DERIVING UPWINDING, MASS LUMPING AND SELECTIVE REDUCED INTEGRATION BY RESIDUAL-FREE BUBBLES

LEOPOLDO P. FRANCA[1] AND ALESSANDRO RUSSO[2]

[1] *Department of Mathematics, University of Colorado at Denver,*
P.O.Box 173364, Campus Box 170, Denver, CO 80217-3364 (USA)
[2] *Istituto di Analisi Numerica del CNR,*
via Abbiategrasso 209, I-27100 Pavia (Italy)

ABSTRACT. We show that three well-known "variational crimes" in finite elements – upwinding, mass lumping and selective reduced integration – may be derived from the Galerkin method employing the standard polynomial-based finite element spaces enriched with residual-free bubbles.

1. Introduction.

In this note we introduce a finite element method based on enriching the classical polynomial-based finite element spaces with residual-free bubbles. We show that in 1D the classical techniques of upwinding, mass lumping and selective reduced integration can be derived by the Galerkin method based on the enriched space.

2. An abstract presentation.

Let $\Omega \subset \mathbf{R}^n$ be a regular domain, $f \in L^2(\Omega)$ and

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

be a linear elliptic boundary value problem which can be given a classical variational formulation as follows:

$$\text{find } u \in V \text{ such that} \quad a\,(u,v) = F(v) \quad \text{for all } v \in V \tag{2.2}$$

where $a\,(\cdot,\cdot)$ is a bilinear form on $V = H_0^1(\Omega)$ which is continuous and coercive with respect to the usual norm on V and $F(v) = \int_\Omega fv\,dx$. Let $V_h \subset V$ be a finite dimensional subspace of V; then the Galerkin approximation for problem (2.2) is

$$\text{find } u_h \in V_h \text{ such that} \quad a\,(u_h,v_h) = F(v_h) \quad \text{for all } v_h \in V_h. \tag{2.3}$$

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The classical finite element method consists basically in taking a partition $\mathcal{T}_h = \{K\}$ of Ω and in defining

$$V_h = V_h^P = \{v_h \in C^0(\bar{\Omega}) : \text{for all } K \in \mathcal{T}_h \; v_{h|_K} \text{ is a polynomial of chosen degree}\}.$$

We wish to enrich the classical polynomial-based space V_h^P by adding a set of bubbles. For each $K \in \mathcal{T}_h$, let B_K be a finite dimensional subspace of $H_0^1(K)$ whose dimension in general will depend on K. The functions of B_K are to be thought as extended to zero outside K. Define then $B = \oplus_{K\in\mathcal{T}_h} B_K$ and $V_h = V_h^P \oplus B$ as the enlarged approximation space; the bubble spaces B_K will be defined later. Any function $v_h \in V_h$ can be split in a unique way as $v_h = v_P + v_B = v_P + \sum_{K\in\mathcal{T}_h} v_{B,K}$ where $v_P \in V_h^P$, $v_B \in B$ and $v_{B,K} = v_{B|_K} \in B_K$. Using this representation, the Galerkin approximation of (2.2) on V_h can be written as follows:

$$(2.4) \qquad \begin{cases} \text{find } u_h = u_P + u_B \in V_h = V_h^P \oplus B \text{ such that} \\ a\,(u_h, v_P) = F(v_P) \quad \text{for all } v_P \in V_h^P \\ a\,(u_h, v_{B,K})_K = F(v_{B,K})_K \quad \text{for all } K \in \mathcal{T}_h, \text{ and for all } v_{B,K} \in B_K \end{cases}$$

where the subscript K means that the integrals involved in $a\,(\cdot,\cdot)$ and $F(\cdot)$ are restricted to K. Let $K \in \mathcal{T}_h$. Then $u_{h|_K} = u_{P|_K} + u_{B,K}$ and the second equation in (2.4) reads as $a\,(u_P + u_{B,K}, v_{B,K})_K = F(v_{B,K})_K$ for all $v_{B,K} \in B_K$ or

$$(2.5) \qquad a\,(u_{B,K}, v_{B,K})_K = -(a\,(u_P,\cdot) - F(\cdot))_K(v_{B,K}) \quad \text{for all } v_{B,K} \in B_K.$$

The residual-free bubble space B_K is defined in such a way that equation (2.5) holds for any test function in $H_0^1(K)$, i.e. we define $u_{B,K}$ as the solution of the variational problem

$$(2.6) \qquad \begin{cases} \text{find } u_{B,K} \in H_0^1(K) \text{ such that} \\ a\,(u_{B,K}, v)_K = -(a\,(u_P,\cdot) - F(\cdot))_K(v) \quad \text{for all } v \in H_0^1(K). \end{cases}$$

Problem (2.6) has always a unique solution which depends linearly on $u_{P|_K}$. Hence, problem (2.6) identifies an affine operator $\mathcal{R}_K$ given by

$$(2.7) \qquad \begin{aligned} \mathcal{R}_K \;:\; V_{h,K}^P &\to H_0^1(K) \\ u_{P|_K} &\mapsto u_{B,K} \end{aligned}$$

where $V_{h,K}^P$ is the linear space of restrictions to K of functions in V_h^P. We define B_K as the image in $H_0^1(K)$ of the operator $\mathcal{R}_K$: $B_K = \mathcal{R}_K(V_{h,K}^P)$. The name "residual-free" appended to these bubbles comes from the fact that the quantity at the right-hand-side of equation (2.6) is the residual with respect to the polynomial part u_P of the solution so that $u_P + u_{B,K}$ solves exactly the equation in the interior of K. A basis for B_K can be constructed as follows. Let $\{\phi_{1,K},\ldots,\phi_{N,K}\}$ be a basis for $V_{h,K}^P$ (N may depend on K) and let $b_{i,K} \in H_0^1(K)$ be the solution of the equation

$$(2.8) \qquad a\,(b_{i,K}, v)_K = -a\,(\phi_{i,K}, v)_K \quad \text{for all } v \in H_0^1(K),\; i = 1,\ldots,N.$$

Let then $b_{f,K} \in H_0^1(K)$ be the solution of the equation

$$(2.9) \qquad a\,(b_{f,K}, v)_K = F(v)_K \quad \text{for all } v \in H_0^1(K).$$

Then, using the linearity (also with respect to f) of the operator $\mathcal{R}_K$, we have

$$B_K = \text{span}\{b_{1,K}, \ldots, b_{N,K}, b_{f,K}\}. \tag{2.10}$$

The dimension of B_K is then bounded by $N+1$. It should be clear at this point that if we go back and define B_K as in (2.10) and we solve problem (2.4), then the bubble part $u_{B,K}$ of the solution u_h also solves problem (2.6). Now we can "eliminate" the bubbles using the operator $\mathcal{R}_K$ in the first equation of (2.4), obtaining a variational problem involving u_P only:

$$\begin{cases} \text{find } u_P \in V_h^P \text{ such that} \\ a\left(u_P + \sum_{K\in\mathcal{T}_h} \mathcal{R}_K(u_P|_K), v_P\right) = F(v_P) \quad \text{for all } v_P \in V_h^P. \end{cases} \tag{2.11}$$

The equation in (2.11) can be re-written in a more meaningful way as

$$a(u_P, v_P) + \underbrace{\sum_{K\in\mathcal{T}_h} a(\mathcal{R}_K(u_P|_K), v_P)_K}_{\textit{bubble modification}} = F(v_P) \quad \text{for all } v_P \in V_h^P, \tag{2.12}$$

highlighting the fact that the procedure of defining the residual-free bubbles and then eliminating them leads to a (consistent!) modification of the classical polynomial-based finite element methods. The problem of determining a set of bubbles (in general, not residual-free) is pursued in [2] with the objective of reproducing a given stabilization operator.

Sometimes, the computation of the operator $\mathcal{R}_K$ could be as difficult as the original problem itself. Hence, in these cases, in order to make effective use of the residual-free bubbles approach, an approximation of the action of $\mathcal{R}_K$ in (2.12) has to be found. An application of this idea to a convection-diffusion operator is given in [3].

In the next Sections we will examine some simple examples that demonstrate in 1D the improvement on the classical methods by using residual-free bubbles.

3. Upwinding.

Let us consider the following one-dimensional boundary value problem:

$$-\varepsilon u'' + u' = f \quad \text{in }]0,1[, \qquad u(0) = u(1) = 0 \tag{3.1}$$

where $\varepsilon > 0$ and f are constants. We partition the interval $[0,1]$ into N subintervals $K_1, \ldots, K_N$ of equal lenght $h = 1/N$ and employ polynomials of degree one. Then $V_h^P = V_h^1 = \{v_1 \in C^0([0,1]) : v_1|_{K_i} \text{ is linear}, i = 1, \ldots, N\}$. All the results can be trivially extended to piecewise constant ε and f and non-uniform partitions. It is simple to see that B_K has dimension one and we can compute directly the map $\mathcal{R}_{K_i}$: if $v_1 \in V_h^1$, we have

$$\mathcal{R}_{K_i}(v_1|_{K_i}) = u_{B,i} = -(v_1' - f)|_{K_i}\,\Phi \tag{3.2}$$

where Φ is a fixed function given in local coordinates by

$$\Phi(s) = -h\,\frac{e^{s/\varepsilon} - 1}{e^{h/\varepsilon} - 1} + s, \quad s \in [0,h]. \tag{3.3}$$

In view of equation (2.12), in order to compute the effect of the residual-free bubbles we have to compute the quantity $\sum_{i=1}^{N} a\,(\mathcal{R}_{K_i}(u_1|_{K_i}), v_1)_{K_i}$ where $a\,(u, v) = \varepsilon \int_0^1 u'v'\,dx + \int_0^1 u'v\,dx$ is the bilinear form associated to problem (3.1) and $v_1 \in V_h^1$. We have

$$
\begin{aligned}
a\,(\mathcal{R}_{K_i}(u_1|_{K_i}), v_1)_{K_i} &= a\,(-(u_1' - f)\,\Phi, v_1)_{K_i} = -(u_1' - f)|_{K_i} a\,(\Phi, v_1)_{K_i} = \\
&-(u_1' - f)|_{K_i} \int_{K_i} \Phi' v_1\,dx = (u_1' - f)|_{K_i} v_1'|_{K_i} \int_{K_i} \Phi\,dx,
\end{aligned} \tag{3.4}
$$

where the first integral in $a\,(\Phi, v_1)_{K_i}$ is zero because v_1 is linear on K_i and Φ is zero on ∂K_i, and the last equality is obtained integrating by parts. By direct inspection, we have

$$
\int_{K_i} \Phi\,dx = \int_0^h \Phi(s)\,ds = h\,(-\varepsilon + \varepsilon\,(h/2\varepsilon)\coth(h/2\varepsilon)) \tag{3.5}
$$

so that the result of (3.4) can be written as

$$
(-\varepsilon + \varepsilon\,(h/2\varepsilon)\coth(h/2\varepsilon)) \int_{K_i} (u_1' - f) v_1'\,dx. \tag{3.6}
$$

Summing over all elements, we have

$$
\sum_{i=1}^{N} a(\mathcal{R}_{K_i}(u_1|_{K_i}), v_1)_{K_i} = (-\varepsilon + \varepsilon\,(h/2\varepsilon)\coth(h/2\varepsilon)) \int_0^1 (u_1' - f) v_1'\,dx. \tag{3.7}
$$

Since f is constant on $[0, 1]$, we have $\int_0^1 f v_1'\,dx = 0$ and then by substituting (3.7) in (2.12) we have the following variational equation to be satisfied by u_1:

$$
\varepsilon\,(h/2\varepsilon)\coth(h/2\varepsilon) \int_0^1 u_1' v_1'\,dx + \int_0^1 u_1' v_1\,dx = \int_0^1 f v_1\,dx \quad \text{for all } v_1 \in V_h^1 \tag{3.8}
$$

which is the well-known optimal artificial diffusion scheme, that gives the exact solution at the nodes. Indeed, in this case we have that $u_1 + u_B$ coincides with the exact solution in $[0, 1]$. If ε is small with respect to h, we have

$$
\varepsilon\,(h/2\varepsilon)\coth(h/2\varepsilon) \approx h/2 \tag{3.9}
$$

which corresponds to classical upwinding. A different presentation of these results has been given in [3], and various extensions to more general situations are contained in [9], [10].

4. Mass lumping.

In this Section we will consider the following one-dimensional boundary value problem:

$$
-\varepsilon u'' + u = f \quad \text{in }]0, 1[, \qquad u(0) = u(1) = 0 \tag{4.1}
$$

where $\varepsilon > 0$ and f are constants. It is well known that the finite element discretization of (4.1) employing continuous piecewise linear functions yields a scheme for which the discrete maximum principle does not hold. The maximum principle property can be recovered for instance by using mass-lumping, which amounts to approximating integrals with the trapezoidal rule.

We will use the same discretization framework of the previous Section. As before, all the results can be trivially extended to piecewise constant ε and f and non-uniform partitions. It can be seen that B_K has dimension two and a long but simple computation reveals that the final equation to be satisfied by the nodal values $\{u_i\}$ of u_1 is the following:

$$-\varepsilon \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + M_h u_i = M_h f \quad i = 1, \ldots, N-1 \tag{4.2}$$

where

$$M_h = \frac{\cosh(h/\sqrt{\varepsilon}) - 1}{\frac{1}{2}(h/\sqrt{\varepsilon})^2}. \tag{4.3}$$

Equation (4.2) gives the exact solution at the nodes for all values of ε, h and f. If h is small with respect to $\sqrt{\varepsilon}$, we have

$$M_h \approx 1, \tag{4.4}$$

which corresponds to mass lumping; if h is kept fixed and we let ε going to zero, it's easy to see that the solution obtained from mass lumping and the solution of (4.2) tend to the same limit.

A more complete treatment of mass lumping emanating from residual-free bubbles is contained in [4].

5. Selective reduced integration.

The deflection of a beam taking into account bending and shear deformations is described by the Timoshenko model. The standard Galerkin finite element method using equal-order piecewise linear approximations for the unknowns rotation and displacement yields "locking" and spurious oscilations for the shear forces. Selective reduced integration has been suggested to cure some of these pathologies and has been justified resorting to an equivalent mixed variational formulation [8],[1]. The Timoshenko beam model is governed by the following differential equations (after non-dimensionalization – e.g., see [1], [6]):

$$\begin{aligned} -\theta'' - \frac{1}{\varepsilon^2}(w' - \theta) &= 0 \qquad \text{in } \Omega \\ -\frac{1}{\varepsilon^2}(w'' - \theta') &= f \qquad \text{in } \Omega \end{aligned} \tag{5.1}$$

where prime denotes differentiation with respect to $x \in \Omega = (0,1)$, θ and w are the rotation and displacement variables, f is the load and ε is a non-dimensional parameter proportional to the beam thickness. The variational formulation corresponding to (5.1) with clamped (i.e. zero Dirichlet) boundary conditions is given by:

$$\begin{cases} \text{find } \{\theta, w\} \in H_0^1(\Omega)^2 \text{ such that} \\ (\theta', \psi') + \frac{1}{\varepsilon^2}(w' - \theta, v' - \psi) = (f, v) \quad \text{for all } \{\psi, v\} \in H_0^1(\Omega)^2 \end{cases} \tag{5.2}$$

where $(f, g) = \int_\Omega fg\,dx$ is the usual L^2 inner product. Using continuous piecewise linear elements and residual-free bubbles under piecewise constant loads, it can be seen (see [5]) that after the elimination of the bubbles the equation for $\{\theta_1, w_1\}$ has the following form:

$$(5.3)\quad (\theta_1', \psi_1') + \sum_K \frac{1}{\varepsilon^2 + \frac{h_K^2}{12}} (w_1' - R\theta_1, v_1' - R\psi_1)_K = (f, v_1) + \sum_K f_K (\chi_K, v_1' - \psi_1)_K$$

where, in each K, $R\phi$ equals $\phi(M_K)$ (the value of ϕ at the midpoint of K) and $\chi_K(x) = x - M_K$, $x \in K$. Formulation (5.3) was *derived* using full integration throughout and by construction its solution is nodally exact. The final form is identical to applying the following tricks to the standard variational formulation:

i) Use one-point reduced integration on the shear energy term;

ii) Replace its coefficient $1/\varepsilon^2$ by $1/(\varepsilon^2 + (h^2/12))$ in each element;

iii) Correct the right-hand-side as in equation (5.4) for piecewise constant loads.

To emerge with these collection of "tricks" requires ingenuity and for the first two tricks different arguments have been given before by several authors (see references in [7], [8], [11]). We wish to point out that the residual-free bubbles point-of-view provides us with a systematic approach to construct discretization procedures that may shed some light on existing schemes and possibly improve them.

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