AN OPTIMAL LAGRANGE MULTIPLIER BASED DOMAIN DECOMPOSITION METHOD FOR PLATE BENDING PROBLEMS

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Abstract. We present a new Lagrange multiplier based domain decomposition method for solving iteratively systems of equations arising from the finite element discretization of plate bending problems. The proposed method is essentially an extension of the FETI substructuring algorithm to the biharmonic equation. The main idea is to enforce the continuity of the transversal displacement field at the subdomain crosspoints throughout the preconditioned conjugate gradient iterations. The resulting method is proved to have a condition number that does not grow with the number of subdomains, and grows at most polylogarithmically with the number of elements per subdomain. These optimal properties hold for numerous plate bending elements that are used in practice including the HCT, DKT, and a class of non-locking elements for the Reissner-Mindlin plate models. Computational experiments are reported and shown to confirm the theoretical optimal convergence properties of the new domain decomposition method. Computational efficiency is also demonstrated with the numerical solution in 45 iterations and 105 seconds on a 64-processor IBM SP2 of a plate bending problem with almost one million degrees of freedom.

Key words. Domain decomposition, biharmonic equation, plates and shells, parallel computing.

AMS(MOS) subject classifications. 65N55, 65F10

1. Introduction.

The FETI (Finite Element Tearing and Interconnecting) method is a domain decomposition algorithm derived from a hybrid variational principle and designed for the iterative solution of systems of equations arising from the finite element discretization of self-adjoint elliptic partial differential equations. In this method, a given spatial domain is "torn" into *non-overlapping* subdomains where an incomplete solution of the primary field is first evaluated using a direct solver. Next, intersubdomain field continuity is enforced via Lagrange multipliers applied at the subdomain interfaces. This "gluing" phase generates a smaller size symmetric *dual* problem where the unknowns are the Lagrange multipliers, and which is best solved with a preconditioned conjugate gradient (PCG) algorithm. The FETI method was developed in [7, 10, 11], and discussed in detail in the monograph [12]. In contrast with other related domain decomposition methods using Lagrange multipliers as unknowns [14, 23], the FETI method distinguishes itself with the treatment of the null spaces of the subdomain stiffness matrices (rigid body modes) associated with the so-called floating subdomains, i.e., subdomains without a sufficient number of essential boundary conditions to prevent local stiffness singularities. Resolving the rigid

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body modes leads to a small "coarse" problem that is solved in each PCG iteration. It was recognized in [9] and proved mathematically in [20] that solving this coarse problem accomplishes a global exchange of information between the subdomains and results in a method which, for elasticity problems, has a condition number that grows only polylogarithmically with the number of elements per subdomain, and is bounded independently of the number of subdomains. Extension to time-dependent problems was done in [8]. However for plate bending problems, the condition number was observed to grow fast with the number of elements per subdomain [9]. This is caused by the fact that plate bending is a fourth order problem, while the FETI domain decomposition method "tears" the approximate solution at subdomain crosspoints, which is suitable only for second order problems.

In this paper, we cure this limitation and extend the FETI methodology to obtain an asymptotically optimal non-overlapping domain decomposition method for plate bending problems. The key idea is to enforce the continuity of the approximate solution at the subdomain crosspoints throughout the iterations. This is done by adding to the FETI coarse problem new Lagrange multipliers. A related idea was explored in the Balancing Domain Decomposition method for plates [17], where approximate continuity at crosspoints is enforced adding new basis functions to the original coarse space [18, 19] in order to keep the energy of the approximate solution minimal with respect to displacements that are solutions for point loads at the subdomain crosspoints. The distinguishing features of both the present method and the method from [17] is that they are *non-overlapping* and work for standard finite elements used in everyday engineering practice. For other domain decomposition methods for the biharmonic equation and plate bending see, for example, [3, 24].

The remainder of this paper is organized as follows. We first formulate the domain decomposed problem as a saddle point problem in Section 2. For reference, the original FETI method is then reviewed in Section 3. In Section 4, we present the new algorithm in an abstract form. The selection of the components of the new FETI method and the required properties of the plate bending elements are specified in Section 5. In Section 6, we prove that the condition number of the new FETI method for plate bending problems is bounded polylogarithmically in H/h, where H and h denote respectively the characteristic subdomain and element sizes. The paper is concluded in Section 7 with the discussion of computational results that confirm the optimal convergence properties of the proposed Lagrange multiplier based domain decomposition method and demonstrate its efficiency and parallel performance.

Extensions to shells, implementation issues, and further computational results will be published elsewhere.

2. Domain Decomposition by Lagrange Multipliers. For completeness and in order to introduce the notation, we first derive the decomposed problem for the intersubdomain Lagrange multipliers following the approach presented in [9]. Lemma 2.3 is new, but cf. [12] for related considerations.

Let Ω be a domain in $\mathbb{R}^2$ decomposed into N_s non-overlapping subdomains $\Omega_1, \Omega_2, \ldots, \Omega_{N_s}$. Let u_s be the vector of degrees of freedom (d.o.f.) for subdomain Ω_s

corresponding to a conforming Finite Element discretization of a plate bending problem defined on Ω, such that each subdomain is the union of some of the elements. Denote $u = (u_s)$.

The problem to be solved is the minimization of the strain energy of the system subject to intersubdomain continuity conditions

$$\mathcal{E}(u) = \frac{1}{2}\sum_s u_s^T K_s u_s - f^T u \to \min \tag{1}$$

$$\text{subject to } \sum_s B_s u_s = 0. \tag{2}$$

Here, K_s is the subdomain stiffness matrix, f is the force vector, and B_s are matrices with entries $+1, -1, 0$ such that (2) expresses the displacement continuity condition for the degrees of freedom attached to a node that is common to two or more subdomains. The matrices B_s map the spaces of subdomain degrees of freedom V_s to the space U of vectors with one component per constraint:

$$B_s : V_s \to U. \tag{3}$$

The stiffness matrix K_s is singular for a *floating subdomain*, that is, one without a sufficient number of essential boundary conditions to prevent the subdomain stiffness matrix K_s from being singular. Let Z_s be matrices with linearly independent columns that generate the kernels of K_s:

$$\text{Range } Z_s = \text{Ker } K_s. \tag{4}$$

If K_s is regular, Z_s is a void matrix.

We convert the problem (1), (2) into a dual problem. Let $\lambda \in U$ be the Lagrange multipliers associated with the continuity constraint (2). The Lagrangian of the problem is

$$\mathcal{L}(u,\lambda) = \frac{1}{2}\sum_s u_s^T K_s u_s - f^T u + \lambda^T \sum_s B_s u_s \tag{5}$$

and equations (1), (2) are equivalent to the saddle point problem

$$\inf_u \sup_{\lambda\in U} \mathcal{L}(u,\lambda) = \sup_{\lambda\in U} \inf_u \mathcal{L}(u,\lambda)$$

It is easy to see that

$$\sup_{\lambda\in U} \mathcal{L}(u,\lambda) = \begin{cases} \mathcal{E}(u) \text{ if } \sum_s B_s u_s = 0 \\ +\infty \text{ otherwise} \end{cases}$$

The dual problem is

$$\mathcal{C}(\lambda) = \inf_u \mathcal{L}(u,\lambda) \to \max, \quad \lambda \in U. \tag{6}$$

The functional $\mathcal{C}(\lambda)$ can be interpreted as the *complementary intersubdomain energy.* Minimizing the Lagrangian over u_s, we get

$$0 = \frac{\partial}{\partial u_s}\mathcal{L}(u,\lambda) = K_s u_s - f_s + B_s^T \lambda$$

Hence,

$$\mathcal{C}(\lambda) = \mathcal{L}(u,\lambda), \qquad u_s = K_s^+(f_s - B_s^T\lambda) \tag{7}$$

if all subdomain problems $\frac{\partial}{\partial u_s}\mathcal{L}(u,\lambda) = 0$ are consistent, that is, if $f_s - B_s^T\lambda \perp \text{Ker } K_s$, and $\mathcal{C}(\lambda) = -\infty$ otherwise. Physically, the consistency condition means that all forces on the subdomain including those from the multipliers λ are in balance if the subdomain is floating. In (7) and in the rest of the paper, the superscript $^+$ denotes pseudoinverse, defined as follows.

DEFINITION 2.1. *Let A be a linear operator. A pseudoinverse A^+ is any linear operator such that if $a \in$ Range A then $AA^+a = a$.*

REMARK 2.2. *The pseudoinverse is not in general unique. However, our algorithms will be invariant to a specific choice of the pseudoinverse. If A is symmetric operator on a finite dimensional space, A^+ can be chosen to be also symmetric from the spectral decomposition*

$$A^+ = \sum_{t>0} \frac{1}{t} v_t v_t^T, \qquad A = \sum_t t v_t v_t^T, \quad A v_t = t v_t i, \quad \|v_t\| = 1. \tag{8}$$

Here and in the rest of the paper, denote

$$\begin{aligned}
G &= [B_s Z_s] = [B_1 Z_1, \ldots, B_{N_s} Z_{N_s}] \\
F &= \textstyle\sum_s B_s K_s^+ B_s^T \\
d &= \textstyle\sum_s B_s K_s^+ f_s \\
P &= I - G(G^T G)^{-1} G^T \\
e &= [Z_s^T f_s]^T = [Z_1^T f_1, \ldots, Z_{N_s}^T f_{N_s}]^T
\end{aligned}$$

It can be proved that if the global structure is not a floating one, then G has full column rank [12, Theorem 5.4], hence $(G^T G)^{-1}$ exists. This will be assumed from now on.

It is easy to see that F is symmetric and positive semi-definite on U

$$u^T F v = v^T F u, \quad \forall u, v \in U, \qquad u^T F u \geq 0, \quad \forall u \in U, u \neq 0. \tag{9}$$

Substituting u_s from (7) into $\mathcal{L}(u,\lambda)$, the dual problem (6) becomes

$$-\frac{1}{2}\lambda^T F \lambda + \lambda^T d \to \max \text{ subject to } G^T\lambda = e, \quad \lambda \in U. \tag{10}$$

The constrained quadratic problem (10) can be written as finding the stationary point of the gradient of $-\frac{1}{2}\lambda^T F\lambda + \lambda^T d$ projected on the set $\{\lambda \in U | G^T\lambda = e\}$. This is equivalent to the linear system

$$\begin{aligned} P(F\lambda - d) &= 0 \\ G^T\lambda &= e \end{aligned} \tag{11}$$

The solution λ of (10) is unique up to addition of a vector from Ker $PF \cap$ Ker G^T, which is characterized as follows.

LEMMA 2.3. *It holds that* Ker $PF \cap$ Ker $G^T =$ Ker $F \cap$ Ker $G^T = \bigcap_s$ Ker B_s^T.

Proof. First, $B_s^T\lambda = 0$ for all s implies $F\lambda = 0$ and $G^T = 0$. On the other hand, let $PF\lambda = 0$ and $G^T\lambda = 0$. From $G^T\lambda = 0$, it follows that $P\lambda = \lambda$, so $PF\lambda = 0$ implies $\lambda^T PFP\lambda = 0$, which gives $F\lambda = 0$. Consequently, $B_s^T\lambda \in$ Ker K_s, while $G^T\lambda = 0$ implies $B_s^T\lambda \perp$ Ker K_s. Hence, $B_s^T\lambda = 0$. □

Nonuniqueness of the multipliers λ corresponds to redundant intersubdomain continuity constraints, which occur naturally at crosspoints of more than two subdomains.

Finally, denote the space of feasible increments

$$V = \{\lambda \in U | G^T\lambda = 0\} = \text{Range } P. \tag{12}$$

3. The Original FETI Algorithm. In this section, we formulate a variant the original FETI algorithm that is suitable for the purposes of this paper.

The FETI method is the method of preconditioned conjugate gradients on the subspace V, with the system operator $PF : V \to V$.

We shall use a projected subdomain by subdomain preconditioner

$$PD : V \to V, \qquad D = \sum_s B_s D_s B_s^T, \tag{13}$$

where D_s are symmetric positive semidefinite and specified later. Then, from Lemma 2.3, Ker $PF \cap V \subset$ Ker $D \cap V$, so all steps of the preconditioned conjugate gradients method are invariant to error components in Ker $F \cap V$; in other words, the conjugate gradients run in the factorspace $V/(\text{Ker } F \cap V)$.

ALGORITHM 3.1 (FETI). *For a given initial* λ_0 *satisfying*

$$G^T\lambda_0 = e$$

compute the initial residual

$$r_0 = P(F\lambda_0 - d)$$

and repeat for $k = 1, 2, \ldots$ *until convergence:*

$$\begin{aligned}
z_{k-1} &= D_{k-1} r_{k-1} \\
y_{k-1} &= P z_{k-1} \\
\xi_k &= r_{k-1}^T y_{k-1} \\
p_k &= y_{k-1} + \frac{\xi_k}{\xi_{k-1}} p_{k-1} \qquad (p_1 = y_0) \\
\nu_k &= \frac{\xi_k}{p_k^T PF p_k} \\
\lambda_k &= \lambda_{k-1} + \nu_k p_k \\
r_k &= r_{k-1} - \nu_k PF p_k
\end{aligned}$$

By induction, we get $r_k, p_k, y_k \in V$, for all k, and therefore we have

$$G^T \lambda_k = e, \qquad \forall k \tag{14}$$

REMARK 3.2. *With our definition of the residual* $r_k = P(F\lambda_k - d)$, *we have* $r_k \to 0$ *as the solution is approached. This alleviates numerical instabilities that forced reorthogonalization in previous implementations of the FETI method [23, 12]. Algorithm 3.1 is mathematically equivalent to the previous implementation [9] in the absence of rounding errors.*

4. The Generalized FETI Algorithm. Preserving the condition (14) throughout the iterations can be interpreted as enforcing that every λ_k be optimal with respect to all possible increments of the form $G\alpha$:

$$G^T \lambda_k = e \iff \mathcal{C}(\lambda_k) \geq \mathcal{C}(\lambda_k - G\alpha), \quad \forall \alpha \tag{15}$$

The key to the generalization of the FETI method to plate bending problems is to make all λ_k optimal in more directions. Let C be a given matrix with the same number of rows as G. Each column of C will give rise to an additional variable in the coarse problem. We shall satisfy in each iteration the *coarse optimality property*

$$\mathcal{C}(\lambda) \geq \mathcal{C}(\lambda - G\alpha - C\beta), \qquad \forall \alpha, \beta. \tag{16}$$

with $\lambda = \lambda_k$. To satisfy this property, consider an auxiliary problem: For a given $\tilde{\lambda}$ find α and β so that

$$\mathcal{C}(\lambda) \to \max, \qquad \lambda = \tilde{\lambda} - G\alpha - C\beta \tag{17}$$

Since we only need solutions satisfying $\mathcal{C}(\lambda) > -\infty$, we consider the maximization problem (17) along with the constraint

$$G^T \lambda = e \tag{18}$$

Introducing new Lagrange multipliers μ for (18), we get that α, β, and μ are solution of the saddle point problem

$$\inf_{\alpha,\beta} \sup_{\mu} \tilde{\mathcal{L}}(\tilde{\lambda} - G\alpha - C\beta, \mu) = \sup_{\mu} \inf_{\alpha,\beta} \tilde{\mathcal{L}}(\tilde{\lambda} - G\alpha - C\beta, \mu)$$

where

$$\tilde{\mathcal{L}}(\lambda, \mu) = -\frac{1}{2}\lambda^T F \lambda + \lambda^T d + \mu^T (G^T \lambda - e)$$

From the optimality conditions

$$\frac{\partial \tilde{\mathcal{L}}(\lambda, \mu)}{\partial \alpha} = 0, \qquad \frac{\partial \tilde{\mathcal{L}}(\lambda, \mu)}{\partial \beta} = 0, \qquad \frac{\partial \tilde{\mathcal{L}}(\lambda, \mu)}{\partial \mu} = 0, \qquad \lambda = \tilde{\lambda} - G\alpha - C\beta,$$

we obtain that (17) is equivalent to the block linear system

$$M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \tilde{\lambda} - \begin{bmatrix} G^T d \\ C^T d \\ e \end{bmatrix} \tag{19}$$

where

$$M = \begin{bmatrix} G^T F G & G^T F C & G^T G \\ C^T F G & C^T F C & C^T G \\ G^T G & G^T C & 0 \end{bmatrix} = \begin{bmatrix} G^T & 0 \\ C^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} G & C & 0 \\ 0 & 0 & I \end{bmatrix}. \tag{20}$$

The solution λ of (17) is unique up to the addition of a vector in Ker $F \cap$ Ker G^T, and we write it as

$$\lambda = \tilde{\lambda} - \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \left(\begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \tilde{\lambda} - \begin{bmatrix} G^T d \\ C^T d \\ e \end{bmatrix} \right). \tag{21}$$

Note that since (17) has a solution for any $\tilde{\lambda}$, so does (19), hence

$$\text{Range} \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \subset \text{Range } M. \tag{22}$$

Further, if λ is coarse optimal, then $\lambda + \delta$ is also coarse optimal if and only if

$$\begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \delta \in \text{Ker } F \cap \text{Ker } G^T \tag{23}$$

We are now ready to formulate the generalized FETI method for the solution of the system (11).

ALGORITHM 4.1 (GENERALIZED FETI). *Given an initial $\tilde{\lambda}_0$, compute the initial coarse optimal λ_0 by*

$$\lambda_0 = \tilde{\lambda}_0 - \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \left(\begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \tilde{\lambda}_0 - \begin{bmatrix} G^T d \\ C^T d \\ e \end{bmatrix} \right) \tag{24}$$

and the initial residual

$$r_0 = P(F\lambda_0 - d).$$

Repeat for $k = 1, 2, \ldots$ until convergence:

$$z_{k-1} = D_{k-1} r_{k-1} \tag{25}$$

$$y_{k-1} = z_{k-1} - [G, C, 0] M^{+} \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} z_{k-1} \tag{26}$$

$$\xi_k = r_{k-1}^T y_{k-1} \tag{27}$$

$$p_k = y_{k-1} + \frac{\xi_k}{\xi_{k-1}} p_{k-1} \qquad (p_1 = y_0) \tag{28}$$

$$\nu_k = \frac{\xi_k}{p_k^T P F p_k} \tag{29}$$

$$\lambda_k = \lambda_{k-1} + \nu_k p_k \tag{30}$$

$$r_k = r_{k-1} - \nu_k P F p_k \tag{31}$$

Again, by induction, the projected preconditioned residuals satisfy the optimality condition for increments (23) with y_{k-1} in place of δ. Therefore, all iterates λ_k satisfy the coarse optimality condition (16).

REMARK 4.2. *In an earlier paper [8], we have studied the case of time-dependent problems where the subdomain stiffness matrices K_s are perturbed by the addition of a multiple of the subdomain mass matrix, thus making the new local matrix positive definite. Consequently, all matrices Z_s are void and the natural coarse problem is lost in time-dependent applications. The methodology developed in [8] for reintroducing a coarsening operator in the FETI algorithm for dynamics problems is a special case of the present generalization where C is taken to be the matrix G* before the perturbation, *that is, $C = [B_s \tilde{Z}_s]$ where the columns of $\tilde{Z}_s$ are the basis of the kernel of the local stiffness matrix of the subdomain Ω_s. The selection of C in [8] and the reason why the preconditioner works are quite different from here.*

5. Method Selection for Plate Problems. The columns of C are chosen as vectors with a one at the position of the Lagrange multiplier that enforces the continuity of the transversal displacement at a crosspoint, and zeroes elsewhere. This guarantees that the transversal component of the approximate solution $u = (u_s)$, $u_s = K_s^+(f_s - B_s^T \lambda)$ is continuous at all crosspoints for all $\lambda = \lambda_k$. A crosspoint is defined as an interface node adjacent to at least three subdomains or to two subdomains and the complement of Ω.

The motivation of the algorithm as well as the theory presented in the next section require that the plate bending finite element satisfies the following *approximate parametric variational principle* recently formulated in [17]:

ASSUMPTION 5.1 ([17, 16]). *We consider elements with displacements and rotations at the vertices only, and assume that there exist constants $c_1 > 0$, c_2 such that if the plate thickness t satisfies $0 < t \leq h$, then for each element T, the local stiffness matrix K_T satisfies*

$$c_1 K_T^{HCT} \leq K_T \leq c_2 K_T^{HCT} \tag{32}$$

where K_T^{HCT} is the HCT element level stiffness matrix of the biharmonic equation [5], with the rotations interpreted as derivatives of the transversal displacement in the HCT element.

That is, as the thickness of the plate goes to zero, the stiffness matrix of the element should be spectrally equivalent to that of the HCT element for the biharmonic equation. The HCT element is a C^1 element that uses cubic splines for values on element sides, linear interpolation for normal derivatives on the sides, and piecewise polynomial extension into the element interior [4]. Assumption 5.1 is proved in [16] for the particular case of the DKT element [1]. Assumption 5.1 also holds for the following general class of non-locking $P1$ Reissner-Mindlin elements.

THEOREM 5.2 ([16]). *Assume that the energy functional for an element T is spectrally equivalent to*

$$\int_T |\nabla\theta|^2\,dx + \frac{1}{t^2+h^2}\int_T |\theta - \nabla u|^2\,dx \tag{33}$$

with $u \in P_1(T), \theta \in (P_1(T))^2$, $h = \mathrm{diam}(T)$, u the transversal displacement, and θ the rotation. Then (32) holds.

Elements with the energy functional of the form (33) include the DKT element as restated in [22].

It should be noted that for the related Timoshenko beam element, the thin limit is exactly the discretization by cubic splines of the biharmonic equation [13].

6. Convergence Analysis. The generalized FETI method given by Algorithm 4.1 is a preconditioned conjugate gradient method in a subspace for the equation $PF\lambda = f$ with a preconditioner QD, where

$$Q = I - \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix}. \tag{34}$$

is the projection on the space of increments that satisfy the optimality condition (23).

It is well known [15] that after k iterations of the preconditioned conjugate gradient method, the energy norm or the error $|||e||| = \langle PFe, e\rangle^{1/2}$ is reduced by a factor of at least

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k,$$

where κ is the condition number, given in our case by

$$\kappa = \frac{\lambda_{\max}(QMPF)}{\lambda_{\min}(QMPF)} \tag{35}$$

with $\lambda_{\max}$ and $\lambda_{\min}$ being the maximum and minimum eigenvalues. The purpose of this section is to estimate the eigenvalues.

6.1. Additional Notations and Assumptions. Before proving a bound on the condition number κ of the generalized FETI method, we need to introduce some specific assumptions. In the the rest of the paper, we use the notation $a \approx b$ to indicate that

$ca \leq b \leq Ca$ with some positive generic constants c, C independent of the characteristic mesh size h and the subdomain diameter H.

Consider the biharmonic boundary value problem in a variational form,

$$u \in \mathcal{V}: \quad a(u,v) = f(v), \quad \forall v \in \mathcal{V},$$

where

$$a(u,v) = \int_\Omega \partial_{11} u \partial_{11} v + \partial_{12} u \partial_{12} v + \partial_{22} u \partial_{22} v, \qquad \mathcal{V} = H_0^2(\Omega).$$

The subdomains Ω_s, $s = 1, ..., N_s$, are assumed to form a regular triangulation, i.e. they are generated from a reference domain (square or cube) $\hat{\Omega}$ of unit diameter by mappings F_s, so that $\Omega_s = F_s(\hat{\Omega}_s)$. These mappings are assumed to satisfy

$$\|\partial F_s\| \leq CH, \qquad \|\partial F_s^{-1}\| \leq CH^{-1} \tag{36}$$

where ∂F_i is the mapping Jacobian and $\|.\|$ is the Euclidean $\mathbb{R}^d$ matrix norm. In other words, the subdomains are assumed to have a regular shape and a diameter $O(H)$.

Furthermore, we assume that the problem is discretized using reduced HCT elements. The general case of plate bending then follows from spectral equivalence of the local element stiffness matrices following Assumption 5.1. Let $V_h(\Omega) \subset \mathcal{V}$ denote the corresponding finite element space, and h denote the characteristic element size. Each subdomain Ω_i is assumed to be a union of some of the elements. The degrees of freedom are values of the transversal displacement and its derivatives (rotations) at the nodal points of the discretization. We find it convenient to identify vectors of degrees of freedom, which are in some spaces $\mathbb{R}^n$, with the associated finite element functions. Operators between the spaces are represented as matrices, and we frequently commit an abuse of notations by using matrices and operators interchangeably. The l^2 inner product is denoted by $\langle \cdot, \cdot \rangle$ on all spaces. The associated norm is $\|u\|^2 = \langle u, u \rangle$.

Let

$$W_s = V_h(\partial \Omega_s) \tag{37}$$

be the space of local vectors of degrees of freedom associated with the boundary of Ω_s, and let W be a space of all boundary degrees of freedom on all subdomains,

$$W = \bigotimes_s W_s \tag{38}$$

Introduce the block matrix

$$B : W \to U = (\bar{B}_1, \ldots, \bar{B}_{N_s}), \tag{39}$$

where $\bar{B}_s$ is the restriction of the operator B_s to the interface degrees of freedom, or, equivalently, the submatrix consisting of the columns of B_s that correspond to interface degrees of freedom, and U is the space of block vectors where each block corresponds to continuity constraints on one edge between subdomains (including endpoints), cf., (3).

In this notation,

$$G = BZ, \qquad Z = \text{diag}\,[Z_s]$$

and Lemma 2.3 becomes

$$\text{Ker } PF \cap \text{Ker } G^T = \text{Ker } F \cap \text{Ker } G^T = \text{Ker } B^T. \tag{40}$$

To be specific, we assume that the operator B is defined as follows. For a node x_i on an edge $\partial\Omega_r \cap \partial\Omega_s$, let

$$(Bw)_{rs}(x_i) = \sigma_{rs}(w_r(x_i) - w_s(x_i)), \tag{41}$$

$$(\nabla Bw)_{rs}(x_i) = \sigma_{rs}(\nabla w_r(x_i) - \nabla w_s(x_i), \tag{42}$$

where $\sigma_{rs} = 1$ or $\sigma_{rs} = -1$. Here, $\nabla w_r(x_i)$ means the values of ∂_1 and ∂_2 degrees of freedom at node x_i.

Note that this definition implies that there are redundant constrains at crosspoints of more than two subdomains, and therefore B does not have full column rank in general. Denote by S_s the Schur complements of the subdomain stiffness matrices obtained by elimination of the interior degrees of freedom and define the the symmetric block diagonal matrix

$$S : W \to W, \qquad S = \text{diag}\,[S_s] \tag{43}$$

Assuming the pseudoinverses S^+ and K_s^+ are chosen to be symmetric, we have

$$F = BS^+B^T = \sum_i B_i K_i^+ B_i^T, \tag{44}$$

which allows us to restrict all considerations to subdomain interfaces.

We assume preconditioner D of the form (13) with $D_s = S_s$, that is,

$$D = BSB^T.$$

The evaluation of the matrix-vector product Su can be implemented by solving a Dirichlet problem in each subdomain; therefore, it is called the *Dirichlet preconditioner* [9].

6.2. An Abstract Bound. We shall use the following abstract lemma to find spectral bounds. It is a generalization of a similar result in [20].

LEMMA 6.1. *Let $\tilde{V}$, $\tilde{V}'$ be inner product spaces, $\|\cdot\|_{\tilde{V}}$ denote the norm on $\tilde{V}$, and let $\langle\cdot,\cdot\rangle$ be a duality pairing between $\tilde{V}$, $\tilde{V}'$ so that $\|y\|_{\tilde{V}'} = \sup_{x\in\tilde{V}} \langle y, x\rangle / \|x\|_{\tilde{V}}$ is the norm on $\tilde{V}'$. Let $A : \tilde{V} \to \tilde{V}'$, and $T : \tilde{V}' \to \tilde{V}$ be linear operators that satisfy*

$$\langle \tilde{x}, Ax\rangle = \langle x, A\tilde{x}\rangle \quad \forall x, \tilde{x} \in \tilde{V} \tag{45}$$

$$\langle \tilde{y}, Ty\rangle = \langle y, T\tilde{y}\rangle \quad \forall y, \tilde{y} \in \tilde{V}' \tag{46}$$

and

$$c_A\|x\|_{\tilde V}^2 \le \langle x, Ax\rangle \le C_A\|x\|_{\tilde V}^2 \qquad \forall x\in \tilde V \tag{47}$$

$$c_T\|y\|_{\tilde V'}^2 \le \langle y, Ty\rangle \le C_T\|y\|_{\tilde V'}^2 \qquad \forall y\in \tilde V' \tag{48}$$

with constants $c_A, C_A, c_T, C_T > 0$*. Then,*

$$\|A\|_{\tilde V\to\tilde V'} \;\le\; C_A, \qquad \|A^{-1}\|_{\tilde V'\to\tilde V} \le \frac{1}{c_A} \tag{49}$$

$$\|T\|_{\tilde V'\to\tilde V} \;\le\; C_T, \qquad \|T^{-1}\|_{\tilde V\to\tilde V'} \le \frac{1}{c_T} \tag{50}$$

Proof. Since A satisfies (45), we may write

$$\begin{aligned}\|A\|_{\tilde V\to\tilde V'} &= \sup_{x\in\tilde V}\frac{\|Ax\|_{\tilde V'}}{\|x\|_{\tilde V}} = \sup_{x,\tilde x\in\tilde V}\frac{\langle Ax,\tilde x\rangle}{\|x\|\|\tilde x\|}\\ &= \sup_{x\in\tilde V}\frac{\langle Ax, x\rangle}{\|x\|^2} \le C_A\end{aligned}$$

Also,

$$\begin{aligned}\frac{1}{\|A^{-1}\|_{\tilde V'\to\tilde V}} &= \inf_{x\in\tilde V}\frac{\|Ax\|_{\tilde V'}}{\|x\|_{\tilde V}} = \inf_{x\in\tilde V}\sup_{\tilde x\in\tilde V}\frac{\langle Ax,\tilde x\rangle}{\|x\|\|\tilde x\|}\\ &\ge \inf_{x\in\tilde V}\frac{\langle Ax, x\rangle}{\|x\|^2} \ge c_A\end{aligned}$$

Similarly, we get the bounds for operator T by switching the roles of $\tilde V$ and $\tilde V'$ and using the duality of norms $\|x\|_{\tilde V} = \sup_{y\in\tilde V'}\langle x, y\rangle/\|y\|_{\tilde V'}$. □

We will apply Lemma 6.1 to $A = QD$, $T = PF$, and we will obtain upper bounds on $\|QDPF\|_{\tilde V'\to\tilde V'}$ and $\|(QDPF)^{-1}\|_{\tilde V'\to\tilde V'}$. Then $\lambda_{\min}(QDPF) \ge 1/\|(QDPF)^{-1}\|_{\tilde V'\to\tilde V'}$ and $\lambda_{\max}(QDPF) \le \|QDPF\|_{\tilde V'\to\tilde V'}$, which gives a bound on the condition number (35).

6.3. Definition of spaces. We will define the spaces $\tilde V$, $\tilde V'$ with the help of the following algebraical properties of Q.

LEMMA 6.2. *The operator Q defined by (34) satisfies:*

(i) Q is a projection in the factorspace $U/\operatorname{Ker} B^T$

(ii) Q^T is a projection in $(\operatorname{Ker} B^T)^\perp = \operatorname{Range} B$

(iii) If $\lambda\in\operatorname{Range} Q$*, then* $C^TPF\lambda = 0$ *and* $G^T\lambda = 0$*.*

(iv) If $u\in\operatorname{Range} PFQ$*, then* $C^Tu = 0$ *and* $G^Tu = 0$*.*

(v) If $u\in\operatorname{Range} B$*, then* $u\in\operatorname{Range} Q^T$ *if and only if* $C^Tu = 0$ *and* $G^Tu = 0$*.*

(vi) $(\operatorname{Range} Q^T\cap\operatorname{Range} B)^\perp\cap\operatorname{Range} Q\subset\operatorname{Ker} B^T$

Proof. From the definition of Q in (34) and from (40),

$$(I-Q)\lambda = 0 \tag{51}$$

if $\lambda \in \text{Ker } B^T = \text{Ker } F \cap \text{Ker } G^T$. So Q induces an operator on the factorspace $U/\text{Ker } B^T$. To prove that this induced operator is a projection, we first show that for any α, β, μ,

$$\begin{bmatrix} G & C & 0 \end{bmatrix} M^+ M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} - \begin{bmatrix} G & C & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} \in \text{Ker } F \cap \text{Ker } G^T. \tag{52}$$

Denote

$$\begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\mu} \end{bmatrix} = M^+ M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix}.$$

Then

$$M \begin{bmatrix} \tilde{\alpha} \\ \tilde{\beta} \\ \tilde{\mu} \end{bmatrix} = M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix}$$

and, from the definition of M (20),

$$\begin{bmatrix} G^T & 0 \\ C^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} F & G \\ G^T & 0 \end{bmatrix} \begin{bmatrix} G(\tilde{\alpha} - \alpha) + C(\tilde{\beta} - \beta) \\ \tilde{\mu} - \mu \end{bmatrix} = 0 \tag{53}$$

From the last block component of (53), $G^T(G(\tilde{\alpha} - \alpha) + C(\tilde{\beta} - \beta)) = 0$. Using this to eliminate $\tilde{\mu} - \mu$ from the first two components of (53) gives

$$\begin{bmatrix} G^T \\ C^T \end{bmatrix} PF \begin{bmatrix} G & C \end{bmatrix} \begin{bmatrix} \tilde{\alpha} - \alpha \\ \tilde{\beta} - \beta \end{bmatrix} = 0.$$

Since $G(\tilde{\alpha} - \alpha) + C(\tilde{\beta} - \beta) \in \text{Range } P$, $P = P^T$ is projection, and PFP is symmetric semidefinite, it follows that $G(\tilde{\alpha} - \alpha) + C(\tilde{\beta} - \beta) \in \text{Ker } F$, concluding the proof of (52).

Now, from the definition of Q,

$$\begin{aligned}
(I - Q)^2 &= \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \\
&= \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F G & G^T F C & 0 \\ C^T F G & C^T F C & 0 \\ G^T G & G^T C & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \\
&= \begin{bmatrix} G & C & 0 \end{bmatrix} M^+ M \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix}
\end{aligned}$$

Using (52), it follows that for any $\lambda \in U$ there is $\bar{\lambda} \in \text{Ker } B^T = \text{Ker } F \cap \text{Ker } G^T$ such that

$$(54)\quad (I-Q)^2\lambda = \begin{bmatrix} G & C & 0 \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix} M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \lambda + \bar{\lambda} = (I-Q)\lambda + \bar{\lambda},$$

so $I - Q$ is a projection on the factorspace $U/\text{Ker } B^T$.

To prove (ii), let $u \in (\text{Ker } B^T)^\perp$. From (51), for any $\bar{\lambda} \in \text{Ker } B^T$, it holds that $\langle Q^T u, \bar{\lambda}\rangle = \langle u, Q\bar{\lambda}\rangle = 0$ showing that $Q^T(\text{Ker } B^T)^\perp \subset (\text{Ker } B^T)^\perp$. Let $u \in (\text{Ker } B^T)^T$ and $\lambda \in U$. From (54), we have with some $\bar{\lambda} \in \text{Ker } B^T$,

$$\langle Q^T Q^T u, \lambda\rangle = \langle u, QQ\lambda\rangle = \langle u, Q\lambda + \bar{\lambda}\rangle = \langle u, Q\lambda\rangle = \langle Q^T u, \lambda\rangle.$$

This shows that Q^T is a projection in $(\text{Ker } B^T)^\perp$, concluding the proof of (ii).

Let $\lambda \in \text{Range}(Q)$. Then from (i),

$$M^+ \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \lambda = \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix}$$

with $G\alpha + C\beta \in \text{Ker } F \cap \text{Ker } G^T$ and some μ. From (22) and the definition of M and M^+,

$$(55)\qquad \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix} \lambda = M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = \begin{bmatrix} G^T G\mu \\ C^T G\mu \\ 0 \end{bmatrix}$$

Expressing μ in terms of λ, i.e., $\mu = (G^T G)^{-1} G^T F\lambda$ from the first component of (55), we get $G\mu = (I-P)F\lambda$, so the second component becomes $C^T(F\lambda - (I-P)F\lambda) = 0$, hence $C^T PF\lambda = 0$. This together with the third component $G^T\lambda = 0$ of (55) proves statement (iii).

Since $\text{Ker } G^T = \text{Range } P$, (iv) follows immediately from (iii).

To prove (v), let $u \in \text{Range } B$. Since Q^T is a projection in Range B according to (i), $u \in \text{Range } Q^T$ if and only if for some α, β, μ,

$$(56)\qquad M^+ \begin{bmatrix} G^T \\ C^T \\ 0 \end{bmatrix} u = \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix}, \qquad F^T G\alpha + F^T C\beta + G\mu = 0.$$

Multiplying by M from the left and using (52) we get that (56) implies

$$\begin{bmatrix} G^T \\ C^T \\ 0 \end{bmatrix} u = \begin{bmatrix} 0 \\ 0 \\ G^T(G\alpha + C\beta) \end{bmatrix},$$

which gives $G^T u = 0$, $C^T u = 0$. Conversely, if $G^T u = 0$ and $C^T u = 0$, then (56) holds with $\alpha = 0$, $\beta = 0$, $\mu = 0$, which completes the proof of (v).

Finally, let $\lambda \in (\text{Range } Q^T \cap \text{Range } B)^\perp \cap \text{Range } Q$ and $u \in \text{Range } B$. Since $u \in \text{Range } B$, it follows from (ii) that $Q^T u \in \text{Range } Q^T \cap \text{Range } B$. Then

$$0 = \langle \lambda, Q^T u \rangle = \langle Q\lambda, u \rangle = \langle \lambda + \bar{\lambda}, u \rangle$$

with some $\bar{\lambda} \in \text{Ker } B^T$ by (i), so $\langle \lambda, u \rangle = 0$, which concludes the proof of (vi). □

We equip the space W with the norm and the seminorm

$$|w|_W^2 = \sum_s |\nabla w|^2_{H^{1/2}(\partial\Omega_s)}, \qquad \|w\|_W^2 = |w|_W^2 + \sum_s \frac{1}{H^3} \|w\|^2_{L^2(\partial\Omega_s)}. \tag{57}$$

Note that from the Poincaré inequality,

$$\|w\|_W^2 \approx |w|_W^2 + + \sum_s \frac{1}{H} \|\nabla w\|^2_{L^2(\partial\Omega_s)} + \sum_s \frac{1}{H^3} \|w\|^2_{L^2(\partial\Omega_s)}. \tag{58}$$

Define the space $\tilde{V} = \text{Range } Q^T \cap \text{Range } B \subset U$ equipped with the norm

$$\|v\|_{\tilde{V}} = \|B^T v\|_W, \quad v \in \tilde{V}, \tag{59}$$

and $\tilde{V}' = \text{Range } Q \subset U / \text{Ker } B^T$ equipped with the dual norm

$$\|v\|_{\tilde{V}'} = \sup_{\tilde{v} \in \tilde{V}} \frac{\langle v, \tilde{v} \rangle}{\|\tilde{v}\|_{\tilde{V}}} \tag{60}$$

From Lemma 6.2 (vi), it follows that if $v \in \tilde{V}'$ amd $\langle v, \tilde{v} \rangle = 0$ for all $\tilde{v} \in \tilde{V}$, then $v \in \text{Ker } B^T$, so (60) indeed defines a norm on the factorspace.

We now verify that the spaces $\tilde{V}$, $\tilde{V}'$ can indeed be used in Lemma 6.1.

Lemma 6.3. *It holds that*

$$PF : \tilde{V}' \to \tilde{V} \tag{61}$$

and

$$QD : \tilde{V} \to \tilde{V}' \tag{62}$$

Proof. Let $\lambda \in \tilde{V}' = \text{Range } Q$ and $u = PF\lambda$. From Lemma 6.2 (iv), $C^T u = 0$ and $G^T u = 0$. Further, $u \in \text{Range } P \subset \text{Range } B$, so Lemma 6.2 (v) gives that $u \in \text{Range } Q^T = \tilde{V}$, which proves (61). Since $\tilde{V}' = \text{Range } Q$ by definition, (62) follows immediately. □

6.4. Discrete Norm Bounds. Our norm on $\tilde{V}$ was chosen so that the preconditioner is uniformly coercive and bounded, that is, so that (47) holds for $A = QD$ with c and C independent of H and h. This is shown in the following lemma.

Lemma 6.4. $\langle v, QDv \rangle \approx \|v\|^2_{\tilde{V}}$, $\forall v \in \tilde{V}$.

Proof. For $v \in \tilde{V} = \text{Range } Q^T$, and since Q^T is a projection, we have by definition of the preconditioner D,

$$\langle v, QDv \rangle = \langle Q^T v, BSB^T v \rangle = \langle v, BSB^T v \rangle = \langle B^T v, SB^T v \rangle$$

From a standard result [17], by a summation over the subdomains, we obtain $\langle w, Sw\rangle = |w|_W^2$, and, therefore, $\langle B^T v, SB^T v\rangle \approx |B^T v|_W^2$. Now, by Lemma 6.2 (v), $v \in \tilde{V}$ implies $B^T v \perp \operatorname{Ker} S$, and hence the seminorm is equivalent to the norm by the Poincaré inequality. □

The following lemmas lead to an estimate of coercivity and ellipticity of PF. We first summarize some well-known results and inequalities in a form suitable for our purposes.

LEMMA 6.5 ([17]). *Let x be a vertex of a subdomain Ω_s. For $z \in V_h(\Omega_s)$ such that $z(x) = 0$, define $w \in V_h(\Omega_s)$ by $w(x) = z(x) = 0, \nabla w(x) = \nabla z(x)$, and $w(x) = 0, \nabla w(x) = 0$ on all other nodes of $\partial\Omega_i$. Then*

$$\|\nabla w\|^2_{H^{1/2}(\partial\Omega_s)} \leq C(1 + \log\frac{H}{h})(\|\nabla z\|^2_{H^{1/2}(\partial)}\Omega_i s + \frac{1}{H}\|\nabla z\|^2_{L^2(\partial\Omega_s)}).$$

The following estimate of the trace norm of the extension by zero is proved as in [2, Lemma 3.5].

LEMMA 6.6. *There exists a constant C such that if the support of u is contained in a segment σ of $\partial\Omega_s$ of length τ, and $|\frac{\partial u}{\partial s}|_{L^\infty(\sigma)} \leq \frac{c}{h}|u|_{L^\infty(\sigma)}$, then*

$$|u|^2_{H^{1/2}(\partial\Omega_s)} \leq |u|^2_{H^{1/2}(\sigma)} + C\left(1 + \log\frac{\tau}{h}\right)|u|^2_{L^\infty(\sigma)}.$$

We will also need a straightforward extension of the discrete Sobolev inequality of Dryja [6] to piecewise polynomial functions of order $p > 1$ [21].

LEMMA 6.7. *Let $p \geq 1$. Then there exists a constant $C = C(p)$ such that for every u continuous on $\partial\Omega_s$ such that $u \in P_p$ on the side of every element T,*

$$|\nabla u|^2_{L^\infty(\partial\Omega_s)} \leq C\left(1 + \log\frac{H}{h}\right)\left(|\nabla u|^2_{H^{1/2}(\partial\Omega_s)} + \frac{1}{H}|\nabla u|^2_{L^2(\partial\Omega_s)}\right).$$

The next two lemmas contain the principal technical estimates.

LEMMA 6.8. *For all $\lambda \in \tilde{V}'$ and all $w \in W$ such that $Bw \in \tilde{V}$, there exists a $\tilde{w} \in W$ such that*

$$\frac{\langle\lambda, B\tilde{w}\rangle^2}{\|\tilde{w}\|^2_W} \geq \frac{C}{(1 + \log H/h)^\alpha}\frac{\langle\lambda, Bw\rangle^2}{\|B^T Bw\|^2_W}$$

where $\alpha = 1$, and $\alpha = 0$ if $\frac{1}{2}BB^T = I$, which happens when there are no nodes shared by more than two subdomains.

Proof. Let us first prove that in the general case we obtain $\alpha \leq 1$. Let $w \in W$ and $Bw \in \tilde{V}$ throughout this proof. We take $\tilde{w} = B^T(BB^T)^+Bw$. By the triangle inequality, we may write

$$\|\tilde{w}\|_W \leq \|\tfrac{1}{2}B^T Bw\|_W + \|\tfrac{1}{2}B^T(I - (\tfrac{1}{2}BB^T)^+)Bw\|_W. \tag{63}$$

Denote $z = \frac{1}{2}B^T(I-(\frac{1}{2}BB^T)^{-1})Bw$. From the definition of B in (41), z is zero at all nodes that belong to at most two subdomains. The remaining nodes lie on subdomain crosspoints. From the definition of B, at every such node, $z_i(x)$ is a linear combination of the entries of B^Tw that correspond to the same node x and the coefficients of the linear combinations are bounded only in terms of the number of subdomains the node belongs to. In addition, it holds that $C^Tz = 0$ since $C^TBw = 0$. Using Lemma 6.5 for the subdomain vertices, we obtain

$$\|z\|_W^2 \leq C(1+\log(H/h))\|B^TBw\|_W^2 \tag{64}$$

This together with (63) and the fact that $B\tilde{w} = Bw$ yields the result.

If $\frac{1}{2}BB^T = I$, we choose $\tilde{w} = \frac{1}{2}B^TBw = w$, which concludes the proof. □

The following lemma is the converse of the bound in Lemma 6.8.

LEMMA 6.9. *For all $\lambda \in \tilde{V}'$ and $w \in W$, there is a $\tilde{w} \in W$ such that $Bv \in \tilde{V}$ and*

$$\frac{\langle\lambda, Bw\rangle^2}{\|w\|_W^2} \leq C(1+\log H/h)^2 \frac{\langle\lambda, B\tilde{w}\rangle^2}{\|B^TB\tilde{w}\|_W^2}$$

Proof. Let $\lambda \in \tilde{V}'$, $w \in W$, and consider $\bar{w} = w + z$, $z \in \text{Ker } S$. Since $\lambda \in \tilde{V}' = \text{Range } Q^T$, we have by Lemma 6.2 (v) that $B^T\lambda \perp \text{Ker } S$, hence

$$\langle\lambda, Bw\rangle = \langle\lambda, B\bar{w}\rangle \tag{65}$$

Now we determine $\bar{w}$ from the condition $G^T\bar{w} = 0$, which can also be written as

$$\langle Bz, B\tilde{z}\rangle = -\langle Bw, B\tilde{z}\rangle \quad \forall \tilde{z} \in \text{Ker } S. \tag{66}$$

The bilinear form $\langle B\cdot, B\cdot\rangle$ is an inner product on the factorspace Ker $S/(\text{Ker } S \cap \text{Ker } B)$, so by the Riesz representation theorem we may conclude that there exists $z \in \text{Ker } S$ satisfying (66). Setting $\tilde{z} = z$ and using the Cauchy inequality gives $\|Bz\| \leq \|Bw\|$.

Now that we have met the condition $G^T\bar{w} = 0$, we need to modify $\bar{w}$ to obtain $\tilde{w}$ so that $C^T\tilde{w} = 0$ without disturbing the properties $G^T\bar{w} = 0$ and $\langle\lambda, Bw\rangle = \langle\lambda, B\bar{w}\rangle$. Let $B\tilde{w} = B\bar{w} + PFPC\alpha$, and solve for α from

$$\langle PFPC\alpha, C\tilde{\alpha}\rangle = \langle B\bar{w}, C\tilde{\alpha}\rangle \quad \forall\tilde{\alpha}. \tag{67}$$

By definition of F, and considering that $B\bar{w} = PBw$, we can rewrite (67) as

$$\langle S^+B^TPC\alpha, B^TPC\tilde{\alpha}\rangle = \langle w, B^TPC\tilde{\alpha}\rangle \quad \forall\tilde{\alpha}. \tag{68}$$

The bilinear form on the left hand side is an inner product on the factorspace $(\text{Ker } S)^\perp/((\text{Ker } S)^\perp \cap \text{Ker } B^TPC)$. Again, by a similar argument, (68) yields α that satisfies

$$\langle S^{+^{1/2}}B^TPC\alpha, S^{+^{1/2}}B^TPC\alpha\rangle = \langle S^{1/2}w, S^{+^{1/2}}B^TPC\alpha\rangle$$

and from the Cauchy inequality, $\|S^{+1/2}B^TPC\alpha\| \le \|S^{1/2}w\|$. Therefore,

$$\|S^+B^TPC\alpha\|_S \le \|w\|_S. \tag{69}$$

Next we estimate $\|B^TB\tilde{w}\|_W$. Since $C^TB\tilde{w} = 0$, i.e., $B\tilde{w}$ is zero at all crosspoints, from Lemma 6.6, it follows that

$$\|B^TB\tilde{w}\|_W^2 \le C\sum_j \left(\frac{1}{H^3}|B\tilde{w}|^2_{L^2(\Gamma_j)} + |\nabla B\tilde{w}|^2_{H^{1/2}(\Gamma_j)} + \log\frac{H}{h}|\nabla B\tilde{w}|^2_{L^\infty(\Gamma_j)}\right) \tag{70}$$

where summation is carried out over all edges Γ_i of the intersubdomain interface. By construction, we have

$$B\tilde{w} = Bw + Bz + PFPC\alpha. \tag{71}$$

We now substitute (71) to (70) and use the triangle inequality. We first deal with the $H^{1/2}$ seminorm and L_2 norm terms. First,

$$\sum_j \left(\frac{1}{H^3}|Bw|^2_{L^2(\Gamma_j)} + |\nabla Bw|^2_{H^{1/2}(\Gamma_j)}\right) \le C\sum_i \left(\frac{1}{H^3}|w|^2_{L^2(\partial\Omega_i)} + |\nabla w|^2_{H^{1/2}(\partial\Omega_i)}\right)$$

From $\|Bz\| \le \|Bw\|$, the fact that $z \in \operatorname{Ker} S$, i.e., z is linear, and from the fact that $|\Gamma_j| \ge c_0H \ \forall j$, which follows from (36), we obtain an estimate for the second term of (70),

$$\sum_j \left(\frac{1}{H^3}|Bz|^2_{L^2(\Gamma_j)} + |\nabla Bz|^2_{H^{1/2}(\Gamma_j)}\right) \le C\frac{1}{H^3}\sum_i ||w||^2_{L^2(\partial\Omega_i)}$$

Using (69) and the fact that Range $S^+ \perp \operatorname{Ker} S$, we get

$$\begin{aligned}\sum_j &\left(|\nabla(PBS^+B^TPC\alpha|^2_{H^{1/2}(\Gamma_j)} + \frac{1}{H^3}\|PBS^+B^TPC\alpha\|^2_{L^2(\Gamma_j)}\right)\\ &\le C_0C\|S^+B^TPC\alpha\|^2_W \approx C_0C\|S^+B^TPC\alpha\|^2_S \le C_0C\|w\|_S \approx C_0C\|w\|^2_W,\end{aligned}$$

where C_0 is given by

$$C_0 = \sup_{u\in W}\frac{\sum_j\left(|\nabla PBu|^2_{1/2,\Gamma_j} + \frac{1}{H^3}\|PBu\|_{0,\Gamma_j}\right)}{\|u\|^2_W} \le C$$

since $\|B\| \le C$, $\|P\| = 1$, and if $g \in$ Range G, then $|\nabla g|_{1/2,\Gamma_j} = 0$. Finally, for the L^∞ norm, by Lemma 6.7,

$$\sum_j |\nabla B\tilde{w}|^2_{L_\infty(\Gamma_j)} \le C(1+\log\frac{H}{h})\sum_j\left(\frac{1}{H^3}|B\tilde{w}|^2_{0,\Gamma_j} + |\nabla B\tilde{w}|^2_{1/2,\Gamma_j}\right)$$

Putting all of the above together gives $\|B^TB\tilde{w}\|_W \le C(1+\log H/h)\|w\|_W$. □
We have now everything ready to prove the estimate (48) with $T = PF$.

LEMMA 6.10. *It holds that*

$$c(1+\log(H/h))^{-\alpha}\|\lambda\|_{\tilde{V}'}^2 \le \langle\lambda, PF\lambda\rangle \le C(1+\log(H/h))^2\|\lambda\|_{\tilde{V}'}^2, \qquad \forall\lambda\in\tilde{V}',$$

with α defined in Lemma 6.8.

Proof. From the spectral decomposition (8), define $S^{-1/2}=\sum_{t>0}t^{-1/2}v_tv_t'$. Then $S^+=S^{-1/2}S^{-1/2}$, and for $\lambda\in\tilde{V}'$,

$$\begin{aligned}
\langle\lambda,F\lambda\rangle &= \langle S^+B^T\lambda, B^T\lambda\rangle = \langle S^{-1/2}B^T\lambda, S^{-1/2}B^T\lambda\rangle\\
&= \|S^{-1/2}B^T\lambda\|^2 = \sup_{x\in W}\frac{\langle S^{-1/2}B^T\lambda, x\rangle^2}{\|x\|^2} = \sup_{\substack{x\in W,\ x=x_1+x_2\\ x_1\in\text{Ker }S\ x_2\perp\text{Ker }S}}\frac{\langle B^T\lambda, S^{-1/2}x\rangle^2}{\|x_1+x_2\|^2}\\
&= \sup_{x_2\in W,\ x_2\perp\text{Ker }S}\frac{\langle B^T\lambda, S^{-1/2}x_2\rangle^2}{\|x_2\|^2}
\end{aligned}$$

since $S^{-1/2}x_1=0$ and $\|x\|^2=\|x_1\|^2+\|x_2\|^2$. Now write any $w\in W$ as

$$w=w_1+w_2,\qquad w_1\in\text{Ker }S,\qquad w_2=S^{-1/2}x_2\perp\text{Ker }S.$$

From the definition of $\tilde{V}'$, $\lambda\in\tilde{V}'$ implies that

$$\langle B^T\lambda, w_1\rangle = 0, \tag{72}$$

hence $\langle B^T\lambda, w_2\rangle=\langle B^T\lambda, w\rangle=\langle\lambda, Bw\rangle$. Since $w_2\perp\text{Ker }S$, the W norm and seminorm are equivalent

$$\|x_2\|^2=\langle x_2,x_2\rangle=\langle w_2, Sw_2\rangle\approx|w_2|_W^2\approx\|w_2\|_W^2,$$

and using (72), it follows that

$$\langle\lambda,F\lambda\rangle=\sup_{w_2\in W,\ w_2\perp\text{Ker }S}\frac{\langle B^T\lambda,w_2\rangle^2}{\langle w_2,Sw_2\rangle}\approx\sup_{w\in W}\frac{\langle\lambda,Bw\rangle^2}{\|w\|_W^2}.$$

Lemma 6.8 shows that

$$\sup_{w\in W}\frac{\langle\lambda,Bw\rangle^2}{\|w\|_W^2}\ge\frac{1}{C(1+\log H/h)^\alpha}\sup_{\substack{w\in W\\ Bw\in\tilde{V}}}\frac{\langle\lambda,Bw\rangle^2}{\|B^TBw\|_W^2}.$$

Lemma 6.9 yields an upper bound

$$\sup_{w\in W}\frac{\langle\lambda,Bw\rangle^2}{\|w\|_W^2}\le C(1+\log H/h)^2\sup_{\substack{w\in W\\ Bw\in\tilde{V}}}\frac{\langle\lambda,Bw\rangle^2}{\|B^TBw\|_W^2}.$$

Finally, by definition of the norm $\|\cdot\|_{\tilde{V}'}$,

$$\sup_{\substack{w\in W\\ Bw\in\tilde{V}}}\frac{\langle\lambda,Bw\rangle}{\|B^TBw\|_W}=\sup_{v\in\tilde{V}}\frac{\langle\lambda,v\rangle}{\|B^Tv\|_W}=\|\lambda\|_{\tilde{V}'}$$

since B spans $\tilde{V}$. ◻

6.5. Condition Number Estimate. The final result now follows from the abstract estimate in Lemma 6.1 with the assumptions verified by Lemma 6.4 and Lemma 6.10.

THEOREM 6.11. *The condition number of the generalized FETI method with the Dirichlet preconditioner satisfies*

$$\kappa = \frac{\lambda_{max}(QDPF)}{\lambda_{min}(QDPF)} \leq C(1 + \log \frac{H}{h})^{\gamma}$$

with $\gamma = 3$*, and* $\gamma = 2$ *if there are no crosspoints between more than two subdomains.*

7. Computational Results. In order to illustrate the potential of the new FETI method for the iterative solution of discretized fourth order partial differential equations, we consider the plate bending problem on a unit square, with thickness equal to 10^{-3}. The Young modulus is $E = 10^6$ and the Poisson coefficient is $\nu = 0.3$. The plate is discretized by a uniform mesh of three-noded triangular DKT plate elements and subjected to a uniform pressure load. Several meshes with different parameters h and several mesh partitions with different parameters H are constructed for assessing the performance of the new FETI method described in this paper. In all cases, the FETI and new FETI methods are preconditioned with the Dirichlet preconditioner [9, 12], and the following stopping criterion is used

$$\frac{\|z_{k-1}\|}{\|f\|} \leq \varepsilon = 10^{-3} \tag{73}$$

where z_{k-1} is the preconditioned residual in Algorithms 3.1 and 4.1. It was shown in [12] that (73) is a good estimator for the global error criterion

$$\frac{\|Ku - f\|}{\|f\|} \leq \varepsilon,$$

where K is the global stiffness matrix and u is the approximate solution averaged between subdomains, and $\|\cdot\|$ is the l^2 norm. This stopping criterion is more adequate and more stringent than a local stopping criterion based on the jump of the solution across the subdomain interfaces [12]. It also eliminates the need for constructing the global solution u at every PCG iteration and bypasses the computation of Ku.

Three series of computational experiments are reported. First, the number of subdomains is fixed to 4, 16, and 64, and in each case, three different meshes corresponding to $h = 1/10$, $h = 1/20$, and $h = 1/40$ are generated. The corresponding performance results (number of iterations and condition number of the preconditioned interface problem) of the FETI and new FETI methods are summarized in Table 1. These results confirm that for a given H, the interface problem associated with the original FETI method has a condition number that grows fast with the mesh size h, while that of the new FETI method has a condition number that is much smaller and grows only weakly with h. For large number of subdomains ($H = 1/8$, $N_s = 64$) the new FETI method is reported to converge about 7 times faster than the original one.

However, it is also interesting to note that when the mesh size h is decreased, the number of iterations of the original FETI method starts from a value that is higher than that of the new FETI algorithm for plates, but grows only weakly and almost at the same slow rate as in the new FETI method. We believe that this observation is a result of the superconvergence properties of the original FETI method [12]. Namely, a non-optimal preconditioner applied to the dual problem gives a bad condition number but good separation of eigenvalues as long as there are not too many subdomains.

Next, the mesh size is fixed to $h = 1/120$ (28800 elements and 86400 d.o.f.), and H is varied between $H = 1/2$ (4 subdomains) and $H = 1/12$ (144 subdomains). The obtained performance results are depicted in Table 2. In that case, the condition numbers of both FETI methods are shown to decrease with the number of subdomains. This is an expected result because when h is fixed and H is decreased, the size of the coarse problem increases for both algorithms. Both FETI methods are also shown to achieve convergence in a number of iterations that is asymptotically independent of the number of subdomains. However, the new FETI method reaches the asymptotic behavior much faster than the original one, and for large number of subdomains ($H \leq 1/8$), the new FETI method is reported to converge about 8 times faster than the original one.

Finally, the subdomain problem size is fixed to $h/H = 1/15$, and the number of subdomains as well as the size of the global problem are increased. The performance results reported in Table 3 show that in that case too, the new FETI method outperforms significantly the original one.

Since solution time is ultimately the most important criterion for assessing performance, we have also benchmarked both FETI methods for the same plate bending problem described above with 960000 d.o.f. and 64 subdomains. The performance results obtained on a 64-processor IBM SP2 are summarized in Table 4. They show that even though the new FETI method consumes an amount of CPU time (55.5 s.) equivalent to that of 50 of its iterations to set up and preprocess the coarse problem (19), and even though each of its iterations is 1.3 times more expensive than an iteration of the original FETI method, the new FETI method is 2.5 times faster than the original one at solving the system of 960000 plate bending equations on a 64-processor IBM SP2.

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Table 1

Fixed number of subdomains, series of refined meshes

2x2 subdomains, H=$\frac{1}{2}$

$\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{10}$	18	2578	12	7.6
$\frac{1}{20}$	22	30101	15	12.6
$\frac{1}{40}$	26	409987	17	18.6

4x4 subdomains, H=$\frac{1}{4}$

$\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{10}$	61	6795	21	11.5
$\frac{1}{20}$	86	84199	27	17
$\frac{1}{40}$	119	1038120	36	24.4

8x8 subdomains, H=$\frac{1}{8}$

$\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{10}$	172	21707	25	13
$\frac{1}{20}$	247	275004	34	19.4
$\frac{1}{40}$	323	3920613	42	27.6

Table 2

Fixed global mesh, series of refined mesh partitions

Global mesh 120x120, h=$\frac{1}{120}$

splitting H	splitting $\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{2}$	$\frac{1}{60}$	27	2079032	18	23.2
$\frac{1}{3}$	$\frac{1}{40}$	64	839240	29	22.4
$\frac{1}{4}$	$\frac{1}{30}$	104	391470	32	21
$\frac{1}{5}$	$\frac{1}{24}$	135	234504	33	19.9
$\frac{1}{6}$	$\frac{1}{20}$	164	160173	32	18.6
$\frac{1}{8}$	$\frac{1}{15}$	222	94285	31	16.6
$\frac{1}{10}$	$\frac{1}{12}$	255	63896	29	14.9
$\frac{1}{12}$	$\frac{1}{10}$	245	46921	27	13.6

TABLE 3
Fixed local mesh, series of refined meshes and mesh partitions

Local mesh 15x15, $\frac{h}{H}=\frac{1}{15}$

splitting		FETI		NEW FETI	
H	h	Iterations	Cond. Numb.	Iterations	Cond. Numb.
$\frac{1}{2}$	$\frac{1}{30}$	20	11088	13	10
$\frac{1}{3}$	$\frac{1}{45}$	49	19004	21	13.4
$\frac{1}{4}$	$\frac{1}{60}$	74	29041	25	14.6
$\frac{1}{5}$	$\frac{1}{75}$	109	40120	28	15.4
$\frac{1}{6}$	$\frac{1}{90}$	145	55068	29	15.9
$\frac{1}{8}$	$\frac{1}{120}$	222	94285	31	16.6
$\frac{1}{10}$	$\frac{1}{150}$	318	144556	32	16.9

TABLE 4
Performance results for 960000 d.o.f. and 64 subdomains

FETI			NEW FETI		
Iterations	Total time	Time per iter.	Iterations	Total time	Time per iter.
314	265 s.	0.8 s.	45	105 s.	1.1 s.

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