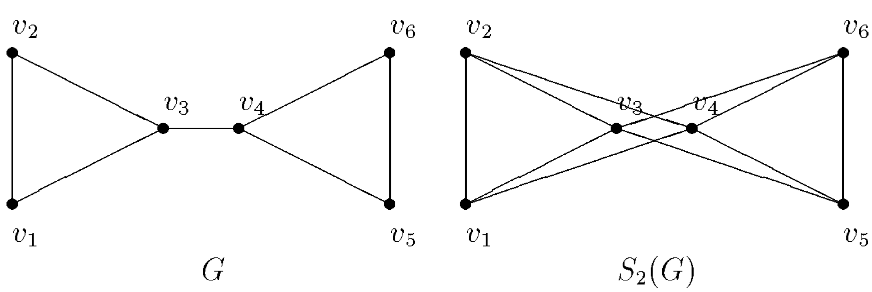


FIG. 5.
An interval graph with a noninterval two-step graph.

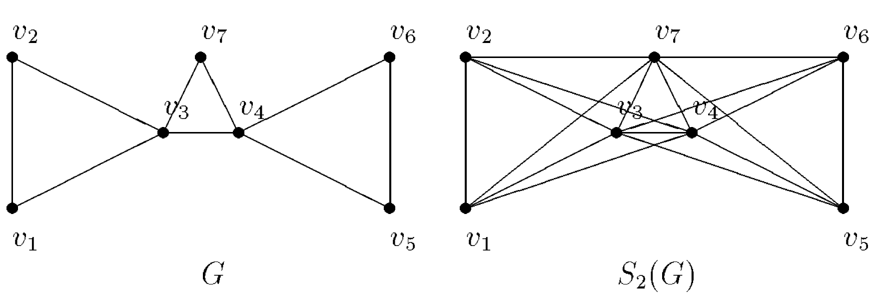


FIG. 6.
An interval graph with an interval two-step graph.

Observe that Theorem 4.5 characterizes some graphs which have an interval two-step graph and some which do not. For example, the graph in Figure 5 is interval while

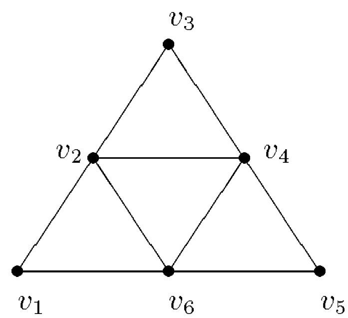


FIG. 7.
$S'(G) = \{\{v_1, v_2, v_3, v_4, v_6\}, \{v_2, v_3, v_4, v_5, v_6\}, \{v_1, v_2, v_4, v_5, v_6\}\}$ does not have a consecutive ranking although $S_2(G) = K_6$ is interval.

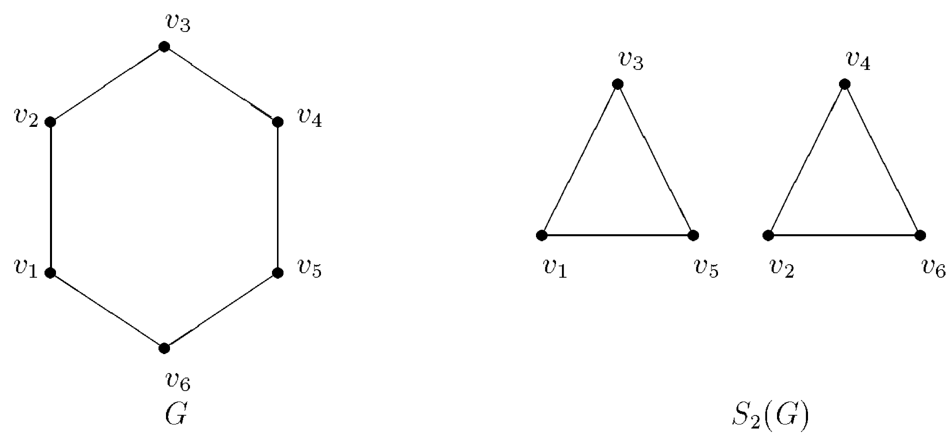


FIG. 8.
$S'(G) = \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_3, v_5\}, \{v_4, v_6\}, \{v_1, v_5\}, \{v_2, v_6\}\}$ does not have a consecutive ranking although $S_2(G)$ is interval.

its two-step graph is not. The graph in Figure 6 is just one example of an interval graph with an interval two-step graph. The graphs shown in Figures 7 and 8 are useful examples demonstrating that Proposition 4.3 does not necessarily hold if G is not interval or connected. In both cases the sets of $S'(G)$ do not have a consecutive ranking, while $S_2(G)$ is interval. Both examples also contain 6-cycles.

5. A Characterization for 6-cycle-free Graphs. We now show that by considering the maximal nonsimplicial competition cover we can characterize a large class of graphs with interval two-step graphs. 6-cycles must be forbidden.

THEOREM 5.1. *Let $G = (V, E)$ be a connected, noncomplete, 6-cycle-free graph. Then $C \subseteq V$ such that $|C| \geq 2$ is a maximal clique in $S_2(G)$ if and only if $C \in S'(G)$.*

Proof. ($\Rightarrow$) Let C be a maximal clique in $S_2(G)$. By an induction argument similar to that used in the proof of Theorem 3.4, we can show there exists a nonsimplicial vertex z such that $C \subseteq N[z]$. First we show there exists z such that $C \subseteq N[z]$. Clearly this is true if $|C| = 2$. By Lemma 3.3 it is true if $|C| = 3$, so assume $|C| \geq 4$. The induction is on $|R|$ where $R \subseteq C$. Assume there exists z such that $R \subseteq N[z]$ for R such that $|R| < k \leq |C|$ and assume $|R| = k \leq |C|$. Pick $x \in R$. Let $R' = R - \{x\}$. By induction hypothesis there exists z_1 such that $R' \subseteq N[z_1]$. Pick $y \in R$ such that $y \neq x$. Let $R'' = R - \{y\}$. By induction hypothesis there exists z_2 such that $R'' \subseteq N[z_2]$. If $z_1 = z_2$ we are done so assume not. Then x and y are joined by a path of length two implies there exists z such that $x, y \in N(z)$. If there exists $w \in R$ such that $w \neq x, y, z_1, z_2, z$ we are done since $w \in N[z_1]$ and $w \in N[z_2]$ implies $xzyz_1wz_2x$ is a 6-cycle, so assume not. Then $R \subseteq \{x, y, z_1, z_2, z\}$. Since $|R| \geq 4$, at least one of the set $\{z_1, z_2\}$ is in R. Without loss of generality assume $z_1 \in R$. Then $z_1 \in N[z_2]$. If $z_1 \notin N[z]$ we are done since z_1 and y are joined by a path of length two implies there exists w such that $z_1, y \in N[w]$ implies $zwyz_1z_2xz$ is a 6-cycle. Therefore assume z_1 and z are adjacent. If $z_2 \notin R$ we are done since $R \subseteq N[z]$ so assume $z_2 \in R$. Similarly, if $z_2 \in N[z]$ we are done so assume not. Then x and z_2 are joined by a path of length two implies there exists w such that $x, z_2 \in N[w]$ implies xwz_2z_1yzx is a 6-cycle. Therefore there exists z such that $R \subseteq N[z]$, completing the proof of our claim.

If z is simplicial then C is a clique in G. Since G is connected and not complete, there exists a vertex $x \notin C$ such that x is adjacent to a vertex $y \in C$. If y is nonsimplicial we are done since $C \subseteq N[y]$ so assume y is simplicial. Then $\{x\} \cup C$ is a clique in $S_2(G)$ containing C, a contradiction. Therefore there exists nonsimplicial z such that $C \subseteq N[z]$. If $z \in C$, since C is a maximal clique in $S_2(G)$ and z is joined to every vertex in C by a path of length two, $C = \mathcal{C}_z$. If $z \notin C$, since C is a maximal clique in $S_2(G)$, $C = N(z)$.

($\Leftarrow$) Let $C \in S'(G)$. By definition there exists a nonsimplicial vertex z such that $C \subseteq N[z]$. We then have two cases. <u>Case 1:</u> There exists nonsimplicial z such that $C = \mathcal{C}_z$. Observe that z may be either Type I or Type III. Clearly C is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists $w \notin C$ such that w is joined to every $s \in C$ by a path of length two. So there exists x such that $z, w \in N(x)$. Observe that $w \notin C$, $C = \mathcal{C}_z$, and w and x adjacent implies w and z are not adjacent. Since $C \not\subseteq N[x]$, there exists a vertex $y \in C$ such that x and y are not adjacent. Then y and w are joined by a path of length two so there exists a vertex u such that $w, y \in N(u), (u \neq z, u \neq x)$. If $z \notin N(u)$ we are done since y and z are contained in a triangle implies there exists a vertex $t (t \neq x, t \neq w)$ such that $y, z \in N(t)$. Then $ztyuwxz$ is a 6-cycle, a contradiction, so $z \in N(u)$. Then $u \in C$. Suppose $x \in C$. If $x \notin N(u)$, we are done since x and z contained in a triangle implies there exists $t (t \neq u, t \neq w, t \neq y)$ such that $x, z \in N(t)$. Then $ztxwuyz$ is a 6-cycle. Therefore $x \in C$ implies $x \in N(u)$. But $C \not\subseteq N[u]$ so there exists $v \in C$ such that u and v are not adjacent. If v and y are adjacent we are done since $zvyuwxz$ is a 6-cycle. So v and y are not adjacent. Then there exists s such that $w, v \in N(s)$ where s is possibly x but $s \neq u, s \neq y$ and $s \neq z$. Then $zvswuyz$ is a 6-cycle. Therefore $x \notin C$. This implies x and u are not adjacent.

But $C \not\subseteq N[u]$ so there exists a vertex $v \in C$ such that u and v are not adjacent. If v and y are adjacent we are done since $zvyuwxz$ is a 6-cycle, so assume not. Then there exists a vertex s (possibly x, but $s \neq y, s \neq u$) such that $w, v \in N(s)$. Then $zvswuyz$ is a 6-cycle, a contradiction. Therefore C is a maximal clique in $S_2(G)$.

Case 2: There exists nonsimplicial z such that $C = N(z)$. Observe that z may be Type II or Type III. Clearly C is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists a vertex $w \notin C$ joined to every $s \in C$ by a path of length two. Let $x \in C$. Then there exists y such that $x, w \in N(y)$. Since $C \not\subseteq N(y)$ there exists $v \in N(z)$ such that y and v are not adjacent. Then w and v must be adjacent to x since otherwise there exists a distinct vertex t such that $w, v \in N(t)$ and $wtvzxyw$ is a 6-cycle. If z is Type II we are done because z is contained in a triangle, namely $vxzv$, a contradiction. So assume z is Type III. Then $C \not\subseteq N[x]$ implies there exists $u \in C$ such that x and u are not adjacent. If u and y are adjacent we are done since $wxvzuyw$ is a 6-cycle. So assume u and y are not adjacent. Then w and u must be adjacent to v since otherwise there exists a distinct vertex t such that u and w are adjacent to t and $utwyxzu$ is a 6-cycle. Then $wvuzxyw$ is a 6-cycle, a contradiction. Therefore C is a maximal clique in $S_2(G)$. □

COROLLARY 5.2. *Let G be a connected, noncomplete 6-cycle-free graph. Then $S_2(G)$ is interval if and only if the maximal nonsimplicial competition cover of G has a consecutive ranking.*

6. Graphs with Sparse 6-cycles. Since the maximal cliques in the two-step graph of a 6-cycle are easily found, it may be possible to find the maximal cliques of the two-step graph in the original graph if we require that the 6-cycles be sparsely arranged. First, a definition. Let $H = abcdefa$ denote a 6-cycle. We then say *the alternating triples* of H are $\{a, c, e\}$ and $\{b, d, f\}$. Figure 8 illustrates that the family of maximal cliques in the two-step graph of a 6-cycle is precisely the set of alternating triples. We can apply this idea to the following large class of graphs. The graph in Figure 4 illustrates the difficulty when 6-cycles overlap by more than a single edge: the set $\{x, x_1, x_2, x_3\}$ forms a maximal clique in $S_2(G)$, but is neither a set in $S'(G)$ nor an alternating triple.

THEOREM 6.1. *Let G be a connected, noncomplete, triangle-free graph such that no two 6-cycles in G have more than a single edge in common. Let C such that $|C| \geq 2$ be a maximal clique in $S_2(G)$. Then either $C = N(z)$ for some nonsimplicial vertex z in G or C is an alternating triple from a 6-cycle in G.*

Proof. If $|C| = 2$ clearly C must be the open neighborhood of a nonsimplicial vertex with precisely two neighbors so the statement is true. If $|C| = 3$ by Lemma 3.3 and maximality of C, we observe the statement is true. So assume $|C| \geq 4$. Let R denote a subset of C. We will prove by induction on $|R|$ that there exists a vertex z such that $C \subseteq N[z]$.

Let $|R| = 4$. Pick arbitrary $x \in R$. Let $R' = R - \{x\}$. Then by Lemma 3.3 there exists y such that $R' \subseteq N[y]$ or R' is the set of alternating triples from a 6-cycle in G. Assume there exists y such that $R' \subseteq N[y]$. Since G is triangle-free, $y \notin C$ (otherwise y would be joined by a path of length two to one of its neighbors) and hence $y \notin R'$. If

$x \in N[y]$ we are done so assume not. Further assume there does not exist z such that $R \subseteq N[z]$. Since $|R| = 4$ there exists a vertex $a \in R'$. Then there exists t such that $x, a \in N(t)$. Since there does not exist z such that $R \subseteq N[z]$ and $|R| = 4$ there exists $b \in R'$ such that $b \neq a$ and there exists $u \neq t$ such that $x, b \in N(u)$. Since there are no triangles in G, t and u are not adjacent to y. Furthermore, $y \notin R$. Let c denote the remaining vertex in R'. If there exists a distinct vertex s such that $x, c \in N(s)$ we are done since $xtaybux$ and $xscybux$ are two 6-cycles with more than a singe common edge. So assume no such s exists. Then c must be adjacent to t or u. WLOG, assume c and u are adjacent. Then $xtaybux$ and $xtaycux$ are two 6-cycles with more than a single common edge, a contradiction. Thus there must exist z such that $R \subseteq N[z]$.

We now assume R' is an alternating triple from a 6-cycle in G. Once again, let a, b, c denote the vertices of R'. Then there exist vertices p, q, r such that $bpaqcrb$ forms a 6-cycle in G. If there exists a vertex z such that three elements of R are in the open neighborhood of z we can let R' be the set of these vertices and we are in the former case. So assume no such z exists. Then x is not adjacent to p, q nor r. So there exist distinct vertices w and y such that $x, b \in N(w)$ and $x, a \in N(y)$. Then $wxyapbw$ and $apbrcqa$ are two 6-cycles with more than a common edge, a contradiction. So there must exist a vertex z such that $R \subseteq N[z]$.

This verifies the statement for $|R| = 4$. Assume the statement is true for all R such that $|R| < k \leq |C|$ and let $|R| = k \leq |C|$. By assumption $|R| > 4$. Pick arbitrary $x \in R$ and let $R' = R - \{x\}$. By induction hypothesis there exists z_0 such that $R' \subseteq N[z_0]$. Suppose there does not exist z such that $R \subseteq N[z]$. Then x and z_0 are not adjacent and there must exist distinct vertices $y, z \in R'$ such that there exist distinct vertices a and b such that $x, z \in N(a)$ and $x, y \in N(b)$. Since there are no triangles in G no two elements of R are adjacent and $a, b \notin R'(a, b \notin N(z_0))$. Furthermore, $z_0 \notin R$. Since $|R| > 4$ there exists another distinct vertex $w \in R'$. If there exists a distinct vertex c such that $w, x \in N(c)$ we are done since $xazz_0ybx$ and $xcwz_0ybx$ are two 6-cycles with more than a single common edge. Thus w must be adjacent to a or b. WLOG assume w and b are adjacent. Then $xazz_0ybx$ and $xbwz_0zax$ are two 6-cycles with more than a single common edge, a contradiction. This proves for all subsets R of C, there must exist z such that $R \subseteq N[z]$; in particular there exists z such that $C \subseteq N[z]$. Since there are no triangles in G, $z \notin C$. Since C is a maximal clique in $S_2(G)$, $C = N(z)$, completing the proof. □

Let $T(G)$ denote the set of alternating triples for all 6-cycles found in the graph G. Define $R(G)$ as $S'(G) \cup T(G)$. Define $R'(G)$ as the set of all sets in $R(G)$ such that no set is properly contained in any other. Figure 9 illustrates that an element of $T(G)$ may be properly contained in an element of $S'(G)$ and vice versa.

THEOREM 6.2. *Let G be a connected, noncomplete, triangle-free graph such that no two 6-cycles have more than a single edge in common. Let $C \in R'(G)$. Then C is a maximal clique in $S_2(G)$.*

Proof. Since G is triangle-free, every nonsimplicial vertex is of Type II. Thus we need only consider two cases: C is the open neighborhood of a nonsimplicial vertex or

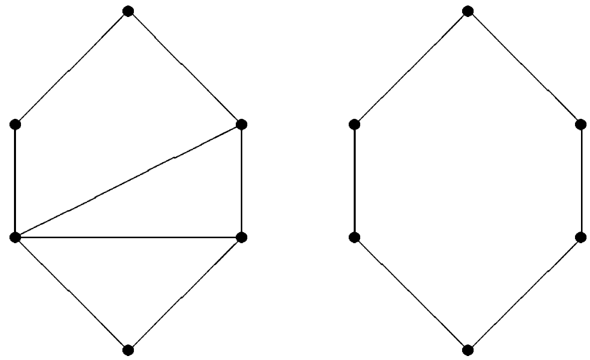


FIG. 9.
The graph on the left contains an element of $T(G)$ which is properly contained in a set of $S'(G)$. The graph on the right contains an element of $S'(G)$ which is properly contained in a set of $T(G)$.

C is an alternating triple.

Assume C is the open neighborhood of a nonsimplicial vertex z. Then $|C| \geq 2$. Clearly C forms a clique in $S_2(G)$. Suppose it is not maximal. Then there exists $w \notin N(z)$ such that w is joined to every vertex in $N(z)$ by a path of length two. Observe there does not exist a vertex p such that $\{w\} \cup N(z) \subseteq N(p)$ since $N(z)$ is not properly contained in $N(p)$. Thus there exist $x, y \in N(z)$ such that there exist distinct a and b (not in $N(z)$ since G is triangle- free) such that $x, w \in N(a)$ and $y, w \in N(b)$. There must exist another distinct vertex $u \in N(z)$ since $N(z)$ is not properly contained in an alternating triple. If there exists a distinct vertex c such that $w, u \in N(c)$ we are done, as we have two 6-cycles in G with more than a single edge in common. Since G is triangle-free $x, y \notin N(w)$. Thus u must be adjacent to a or b, in either case creating two 6-cycles with more than a single edge in common, a contradiction. Thus no such w can exist, i.e. $N(z)$ is a maximal clique in $S_2(G)$.

Alternatively, assume C is an alternating triple $\{x, y, z\}$. Then there exist vertices a, b, c such that $xaybzcx$ is a 6-cycle in G. Clearly C is a clique in $S_2(G)$. Suppose it is not maximal. Then there exists a vertex w joined to x, y and z by a path of length two. Since G is triangle-free $x, y, z \notin N(w)$. Suppose w is adjacent to more than one element of the set $\{a, b, c\}$. One can easily show we have two 6-cycles with more than a single edge in common. Suppose w is adjacent to one element of the set $\{a, b, c\}$. WLOG, assume w and c are adjacent. Then w and y are joined by a path of length two implies there exists a new vertex d such that $w, y \in N(d)$ and we have two 6-cycles with more than a single edge in common. Thus w is not adjacent to a, b or c. Then there must exist new vertices s and t such that $w, x \in N(s)$ and $w, z \in N(t)$. If $s = t$ then $xaybzsx$ and $xaybzcx$ are two 6-cycles with more than a single common edge. Therefore $s \neq t$. But then $xczbyax$ and $xcztwsx$ are two 6-cycles with more than a single common edge. Thus C is a maximal clique in $S_2(G)$, completing the proof. □

Then by Theorems 6.1, 6.2 and the Gilmore-Hoffman characterization of interval graphs we conclude:

COROLLARY 6.3. *Let G be an connected, noncomplete, triangle-free graph such that no two 6-cycles in G share more than one edge. Then $S_2(G)$ is interval iff $R'(G)$ has a consecutive ranking.*

7. Conclusions and Directions for Further Research. The following open questions may be of interest in characterizing graphs with interval two-step graphs.

1. Which graphs have complete two-step graphs or two-step graphs consisting of complete components? For example, the two-step graph of the complete bipartite graph $K_{1,m}$ is $K_1 \cup K_m$.
2. Which graphs have chordal two-step graphs? This is related to characterizing graphs with chordal squares. These problems have been considered by Phelps [18], Harary and McKee [10], and Lundgren and Merz [14]. Also related is the problem of characterizing graphs with interval squares (see [15, 14]).

Results in these areas are potentially useful with regard to the channel assignment problem. Lundgren, Maybee, and Rasmussen [13] discuss this application in greater detail. Optimal colorings or T-colorings are desired in making frequency assignments. Raychaudhuri [19] extended a result of Cozzens and Roberts [7] to give an $O(n^2)$ algorithm for finding a T-coloring of an interval graph. Rose, Tarjan, and Leuker [21] showed that a chordal graph can be recognized in linear time. A linear time algorithm developed by Fulkerson and Gross [8] can then be used to find the maximal cliques of a chordal graph. Booth and Leuker [3] showed that a family of sets, the maximal cliques in this case, can be tested for a consecutive ranking in linear time, thus proving that interval testing can be done in linear time. $S'(G)$ can be found in $O(|V|^2)$ time. The algorithm due to Booth and Leuker can then be used to test $S'(G)$ for a consecutive ranking. Thus given an incomplete connected graph with no 6-cycle, we can perform interval testing in time proportional to $|V|^2$.

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