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A NEW TECHNIQUE FOR CONSTRUCTION OF IMAGE PYRAMIDS

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ABSTRACT

A systematic graph-theoretic approach to image pyramid generation is proposed in this article that does not assume any specific pixel geometry. It is argued that requirements for construction of image pyramids is very similar to the assumptions of multigrid convergence theory. Smoothed aggregation based basis functions are employed as pyramid generating kernels. Experimental results are shown to demonstrate the effectiveness of the the proposed approach.

1. INTRODUCTION

Image pyramids are data structures for representing image data obtained by an imaging sensor in multiple resolutions. Such data structures are motivated by both biological and computational evidences, and useful for multiresolution image processing and analysis, image compression, fusion and coding, as well as for the design of computer architecture. The generation of an image pyramid can be viewed as the application of a bank of low-pass filters with successively narrower bandwidths, followed by subsampling with wider steps. The design of a good pyramid generation process becomes the problem of finding an efficient and effective way to perform such filtering operation with finite-width kernels. In this article, we formulate the problem of construction of pyramids in the framework of regularity-free abstract multigrid convergence theory, and propose a systematic computational approach for building successive levels of pyramid without assuming any specific pixel geometry (shape or size).

Several different pyramidal implementations have been proposed in the literature. Perhaps, the most popular and well-known are the Gaussian and Laplacian pyramids [1]. The construction of Gaussian pyramids involves recursive application of a REDUCE function that can be implemented by a combination of low-pass filter (Gaussian-like impulse response) and sampling rate converter. The reverse of this mapping from coarse-level to fine-level is obtained by extrapolation onto a finer sampling grid. The filters are identical in both mappings. Laplacian pyramids are obtained by taking the difference between two successive levels in the Gaussian pyramid. The fine-level image can be obtained by summing the approximations of levels in Laplacian pyramid into the fine level. In recent years, wavelet–based and polynomial spline–based digital filters have been reported for construction of image pyramids, that offer certain advantages over the Gaussian and Laplacian pyramids.

In order to maintain compatibility between image processing operations performed at different levels in the pyramid, it is essential that fine-level approximation obtained from any coarse-level should be as close as possible to the original fine-level image. As pointed out in [6], the combination of Gaussian kernels (interpolation/prolongation) is not consistent with this requirement. On the other hand, wavelet and spline basis functions do satisfy this important requirement, but are essentially geometric in nature. It is not at all clear how their approximation properties are retained in the case of irregular pixel geometries, e.g., in log-polar foveated sensors, which are great candidates for real time active vision owing to low data rate. For such sensors, it is necessary to represent the image in a graph-theoretic fashion and generate pyramids without assuming any particular pixel geometry.

Almost all pyramid generating kernels are based on geometric considerations, e.g., unimodality, symmetry, equal contribution etc. Here, we put the problem of generation of image pyramid in the framework of regularity-free abstract multigrid convergence theory. Construction of pyramid can exploit the idea of coarse space generation in multigrid methods. It turns out that the requirements for building multiresolution copies of an image is very closely related to the assumptions of regularity-free abstract multigrid convergence theory presented in [2]. The proposed framework utilizes the concept of energy of an image in H^1-space. Now, a hierarchy of coarse spaces can be associated with different levels of the image pyramid. Smoothed aggregation is used to build the coarse space basis functions that meet the desired properties of the image pyramid. Coarse level images are obtained as L^2-projection from the fine space to the corresponding coarse space. This requires the solution of system of linear equations involving the so-called *mass matrix*. Mass matrix is well-conditioned and of order of the coarse-level image. Conjugate gradient method without preconditioning can be used to solve the system of equations in $O(n_1)$ operations, where n_1 is the number of pixels in the original image. Experimental results are included to demonstrate the effectiveness of this approach for image pyramid generation.

The rest of this paper is organized as follows. We recall some important definitions in section 2 and also introduce the concept of energy functional of an image. Mathematical

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requirements for building image pyramids are presented in section 3 and smoothed aggregation based image coarsening strategy is utilized for creating appropriate basis functions. Finally a new algebraic approach is proposed for generating image pyramids. Section 4 deals with experimental results. Conclusions of this study are summarized in section 5.

2. PRELIMINARIES

In this section we first present useful mathematical notations and definitions, and then introduce the idea of an energy functional of the image.

2.1. Notations

Some useful norms and inner products are defined in this subsection. On the vector space R^m, where m is positive integer, l^2 inner product and norms are defined as,

$$\langle x, y\rangle_{R^m} = \sum_{i=1}^{n} x_i y_i, \qquad \| x \|_{R^m} = \langle x, x\rangle_{R^m}^{1/2}$$

Let Ω be the domain where the image is defined. We recall definitions of L^2 inner product and norm, H^1 inner product, norm and semi-norm. In the following definitions, all functions x, and y are taken into account for which the corresponding integrals written below are defined. Derivatives are understood in distributive sense.

$$(x, y)_{L^2(\Omega)} = \int_\Omega xy d\Omega, \qquad \| x \|_{L^2(\Omega)} = (x, x)_{L^2(\Omega)}^{1/2},$$

$$(x, y)_{H^1(\Omega)} = \int_\Omega \nabla x \nabla y d\Omega, \qquad | x |_{H^1(\Omega)} = (x, x)_{H^1(\Omega)}^{1/2},$$

$$\| x \|_{H^1(\Omega)} = \| x \|_{L^2(\Omega)} + | x |_{H^1(\Omega)} .$$

Finally, L^2 and H^1 spaces are defined as,

$$L^2(\Omega) = \{x : \| x \|_{L^2(\Omega)} < \infty\},$$

$$H^1(\Omega) = \{x : \| x \|_{H^1(\Omega)} < \infty\}.$$

2.2. Energy of An Image

Let us represent an $M \times N$ image in the lexicographic form as a vector $\hat{v} \in R^{n_1}$, where its i-th entry $\hat{v}_i$ is the intensity at i-th pixel. So, R^{n_1} can be understood as the space of all admissible images.

We construct a triangular mesh by connecting any image pixel to its N, E and NE neighbors. We define the standard P^1-finite element (FE) space $V_1 \subset H^1(\Omega)$,

$$V_1 = span\{\phi_i^1\}_{i=1}^{n_1}$$

The basis function ϕ_i^1 is a continuous function, linear at each triangle, and satisfying

$$\phi_i^1(x_j, y_j) = \begin{cases} 1 & \text{iff } j = i \\ 0 & \text{otherwise} \end{cases}$$

where $(x_i, y_i)^T$ denotes the coordinates of the i-th pixel. The interpolation operator, $\Pi_1 : R^{n_1} \rightarrow V_1$ is defined in the usual way,

$$\Pi_1 \hat{v} = \sum_{i=1}^{n_1} \hat{v}_i \phi_i^1. \qquad (1)$$

The representation of the function $v \in V_1$ w.r.t. the basis $\{\phi_i^1\}_{i=1}^{n_1}$ is denoted by $\hat{v}$, i.e.,

$$v = \Pi_1 \hat{v}.$$

Let us define the relative *energy* of the image as,

$$E(v) = \frac{| v |_{H^1(\Omega)}^2}{\| v \|_{L^2(\Omega)}^2}. \qquad (2)$$

The numerator attains a high value for strong spatial variation of intensities in an image. The L^2 norm in the denominator assures the *scaling invariance*, i.e.,

$$E(tv) = E(v)$$

where $t \in R$. The numerator can be rewritten as,

$$| v |_{H^1(\Omega)}^2 = | \Pi_1 \hat{v} |_{H^1(\Omega)}^2 = \langle A_1 \hat{v}, \hat{v}\rangle_{R^{n_1}} \qquad (3)$$

where, $A_1 = \{a_{ij}^1\}_{i,j=1}^{n_1}$ is the *stiffness matrix*, with entries $a_{ij} = (\phi_i^1, \phi_j^1)_{H^1(\Omega)}$, and denominator as,

$$\| v \|_{L^2(\Omega)}^2 = \| \Pi_1 \hat{v} \|_{L^2(\Omega)}^2 = \langle M_1 \hat{v}, \hat{v}\rangle_{R^{n_1}} \qquad (4)$$

where, $M_1 = \{m_{ij}^1\}_{i,j=1}^{n_1}$ is the *mass matrix* with entries $m_{ij} = (\phi_i^1, \phi_j^1)_{L^2(\Omega)}$. Thus,

$$E(v) = \frac{\langle A_1 \hat{v}, \hat{v}\rangle_{R^{n_1}}}{\langle M_1 \hat{v}, \hat{v}\rangle_{R^{n_1}}}. \qquad (5)$$

The stiffness matrix A_1 is symmetric, positive semi-definite with stencil corresponding to 5-point scheme. It can be easily seen that entries, $a_{ii} = 4$ for interior pixel i, $a_{ii} = 3$ for non-corner boundary pixel i, and $a_{ii} = 2$ for corner pixel i. $a_{ij} = -1$ if pixels i and j are the four-connected neighbors. a_{ij}'s are zero otherwise.

The energy defined in (2) is related to the Fourier representation of images. Eigenvectors of the matrix A_1 are of the form $sin(\omega_x x_i) sin(\omega_y y_i)$ for a certain frequency-pair (ω_x, ω_y). The corresponding eigenvalues increase with (ω_x, ω_y). The energy of eigenvectors tends to increase with an increase in frequency (ω_x, ω_y).

3. CONSTRUCTION OF IMAGE PYRAMIDS

The image pyramid will be represented with a hierarchy of coarse spaces $V_h \equiv V_L \subset V_{L-1} \subset .... \subset V_1$, where, L is maximum number of levels in the pyramid. We require the space V_l be spanned by the basis $\{\phi_i^l\}_{i=1}^{n_l}$, whose characteristic diameter of support is h_l, i.e.,

$$ch_l \leq \text{diam}(\text{supp} \ \ \phi_i^l) \leq Ch_l,$$

where c and C are constants independent of the level. h_l is required to increase with the number of level l geometrically.

With each coarse space V_l, a coordinate space $\hat{V}_l = R^{n_l}$ is associated. The interpolator $\Pi_l : \hat{V}_l \rightarrow V_l$ is given by,

$$v \equiv \Pi_l \hat{v} = \sum_{i=1}^{n_l} \hat{v}_i \phi_i^l.$$

As with the full resolution level, coordinate vectors are denoted by $\hat{}$. The coarsening of the image from level 1 to level l will be performed by the linear mapping

$$Q_l : V_1 \rightarrow V_l.$$

For computational purpose, we will use discrete representation $\hat{Q}_l$ of Q_l defined by:

$$\hat{Q}_l \hat{u} = \hat{w} \Longleftrightarrow Qu = w, \quad \text{for all } u \in V_1$$

where

$$u = \Pi_1 \hat{u}, \qquad w = \Pi_l \hat{w}.$$

3.1. Requirements for Image Coarsening

Construction of image pyramids is, in essence, similar to the generation of coarse spaces for multigrid cycling. Following requirements are very closely related to assumptions of regularity-free multigrid convergence theory [2]:

I. Small Reconstruction Error: Vision psychology research confirms that human vision system is more sensitive to low-frequency information rather than the high frequency one. So, we require that small energy in full resolution image should be approximated well by its coarse level representation. This is assured by,

$$\| u - Q_l u \|_{L^2(\Omega)} \leq c h_l \mid u \mid_{H^1(\Omega)}, \quad u \in V_1. \quad (6)$$

In terms of energy of the image given by (2),

$$\frac{\| u - Q_l u \|^2_{L^2(\Omega)}}{\| u \|^2_{L^2(\Omega)}} \leq C h_l^2 E(u) \quad (7)$$

The inequality (6) is usually called *approximation property* in multigrid theory.

II. Smooth Low resolution Image: The energy of low-resolution copies of the full resolution image should be progressively smaller. This ensures that representation of images becomes smoother as we go up the image pyramid, and global image features get increasingly localized spatially. This requirement can be formulated mathematically as the so-called *H^1-inverse inequality*:

$$E(u) \leq \frac{C}{h_l^2} \quad \text{for } u \in V_l. \quad (8)$$

III. Bandpass decomposition: The difference of full resolution approximation obtained from two adjacent levels in the pyramid falls in a certain frequency range. More precisely, for $w \in \text{Rng}(Q_l - Q_{l+1})$,

$$\frac{c}{h_{l+1}^2} \leq E(w) \leq \frac{C}{h_l^2}. \quad (9)$$

This is equivalent to band-pass filtering of the image and ensures that difference of approximations between any two levels carries information in a certain frequency range.

IV. Small Computational Expenditure: For computational efficiency, the operation $\hat{Q}_l \hat{x}$ for $\hat{x} \in \hat{V}_1$ can be performed in $O(n_1)$, where n_1 is the number of pixels in the full resolution image.

The approximation property (6) leads to the choice of Q_l satisfying

$$\| u - Q_l u \|_{L^2(\Omega)} \leq \min_{w \in V_l} \| u - w \|_{L^2(\Omega)} . \quad (10)$$

It is well-known that such Q_l exists and is unique. This so-called *L^2-orthogonal projection* is usually in literature defined by

$$(u - Q_l u, w)_{L^2(\Omega)} = 0 \quad \text{for all } u \in V_l, \ w \in V_l. \quad (11)$$

It turns out that as a result of choosing Q_l as the orthogonal projection, the bandpass decomposition requirement is a consequence of the approximation property (6) and the inverse inequality (8).

In the next subsection, we present an image coarsening algorithm that utilizes coarse-space basis functions, satisfying the requirements **I–IV**.

3.2. Algorithm for Pyramid Construction

We present a new algorithm for constructing image pyramids based on the concept of *smoothed aggregation*, analyzed recently in [5] in the framework of algebraic multigrid method. Such a technique has also been used for developing a scalable algorithm for estimating discontinuous optical flow field [4]. Here an algorithm for successively generating low resolution copies of an image is proposed that utilizes coarse–space basis functions generated by smoothed aggregation. Images are treated as vectors in lexicographic form. The computational steps can be described as follows.

Create Stiffness Matrix: Create the fine-level (full resolution) stiffness matrix A_1, based on the definition of energy of an image $E(\hat{v})$. A_1 is typically a symmetric, positive definitive sparse matrix of order $N \times N$, where N is the number of pixels in full resolution.

Generation of Prolongators: Generate a hierarchy of prolongators $\{p_l\}_{l=1}^{L-1}$, where L is the maximum number of levels. p_l is a $n_l \times n_{l+1}$ matrix, whose columns are basis functions $\{\phi_i^l\}$. Stiffness matrices A_{l+1} corresponding to the $(l+1)$-th level in the image pyramid is obtained variationally

$$A_{l+1} = p_l^T A_l p_l. \quad (12)$$

Prolongators are obtained in two steps: (a) *Aggregation* to generate auxiliary prolongators, followed by (b) *smoothing* of auxiliary prolongators. Aggregation generates prolongators that satisfy the requirements of constant preservation, and the shape of supports of basis functions. However, these auxiliary prolongators do not satisfy the requirement of smoothness of coarse-level image vector. Smoothing is performed to ensure small energy of coarse-level image vectors. Unknown aggregation technique generates the disjoint covering of the set of nodes $\{C_i\}_{i=1}^m$, where each set C_i contains strongly coupled nodes only. Entries of the stiffness matrix provides the information about the coupling of nodes. The strongly coupled neighborhood, N_i^l of node i is defined as,

$$N_i^l = \{\ j :\ \ |a_{ij}| \geq \epsilon\ \sqrt{a_{ii}a_{jj}}\ \}.$$

If the stiffness matrix is diagonally dominant, it is more appropriate to use,

$$N_i^l = \{\ j :\ \ |a_{ij}| \geq \epsilon\ \sqrt{q_i q_j}\ \}.$$

where

$$q_i = \max_{k \neq i} |a_{ik}|, \quad q_j = \max_{k \neq j} |a_{jk}|,$$

and $\epsilon \in (0, 1)$. These sets contain only a few nodes. To each set C_i, one of its nodes (the so-called C-point) is assigned to play the role of coarse-level image point. The prolongator is built by simply copying the degree of freedom associated with the C-point to all nodes in the corresponding $\{C_i\}$. Thus, auxiliary prolongators can be expressed as,

$$\hat{p}_{ij} = \begin{cases} 1 & \text{if } i \in C_j \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

$\hat{p}$ is very sparse and contains one nonzero entry per row. Prolongators $\{p_l\}$ are obtained by smoothing the auxiliary $\{\hat{p}_l\}$ as following:

- Smooth the auxiliary prolongators:
  - Define the smoother

$$S_l = I - \omega D^{-1} A_l \tag{14}$$

where $D = \text{diag}(A_l)$, and compute the new prolongator p_l by

$$p_l = S_l \hat{p}_l. \tag{15}$$

Build Coarsening Operators: Compute prolongator between levels l and 1 as

$$P_l \leftarrow p_1 p_2 .. p_{l-1}, \quad R_l = P_l^T, \quad M_l = R_l P_l. \tag{16}$$

M is the so-called *mass matrix* of the order of the coarse image, and well-conditioned.

Generate Coarse Image: Compute

$$f^l = R_l x^1 \tag{17}$$

from fine-level image vector x^1, and obtain the coarse-level image vector x^l by

$$M_l x^l = f^l. \tag{18}$$

M_l is well-conditioned and conjugate gradient without preconditioning can be used to solve for x^l in less than $O(n_1)$ operations.

If A is chosen as the 5-point finite-difference discretization of Laplacian, the proposed aggregation based technique inherently supports 3×3 coarsening, rather than the conventional 2×2 coarsening. In order to maintain 2×2 coarsening between adjacent levels, it is sufficient to apply Jacobi smoothers only when building the prolongator matrix, while the restriction operator can be generated as,

$$R_l = (\hat{p}_1 \hat{p}_2 ... \hat{p}_{l-1})^T.$$

Fig. 1. 3×3 Smoothed aggregation based pyramid generation. (Top) Original image. (Bottom Left) First level approximation. (Bottom Right) Second level approximation.

	Aggregation 3×3	Aggregation 2×2	Gaussian	Cubic
Level 1	25.45	28.10	25.08	27.85
Level 2	20.45	24.17	21.37	23.89
Level 3		20.94	18.56	20.81

Table 1: Approximation errors (in dB) for different pyramid generating kernels.

4. EXPERIMENTAL RESULTS

Fig. 4(Top) shows the 256×256 original image of Lena. 3×3 smoothed aggregation based technique has been used to create a hierarchy of coarse images. Fine-level approximations obtained from first two levels are shown in Fig. 4(Bottom Left) and (Bottom Right), respectively, for $\omega = 0.85$. Note that for 3×3 aggregation, each coarse level image is $\frac{1}{9}$-th the size of finer level image. Fig. 4–4 show the results obtained using the proposed, Gaussian [1], and cubic fitting [3] based pyramid generation schemes. Approximation errors for different techniques are tabulated in Table I.

Note that the size of pyramid generating kernels in the proposed technique is a little bit smaller than the Gaussian kernels (twenty-one for the proposed compared to twenty-five for the Gaussian) and significantly smaller than cubic-fitting based technique. Moreover, the proposed technique is algebraic in nature, and is easily applicable to unstructured pixel topology.

All these results clearly establish the efficacy of the proposed pyramid generation technique.

5. CONCLUSIONS

An axiomatic graph-theoretic approach to image pyramid generation is proposed in this article based on the assumptions of algebraic multigrid theory. An energetic norm is defined for the image and smoothed aggregation based basis functions are generated which span a hierarchy of coarse spaces, corresponding to different pyramid levels. Experi-

Fig. 2. 2×2 Smoothed aggregation based pyramid generation. (Top) First level approximation. (Bottom Left) Second level approximation. (Bottom Right) Third level approximation.

Fig. 3. Gaussian pyramid generation. (Top) First level approximation. (Bottom Left) Second level approximation. (Bottom Right) Third level approximation.

Fig. 4. Cubic fitting based pyramid generation. (Top) First level approximation. (Bottom Left) Second level approximation. (Bottom Right) Third level approximation.

mental results with conventional pixel topology show that the proposed technique outperforms several existing methods in terms of closeness of fine-level approximation w.r.t. the original image.

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