Elimination Ordering Characterizations of Digraphs with Interval and Chordal Competition Graphs

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Abstract. Competition graphs have appeared in a variety of applications from food webs to communication networks to energy models. The competition graph, $C(D)$, of a digraph D has the same vertex set as D and (x, y) is an edge in $C(D)$ if and only if there is a vertex z such that (x, z) and (y, z) are arcs in D. It has been observed that most actual food webs have interval competition graphs. Subsequent research investigating this phenomenon led to the problem of characterizing acyclic digraphs which have interval competition graphs since food web models are generally acyclic. This evolved into the problem of characterizing digraphs with interval or chordal competition graphs. The problem is difficult because forbidden subgraph characterizations generally don't work. Here we use an elimination ordering approach which leads to useful characterizations which can be generalized to determine which graphs have interval or chordal squares or two-step graphs.

key words: competition graphs, interval graph, chordal graph, elimination ordering, square, two-step graph

1. Introduction. Let $D = (V, A)$ be a digraph. The *competition graph*, $C(D) = (V, E)$ is a graph defined on the same vertex set with $(x, y) \in E$ if and only if there exists $z \in V$ such that (x, z) and (y, z) are in A. The study of competition graphs first arose in the study of food webs by Cohen [2, 3, 4, 5]. Using a digraph to model relationships between species in some environment, a vertex may represent a single species, or class of organisms that share similar kinds of predators and prey. One may then ask the following question: in order to characterize the competition between species, how many variables must be considered? One example of such a variable is diet; in the food web model, we place an arc from x to y if x preys upon y. The *boxicity* of a graph, first introduced by Roberts [15], is defined to be the smallest integer k such that the graph can be represented as an intersection graph of boxes in k-space. Surprisingly, Cohen observed that a large number of competition graphs for actual food webs are interval, i.e., have boxicity one. Thus intervals representing a single variable characterize competition in the community; a nice summary of this problem may be found in Roberts [16]. Since its discovery, an explanation for this phenomenon has been sought. Sugihara [19] later suggested the chordal property (all interval graphs are chordal) was significant with respect to food webs. In this paper, we present a characterization of digraphs with chordal and interval competition graphs which may lead to interpretations of the structure of actual food webs. This characterization is different from previous approaches to the problem, for example, the approach by Lundgren and Maybee [10].

Digraphs may also be used to model radio communication networks. Competition graphs and their applications to such networks have been studied extensively, for example by Raychaudhuri and Roberts [14]. Let each vertex of a digraph represent a station in a radio communication network, with an arc from x to y if y can receive a signal transmitted by x. The competition graph of this digraph shows which stations

[1] This research was partially supported by Research Contracts N00014-93-1-0670 and N00014-91-J-1145 of the Office of Naval Research.

will conflict if they are assigned the same radio frequency. In the interest of efficiently assigning frequencies to the stations, it is desirable to determine whether or not the competition graph of such a digraph is interval.

In a similar model, we have a set of points in Euclidean k-space, where each point represents a station in a radio communication network. A message sent by a transmitter at x can be received at y if and only if the distance between x and y is at most a fixed amount, δ. Note that in this model we have a loop at each vertex and the digraph is symmetric; such a digraph has an underlying graph with a loop at each vertex. Let G be the underlying graph of such a digraph D, with loops removed. It is an easy exercise to then show that the competition graph of the digraph D is the square of G, where two vertices are adjacent in the square of G if and only if they are joined by a simple path of one or two edges in G. Remove the loops from D and call this digraph D'. It is also easy to verify that the two-step graph of G is the competition graph of D', where two vertices are adjacent in the two-step graph of G if and only if they are joined by a simple path on exactly two edges in G. Symmetric digraphs with interval competition graphs have been studied by Lundgren, Maybee, and Rasmussen [11]. The problem of characterizing graphs with interval squares and two-step graphs has been studied, for example by Raychaudhuri [13] and Lundgren, Maybee, Merz, and Rasmussen [12]. In this paper we generalize our elimination ordering characterizations of digraphs with chordal and interval competition graphs to elimination ordering characterizations of graphs with chordal and interval squares and two-step graphs.

2. Digraphs with Chordal Competition Graphs. Chordal graphs have several nice properties. A graph is chordal if and only if it contains no generated k-cycles, $k \geq 4$, that is, all k-cycles, $k \geq 4$, contain a chord. For a comprehensive introduction to chordal (or triangulated) graphs, see Golumbic [7]. The property of chordal graphs we will exploit is the following: a graph is chordal if and only if it has a perfect elimination ordering. Recall that a vertex is *simplicial* if and only if the vertices adjacent to it form a clique. We say an ordering $v_1, v_2, \ldots, v_n$ of the vertices of a graph is a *perfect elimination ordering* if and only if v_i is simplicial in $G_i = G - \{v_1, v_2, \ldots, v_{i-1}\}$, where $G_1 = G$. This characterization is due to Rose [17] and algorithmic aspects of this characterization are well studied [18]. This characterization raises the following question: can we define an elimination ordering of a digraph which corresponds to a perfect elimination ordering of the competition graph of the digraph, provided one exists?

Let $D = (V, A)$ be a digraph. We say $x \in V$ is *di-simplicial* in the digraph D if and only if whenever there are vertices $y, z, u, v \in V$ such that $(x, u), (y, u), (x, v), (z, v) \in A$, there exists a vertex $w \in V$ such that (y, w) and (z, w) are in A. The idea of a di-simplicial vertex in a digraph was first introduced by Hefner, et al. [8].

We say $v_1, v_2, \ldots, v_n$ is a di-simplicial elimination ordering if and only if v_i is di-simplicial in $D_i = (V, A_i)$ where $A_i = A$ minus all outarcs of $v_1, \ldots, v_{i-1}$. We then have the following characterization of digraphs whose competition graph is chordal.

THEOREM 2.1. *Let D be a digraph. Then $C(D)$ is chordal if and only if D has a di-simplicial elimination ordering.*

Proof. ($\Rightarrow$) Assume $C(D)$ is chordal. Then $C(D)$ has a perfect elimination ordering $v_1, v_2, \ldots, v_n$. We claim that $v_1, v_2, \ldots, v_n$ is a di-simplicial elimination ordering for D. Consider v_i which is simplicial in $G_i = G - \{v_1, \ldots, v_{i-1}\}$. Since v_i is simplicial in G_i, the closed neighborhood of v_i in G_i is a clique in G_i. We claim v_i is di-simplicial in D_i. Suppose there exist $y, z, u, v \in V$ such that

$$(y, u), (v_i, u), (z, v), (v_i, v) \in A_i.$$

Observe $y, z \notin \{v_1, \ldots, v_{i-1}\}$ since these vertices have no outgoing arcs in D_i. Thus y and z are vertices in G_i and since y and z compete with v_i at u and v respectively, (y, v_i) and (z, v_i) are edges in G_i, so y and z are adjacent in G_i since v_i is simplicial in G_i. Thus y and z have a common prey, i.e., there exists $w \in V$ such that (y, w) and (z, w) are arcs in A_i. Therefore v_i is di-simplicial in D_i.

($\Leftarrow$) Assume D has di-simplicial ordering $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering for $C(D)$. Consider v_i which is di-simplicial in D_i. We must show v_i is simplicial in G_i. Suppose there exist $y, z \in G_i$ such that $y, z \in N(v_i)$. Since $G_i \subseteq C(D)$, y and z have common prey with v_i call them u and v respectively. Thus

$$(y, u), (v_i, u), (z, v), (v_i, v) \in A.$$

Since $y, z \in G_i, y, z \notin \{v_1, \ldots, v_{i-1}\}$ so (y, u) and (z, v) are arcs in A_i and since v_i is di-simplicial in D_i we conclude there exists $w \in V$ such that $(y, w), (z, w) \in A_i$. Thus y and z are adjacent in $C(D)$ and G_i. Therefore v_i is simplicial in G_i, completing the proof. □

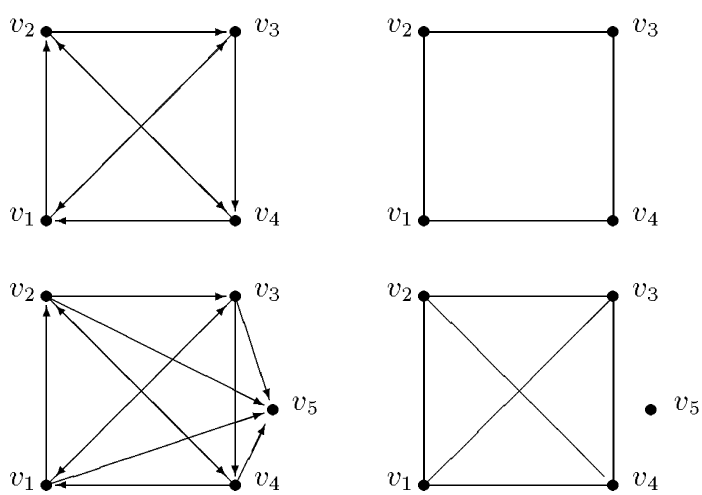

FIG. 1. *The upper digraph D' is a generated subdigraph of the lower digraph D. Their respective competition graphs are given to the right. Observe $C(D')$ is not chordal/interval, while $C(D)$ is chordal/interval. This example illustrates the difficulty in using forbidden subgraph characterizations to find digraphs with chordal/interval competition graphs.*

The lower digraph in Figure 1 illustrates a di-simplicial elimination ordering. Note the di-simplicial elimination ordering v_1, v_2, v_3, v_4, v_5 of the digraph corresponds to a perfect elimination ordering of the competition graph. This characterization is particularly nice because it avoids the forbidden subgraph problem of previous approaches: it is easy to construct a digraph whose competition graph is chordal, containing a generated subgraph whose competition graph is not chordal, as in Figure 1. It is well known that a perfect elimination ordering can be found in linear time [18]. These algorithms can be slightly modified to produced an ordering of the vertices in a digraph which is a di-simplicial elimination ordering if and only if the competition graph is chordal. In the final section of this paper, we discuss possible implications of this characterization to the food web problem.

3. Digraphs with Interval Competition Graphs. A graph is interval if and only if it is the intersection graph of intervals on the real line. Thus given an interval graph, we can assign an interval on the real line to each vertex such that two vertices are adjacent if and only if their corresponding intervals intersect. For a comprehensive introduction to interval graphs see Golumbic [7].

Jamison and Laskar [9] provide an ordering analogous to the perfect elimination ordering of chordal graphs which characterizes interval graphs. An ordering $v_1, v_2, \ldots, v_n$ is an *interval elimination ordering* of a graph $G = (V, E)$ if and only if for all v_i, v_j, v_k such that $i < j < k$,

$$(\mathcal{I}) \qquad (v_i, v_k) \in E \Rightarrow (v_j, v_k) \in E.$$

A graph G has an interval elimination ordering if and only if G is interval. This suggests the following question: can we define an elimination ordering of a digraph which corresponds to an interval elimination ordering in the competition graph, provided one exists?

We say an ordering $v_1, v_2, \ldots, v_n$ is a di-interval elimination ordering of a digraph D if and only if for all v_i, v_j, v_k such that $i < j < k$,

$$(\mathcal{A}) \qquad \exists\ u \in V \ni (v_i, u), (v_k, u) \in A \Rightarrow \exists\ w \in V \ni (v_k, w), (v_j, w) \in A.$$

THEOREM 3.1. *Let D be a digraph. Then $C(D)$ is interval if and only if D has a di-interval elimination ordering.*

Proof. ($\Rightarrow$) Assume $C(D)$ is interval. Then $C(D)$ has an interval elimination ordering $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is a di-interval elimination ordering for D. Consider arbitrary v_i, v_j, v_k such that $i < j < k$ which have property $\mathcal{I}$ in $C(D)$. We must show v_i, v_j, v_k have property $\mathcal{A}$ in D. Suppose there exists u such that $(v_i, u), (v_k, u) \in A$. Then (v_i, v_k) is an edge in $C(D)$. By property $\mathcal{I}$, we conclude (v_j, v_k) is an edge in $C(D)$. Thus v_j and v_k have a common prey in D, i.e., there exists w such that (v_k, w) and (v_j, w) are arcs in D, proving the desired result.

($\Leftarrow$) Assume D has a di-interval elimination ordering $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is an interval elimination ordering for $C(D)$. Consider arbitrary v_i, v_j, v_k such that $i < j < k$. These vertices have property $\mathcal{A}$ in D. We must show they have property $\mathcal{I}$ in $C(D)$. Suppose (v_i, v_k) is an edge in $C(D)$. Then v_i and v_k have common prey in D, i.e., there exists u such that (v_i, u) and (v_k, u) are arcs in D.

Then property $\mathcal{A}$ implies there exists w such that (v_k, w) and (v_j, w) are arcs in D, i.e., v_k and v_j are adjacent in the competition graph, completing the proof. □

The lower digraph in Figure 1 illustrates a di-interval elimination ordering. Observe the di-interval elimination ordering v_1, v_2, v_3, v_4, v_5 corresponds to an interval elimination ordering of the competition graph given to the right. Again, this characterization also avoids the forbidden subgraph problem of previous characterizations, illustrated in Figure 1. Whether or not interval elimination orderings can be used to characterize interval graphs efficiently is undetermined. Whether or not di-interval elimination orderings can be used to efficiently characterize digraphs with interval competition graphs is also undetermined. In the final section of this paper we suggest one possible approach to solving this problem and possible implications of this characterization to the food web problem.

4. Squares and Two-Step Graphs. We now make analogous definitions for the underlying graph of a loopless symmetric digraph. In this paper, the closed neighborhood of a vertex x, denoted $N[x]$, refers to the set containing x and all vertices to which x is adjacent. The open neighborhood of a vertex x, denoted $N(x)$, refers to the set containing all vertices to which x is adjacent. Given a graph G, the two-step graph of G, $S_2(G)$, is a graph defined on the same vertex set such that two vertices x and y are adjacent in $S_2(G)$ if and only if there exists a vertex z such that x and y are in $N(z)$. As stated earlier, the competition graph of a loopless symmetric digraph is identical to the two-step graph of the underlying graph. An ordering $v_1, v_2, \ldots, v_n$ is a *chordal two-step elimination ordering* if and only if for all $y, z \in \{v_i, v_{i+1}, \ldots, v_n\}$ such that there exist u and v such that $y, v_i \in N(u)$ and $z, v_i \in N(v)$, there exists w such that $y, z \in N(w)$.

THEOREM 4.1. *A graph G has a chordal two-step graph if and only if G has a chordal two-step elimination ordering.*

Proof. ($\Rightarrow$) Assume $S_2(G)$ is chordal. Then $S_2(G)$ has a perfect elimination ordering $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is a chordal two-step elimination ordering for G. Suppose there exist y and z in $\{v_i, v_{i+1}, \ldots, v_n\}$ such that there exists u and v such that $y, v_i \in N(u)$ and $z, v_i \in N(v)$ in G for fixed i. Then $y, z \in N(v_i)$ in $S_2(G)$. Since v_i is simplicial in $S_2(G)_i$ and $y, z \in \{v_i, v_{i+1}, \ldots, v_n\}$, y and z are adjacent in $S_2(G)$. Thus there exists w such that $y, z \in N(w)$ in G, completing the proof of the claim.

($\Leftarrow$) Assume G has a chordal two-step elimination ordering $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering for $S_2(G)$. Consider arbitrary v_i. We must show v_i is simplicial in $S_2(G)_i$. Suppose y and z are in $N(v_i)$ in $S_2(G)_i$. Then $y, z \in \{v_i, v_{i+1}, \ldots, v_n\}$. So there exist u and v such that $y, v_i \in N(u)$ and $z, v_i \in N(v)$ in G. Since $v_1, v_2, \ldots, v_n$ is a chordal two-step elimination ordering we conclude there exists w such that $y, z \in N(w)$ in G. Thus y and z are adjacent in $S_2(G)$, i.e., v_i is simplicial in $S_2(G)_i$. So $v_1, v_2, \ldots, v_n$ is a perfect elimination ordering for $S_2(G)$. □

Given a graph G, the square of G, denoted G^2, is a graph defined on the same vertex set with an edge between two vertices x and y if and only if there exists a vertex

z such that x and y are in $N[z]$. Note this leaves the possibility that x and y are adjacent. An ordering $v_1, v_2, \ldots, v_n$ is a *chordal square elimination ordering* if and only if for all $y, z \in \{v_i, v_{i+1}, \ldots, v_n\}$ such that there exist u and v such that $y, v_i \in N[u]$ and $z, v_i \in N[v]$, there exists w such that $y, z \in N[w]$. By an argument similar to the proof of Theorem 4.1 using closed neighborhoods instead of open neighborhoods, we obtain the following result.

THEOREM 4.2. *A graph G has a chordal square if and only if G has a chordal square elimination ordering.*

Observe that a graph may have a chordal square while having a nonchordal two-step graph. The graph in Figure 2 is such an example. Observe that $v_1, v_2, v_3, v_4, v_5, v_6$ is a chordal square elimination ordering, but since the two-step graph is not chordal, this graph has no chordal two-step elimination ordering. Further observe that the corresponding symmetric digraph has no di-simplicial vertices.

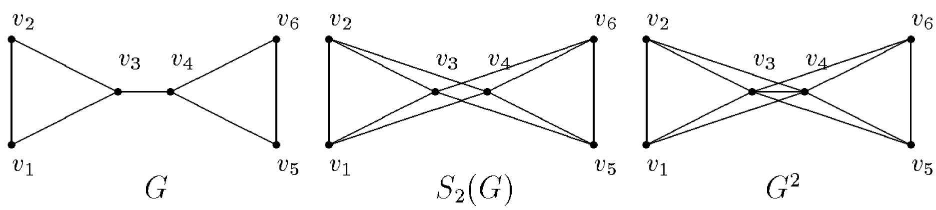

FIG. 2.

A graph with a chordal square and a nonchordal two-step graph.

An ordering $v_1, v_2, \ldots, v_n$ is an *interval two-step elimination ordering* if and only if for all i, j, k such that $i < j < k$, if there exists u such that $v_i, v_k \in N(u)$ then there exists v such that $v_j, v_k \in N(v)$.

THEOREM 4.3. *A graph G has an interval two-step graph if and only if G has an interval two-step elimination ordering.*

Proof. ($\Rightarrow$) Assume $S_2(G)$ is interval. Then $S_2(G)$ has an interval elimination ordering $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is an interval two-step elimination ordering for G. Consider arbitrary i, j, k such that $i < j < k$ and suppose there exists u such that $v_i, v_k \in N(u)$ in G. Then v_i and v_k are adjacent in $S_2(G)$. Since $v_1, v_2, \ldots, v_n$ is an interval elimination ordering we conclude v_j and v_k are adjacent in $S_2(G)$. Thus there exists v such that $v_j, v_k \in N(v)$ in G. So $v_1, v_2, \ldots, v_n$ is an interval two-step elimination ordering for G.

($\Leftarrow$) Assume G has an interval two-step elimination ordering, $v_1, v_2, \ldots, v_n$. We claim $v_1, v_2, \ldots, v_n$ is an interval elimination ordering for $S_2(G)$. Pick arbitrary i, j, k such that $i < j < k$ and suppose v_i and v_k are adjacent in $S_2(G)$. Then there exists u such that $v_i, v_k \in N(u)$ in G. Since $v_1, v_2, \ldots, v_n$ is an interval two-step elimination ordering there exists v such that $v_j, v_k \in N(v)$ in G. Thus v_j and v_k are adjacent in $S_2(G)$. Thus $v_1, v_2, \ldots, v_n$ is an interval elimination ordering for $S_2(G)$, completing the proof. □

An ordering $v_1, v_2, \ldots, v_n$ is an *interval square elimination ordering* if and only if for all i, j, k such that $i < j < k$, if there exists u such that $v_i, v_k \in N[u]$ then there exists v such that $v_j, v_k \in N[v]$. By an argument similar to the proof of Theorem 4.3 using closed neighborhoods instead of open neighborhoods we obtain the following result.

THEOREM 4.4. *A graph G has an interval square if and only if G has an interval square elimination ordering.*

5. Conclusions and Open Problems. The following are related open problems and implications of the results presented in this paper.

> How may interval elimination orderings of graphs and di-interval elimination orderings of digraphs be determined?

The problem of finding an interval elimination ordering directly (if one exists) in an arbitrary graph remains open. There are well-know algorithms that can perform interval testing in linear time. Another useful property of interval graphs is that the maximal cliques can be consecutively ranked, that is, if $C_1, C_2, \ldots, C_r$ are the consecutively ranked maximal cliques of an interval graph, and vertex x is in C_i and C_k, $i < k$, then $x \in C_j$ for all $i \leq j \leq k$. The standard algorithm for interval testing produces the consecutively ranked maximal cliques [1]. Given this information we can produce an interval elimination ordering as follows.

ALGORITHM 5.1. [Given a family of sets $\{C_1, C_2, \ldots, C_r\}$ corresponding to the consecutively ranked maximal cliques in an interval graph G, we find an interval elimination ordering for G.]

1. Consider the r_1 vertices in $C_1 - C_2$. Arbitrarily order them $v_1, v_2, \ldots, v_{r_1}$.
2. Consider the r_2 vertices in $C_2 - C_3$ that have not yet been ordered. Arbitrarily order them $v_{r_1+1}, \ldots, v_{r_1+r_2}$.
3. Similarly order the remaining vertices for $C_3 - C_4$, $\ldots$, $C_{r-1} - C_r$.
4. Arbitrarily order the vertices in $C_{r-1} \cap C_r$.
5. Complete the ordering by arbitrarily ordering the remaining vertices in C_r resulting in $v_1, \ldots, v_n$.

THEOREM 5.1. *The ordering produced in Algorithm 5.1 is an interval elimination ordering for the graph G.*

Proof. Consider arbitrary i, j, k such that $i < j < k$. We claim if v_i and v_k are adjacent then v_j and v_k are adjacent. So suppose v_i and v_k are adjacent. If $v_i, v_k \in C_r$ we are done since v_i, v_j and v_k were chosen in one of the last two steps of the algorithm and hence v_j is also an element of C_r. So assume this is not the case. Then there exists $b < r$ such that $v_i \in C_b - C_{b+1}$ and $d \geq b$ such that $v_k \in C_d - C_{d+1}$ (if $d < r$) or $v_k \in C_d$ (if $d = r$). If $v_j \in C_r$, then $v_k \in C_r$, i.e., v_j and v_k are adjacent. So assume $v_j \in C_f - C_{f+1}$ for $b \leq f \leq d$, where $f < r$. Observe v_k must be contained in C_b since v_i and v_k are adjacent, $v_i \notin C_{b+1}$ and the ranking is consecutive. This consecutive ranking then implies $v_k \in C_f$ since $b \leq f \leq d$. Therefore v_j and v_k are adjacent, completing the proof that the ordering found in Algorithm 5.1 is an interval elimination ordering for G. □

Given an interval elimination ordering for a graph we can also produce the consecutively ranked maximal cliques. Briefly, this is done by consecutive examination of the neighborhoods of the vertices (consecutive with respect to the order of elimination). For example, if k_1 is the largest integer for which v_1 and v_{k_1} are adjacent, then

$$\{v_j | v_1 \text{ and } v_j \text{ adjacent }, 1 \leq j \leq k_1\}$$

is a maximal clique. Furthermore, if k_2 is the largest integer for which v_2 and v_{k_2} are adjacent, then

$$\{v_j | v_2 \text{ and } v_j \text{ adjacent }, 2 \leq j \leq k_2\}$$

is a maximal clique if and only if either v_2 and v_1 are adjacent or $k_2 > k_1$. A similar approach can be used to find the remaining maximal cliques in a consecutive ranking.

Thus given an interval graph, an interval elimination ordering is easy to find, provided you have either the maximal cliques of the graph or the intervals for which the graph is the intersection graph. These techniques do not use interval elimination orderings to determine if a graph is interval. Furthermore, the techniques used to find an interval elimination ordering do not translate to finding a di-interval elimination ordering of a digraph. Short of trying all possibilities, is there a method for determining whether or not a graph has an interval elimination ordering or a digraph has a di-interval elimination ordering. One approach might be to consider the di-interval elimination ordering for special classes of digraphs such as transitive or acyclic.

> Do di-simplicial and di-interval elimination orderings have special applications to the structure of food webs?

The di-simplicial elimination ordering fits in very well with Sugihara's assembly rule for food webs: predators cannot join a community by overlapping, in their diets, with two or more predators that do not overlap in *their* diets. Consider the forced incidence diagram for a di-simplicial elimination ordering in Figure 3. Note that since v_k is earlier in the ordering than v_{n-2} and v_{n-3}, v_k and v_{n-3} have a common prey, and since v_k and v_{n-2} have a common prey, if the ordering shown is a di-simplicial elimination ordering, it follows that v_{n-3} and v_{n-2} must also have a common prey. Is there an analogous tie between di-interval elimination ordering and food web theory. For example, is it possible to linearly order the members of a food web on some scale of adaptability? If so, it makes sense that the competition graphs of actual food web models are interval. Consider the forced incidence diagram in Figure 4 for a di-interval elimination ordering. Such a linear ordering has the following interpretation in the this diagram: since v_k comes after v_i in the di-interval elimination ordering and v_k is "adaptable enough" to share prey with v_i, v_k should also be "adaptable enough" to share prey with all vertices between v_k and v_i. If such a scale for linear ordering exists, the di-interval elimination ordering may be a useful tool in analyzing the correctness of data in a food web.

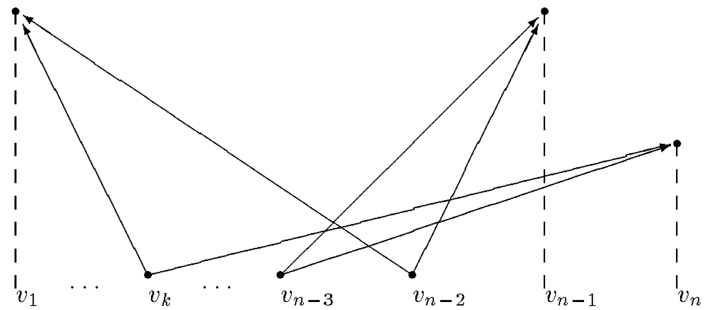

FIG. 3. *Assume $v_1, v_2, \ldots, v_n$ is a di-simplicial elimination ordering for a digraph D. The presence of arcs (v_k, v_n), (v_{n-3}, v_n), (v_k, v_1), and (v_{n-2}, v_1) forces v_{n-2} and v_{n-3} to have common prey.*

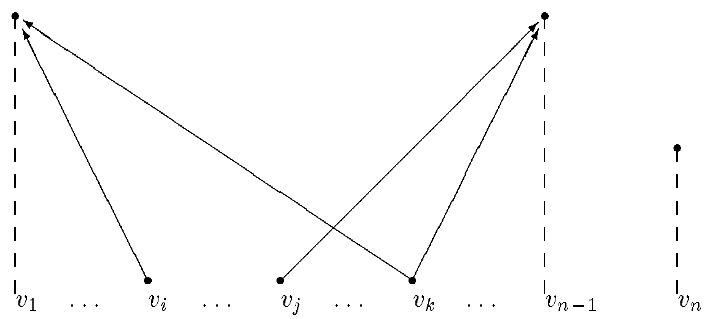

FIG. 4. *Assume $v_1, v_2, \ldots, v_n$ is a di-interval elimination ordering for a digraph D. The presence of arcs (v_k, v_1) and (v_i, v_1) forces v_k to have common prey with vertices $v_{i+1}, \ldots, v_{k-1}$.*

> As mentioned earlier, Cohen's cascade model does not account for all interval food web models. Do either the di-simplicial or di-interval elimination orderings help in explaining this difference?

Cohen and Palka [6] have developed a probabilistic model for food webs, the cascade model, which suggests that the probability the competition graph of a food web is interval approaches 0 as the number of species in the web increases. One important assumption of this model is that competition between two species is independent of competition between other pairs of species. The number of food webs constructed from empirical data having interval competition graphs is still higher than predicted by the cascade model. Do di-interval elimination orderings imply anything about the assumptions of the cascade model?

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