MASS LUMPING EMANATING FROM RESIDUAL-FREE BUBBLES

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ABSTRACT. A nodally exact scheme is derived for a model equation in 1D involving zeroth and second order terms. The method is derived using residual-free bubbles in conjunction with the Galerkin approximation. It is shown that this approach leads to the mass lumping scheme for sufficiently small mesh sizes.

1. The residual-free bubbles approach.

The usage of the Galerkin method enriched with bubble functions has gained new impetus recently [1], [2]. In particular, the observation that this approach gives rise to streamline upwinding [2], [3], finally brought together two apparently distinct discretization procedures: namely, the employment of the standard Galerkin method with, possibly, more complex functions and the practice of combining the Galerkin method with least-squares like terms, viewed as upwind schemes. An abstract theory has been put together constructing a bridge between the stabilized methods and the Galerkin method using standard polynomial finite elements plus "virtual bubbles" [1]. Lately, a particular choice of bubble functions enabled to recover the one-dimensional nodally exact upwind scheme [3], [7], [8]. We will denote this choice by "residual-free" bubbles. The essential idea is that these bubbles are assumed to satisfy strongly the differential equations in each element subjected to homogeneous boundary conditions on the element boundary. Let us consider a boundary value problem given by

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma = \partial\Omega \end{cases} \tag{1.1}$$

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where $\mathcal{L}$ is a second order linear elliptic differential operator and f is a given function defined on Ω (we will omit technicalities, such as the limitations on the shape of Γ, more complicated boundary conditions, etc., so not to encumber the presentation). Assume that problem (1.1) can be given a classical variational formulation as follows:

$$\text{(1.2)} \qquad \begin{cases} \text{find } u \in V \text{ such that} \\ (\mathcal{L}u, v) = a(u, v) = (f, v) \quad \text{for all } v \in V \end{cases}$$

where $a(\cdot,\cdot)$ is a bilinear form on $V = H_0^1(\Omega)$ which is continuous and coercive with respect to the usual norm of V and $(\cdot,\cdot)$ is the usual scalar product in $L^2(\Omega)$. Let $V_h \subset V$ be a finite dimensional subspace of V; then the Galerkin approximation for problem (1.2) is

$$\text{(1.3)} \qquad \begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = (f, v_h) \quad \text{for all } v_h \in V_h. \end{cases}$$

The classical finite element method consists roughly in taking a partition $\mathcal{P}_h = \{K\}$ of Ω and in defining

$$\text{(1.4)} \qquad V_h = V_h^{(k)} = \{v_h \in C^0(\bar{\Omega}) : v_h|_K \text{ is a polynomial of degree} \leq k\}.$$

We wish to enrich the classical polynomial-based space $V_h^{(k)}$ by adding a set of bubbles, i.e. functions whose support is contained within one element. These bubbles will be selected in such a way that the differential equation is satisfied exactly in each element. For each $K \in \mathcal{P}_h$, let B_K be a finite dimensional subspace of $H_0^1(K)$ (to be determined later); the functions of B_K are to be thought as extended to zero outside K. Define then

$$\text{(1.5)} \qquad B = \bigoplus_{K \in \mathcal{P}_h} B_K$$

and

$$\text{(1.6)} \qquad V_h = V_h^{(k)} \oplus B$$

as the enlarged approximation space. Any function $v_h \in V_h$ can then be split in a unique way as

$$\text{(1.7)} \qquad v_h = v_k + v_b = v_k + \sum_{K \in \mathcal{P}_h} v_{b,K}$$

where $v_k \in V_h^{(k)}$, $v_b \in B$ and $v_{b,K} = v_{b|_K} \in B_K$. Using this representation, the Galerkin approximation of (1.2) on V_h can in turn be split as follows:

$$
\left\{
\begin{array}{l}
\text{find } u_h = u_k + u_b \in V_h = V_h^{(k)} \oplus B \text{ such that} \\
a\,(u_h, v_k) = (f, v_k) \quad \text{for all } v_k \in V_h^{(k)} \\
a\,(u_h, v_{b,K})_K = (f, v_{b,K})_K \quad \text{for all } K \in \mathcal{P}_h, \text{ and for all } v_{b,K} \in B_K
\end{array}
\right.
\tag{1.8}
$$

where the subscript K means that the integrals involved in $a\,(\cdot,\cdot)$ and $(\cdot,\cdot)$ are restricted to the element K. We have $u_{h|_K} = u_{k|_K} + u_{b,K}$ so that the second equation in (1.8) can be written as

$$
a\,(u_k + u_{b,K}, v_{b,K})_K = (f, v_{b,K})_K \text{ for all } v_{b,K} \in B_K
\tag{1.9}
$$

or, in "residual-free" form,

$$
a\,(u_{b,K}, v_{b,K})_K = -\underbrace{[a\,(u_k,\,\cdot\,)_K - (f,\,\cdot\,)_K]}_{residual \,\in H^{-1}(K)}(v_{b,K}) \quad \text{for all } v_{b,K} \in B_K.
\tag{1.10}
$$

The residual-free bubble space B_K is defined in such a way that in the element K the bubble part $u_{b,K}$ of the solution u_h of (1.8) satisfy equation (1.10) *for any test function* $v \in H_0^1(K)$:

$$
a\,(u_{b,K}, v)_K = -\left[a\,(u_k, v)_K - (f, v)_K\right] \quad \text{for all } v \in H_0^1(K).
\tag{1.11}
$$

In other words, $u_{b,K}$ satisfies *strongly* the following boundary value problem:

$$
\left\{
\begin{array}{ll}
\mathcal{L}u_{b,K} = -(\mathcal{L}u_k - f) & \text{in } K \\
u_{b,K} = 0 & \text{on } \partial K.
\end{array}
\right.
\tag{1.12}
$$

Problem (1.12) can in principle be solved in the following way. Let n_{en} be the number of polynomial degrees of freedom of the element K; then $u_{b,K}$ can be obtained as a linear combination of the solutions of the following auxiliary problems:
i) for each $i = 1, \ldots, n_{\text{en}}$ find the bubble shape functions $\phi_{i,K}$ such that

$$
\left\{
\begin{array}{ll}
\mathcal{L}\phi_{i,K} = -\mathcal{L}\psi_{i,K} & \text{in } K \\
\phi_{i,K} = 0 & \text{on } \partial K
\end{array}
\right.
\tag{1.13}
$$

where $\psi_{i,K}$ is the polynomial shape function associated with node i;

ii) find the bubble shape function $\phi_{f,K}$ such that

$$\begin{cases} \mathcal{L}\phi_{f,K} = f & \text{in } K \\ \phi_{f,K} = 0 & \text{on } \partial K. \end{cases} \tag{1.14}$$

Note that if $u_k|_K = \sum_{i=1}^{n_{\text{en}}} c_{i,K}\psi_{i,K}$, then by (1.13) and (1.14) we have

$$u_{b,K} = u_b|_K = \sum_{i=1}^{n_{\text{en}}} c_{i,K}\phi_{i,K} + \phi_{f,K} \tag{1.15}$$

with the same coefficients $c_{i,K}$'s. Thus by (1.12) we have the following representation:

$$u_h|_K = \sum_{i=1}^{n_{\text{en}}} c_{i,K}(\psi_{i,K} + \phi_{i,K}) + \phi_{f,K}. \tag{1.16}$$

It should be clear at this point that if we go back and define the bubble space B_K as

$$B_K = \text{span}\,\{\phi_{1,K}, \dots, \phi_{n_{\text{en}},K}, \phi_{f,K}\} \tag{1.17}$$

and we solve problem (1.8), then the bubble part $u_{b,K}$ of the solution u_h also solves problem (1.11). Now we can eliminate the residual-free bubbles in (1.8) to obtain an equation involving only u_k. Since the bubbles are residual-free, the second equation in (1.8) is automatically satisfied, and we are led to the problem on $V_h^{(k)}$

$$\begin{cases} \text{find } u_k \in V_h^{(k)} \text{ such that} \\ a(u_k, v_k) + \underbrace{\sum_{K \in \mathcal{P}_h} a(u_{b,K}, v_k)_K}_{\textit{bubble modification}} = (f, v_k) \quad \text{for all } v_k \in V_h^{(k)} \end{cases} \tag{1.18}$$

where $u_{b,K}$ is given by (1.15) in terms of the unknown u_k. Equation (1.18) highlights the fact that the procedure of defining the residual-free bubbles and then eliminating them leads to a (consistent!) modification of the classical polynomial-based finite element methods.

Sometimes, the computation of $u_{b,K}$ in (1.18) could be as difficult as the original problem itself. Hence, in these cases, in order to make effective use of the residual-free bubbles approach, an approximation of $u_{b,K}$ has to be found (see e.g. [4], [5]). In this work we examine the numerical scheme (1.18) for a model problem in one dimension involving a zeroth and a second order differential operator. In this particular situation, we can compute explicitly $u_{b,K}$; we show that in the limit when the mesh is refined, the numerical method coincides with the mass lumping scheme. As far as we are aware of, this is the first derivation of mass lumping from the Galerkin method without any tricks, by simpling enriching it with special functions (residual-free bubbles) and computing integrals exactly. In our viewpoint this brings more legitimacy to a scheme viewed as a variational crime heretofore.

2. "Legal" mass lumping.

The one-dimensional model problem we wish to solve is given by:

$$\begin{cases} \sigma u - \kappa u'' = f & \text{in } [0, L] \\ u(0) = 0 & \\ u(L) = 0 & \end{cases} \tag{2.1}$$

where σ and κ are positive and smooth functions on $[0, L]$ and $f \in L^2(0, L)$. Let $\mathcal{P}_h = \{K\}$ be a decomposition of $[0, L]$; we will employ continuous, piecewise linear elements. In this case we have $n_{\text{en}} = 2$ and in each element K we need to solve for $i = 1, 2$ (cf. eqs. (1.12), (1.13)):

$$\begin{cases} \sigma\phi_{i,K} - \kappa\phi''_{i,K} = -\sigma\psi_{i,K} & \text{in } K \\ \phi_{i,K} = 0 & \text{on } \partial K \end{cases} \tag{2.2}$$

and

$$\begin{cases} \sigma\phi_{f,K} - \kappa\phi''_{f,K} = f & \text{in } K \\ \phi_{f,K} = 0 & \text{on } \partial K \end{cases} \tag{2.3}$$

where $\psi_{1,K}$ and $\psi_{2,K}$ are sketched in Figure 1; we used the fact that $\psi''_{i,K} \equiv 0$ in K.

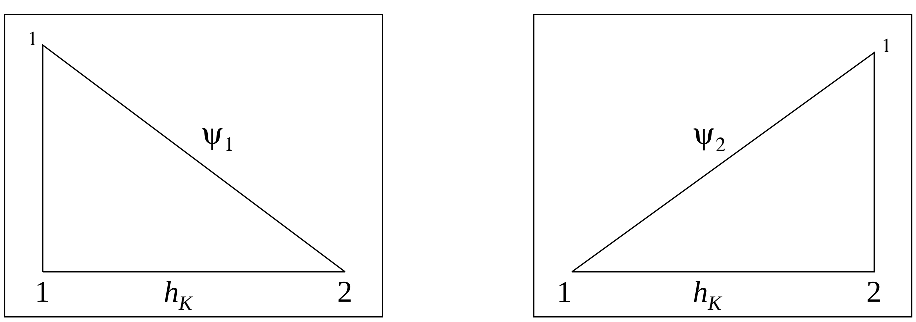

Figure 1: The linear basis functions ψ_1 and ψ_2 in an element K.

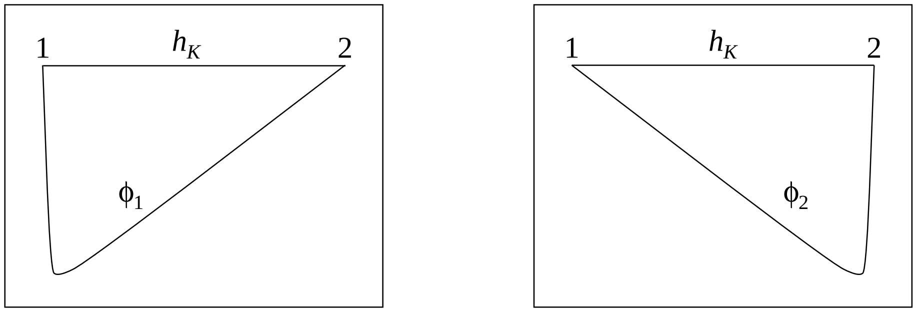

Figure 2: The residual-free bubble shape functions $\phi_{1,K}$ and $\phi_{2,K}$ for $\kappa \ll \sigma$.

From now on, we will assume that σ, κ and f are piecewise constants on $\mathcal{P}_h$. Hence, $\phi_{f,K}$ is a linear combination of $\phi_{1,K}$ and $\phi_{2,K}$ so that $\dim B_K = 2$. In the case when $\kappa \ll \sigma$ (the numerically difficult case) the solutions of (2.2) for $i = 1, 2$ are sketched in Figure 2.

To solve (2.2) analytically, we make the substitution

$$\phi_{i,K} = \lambda_{i,K} - \psi_{i,K} \tag{2.4}$$

into (2.2) to get

$$\begin{cases} \sigma\lambda_{i,K} - \kappa\lambda''_{i,K} = 0 & \text{in } K \\ \lambda_{i,K} = \psi_{i,K} & \text{on } \partial K \end{cases} \tag{2.5}$$

which can be rewritten with respect to a local coordinate $\xi \in [0, h_K]$ into the problems

$$\begin{cases} \sigma\lambda_1 - \kappa\lambda_{1,\xi\xi} = 0 & \text{for } \xi \in [0, h_K] \\ \lambda_1(0) = 1 \\ \lambda_1(h_K) = 0 \end{cases} \tag{2.6}$$

$$\begin{cases} \sigma\lambda_2 - \kappa\lambda_{2,\xi\xi} = 0 & \text{for } \xi \in [0, h_K] \\ \lambda_2(0) = 0 \\ \lambda_2(h_K) = 1. \end{cases} \tag{2.7}$$

The solutions of (2.6) and (2.7) are given by

$$\lambda_1(\xi) = \frac{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,(h_K - \xi)\right)}{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h_K\right)}$$

(2.8)

$$\lambda_2(\xi) = \frac{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,\xi\right)}{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h_K\right)}.$$

For $\kappa \ll \sigma$, λ_1 and λ_2 are sketched in Figure 3. The solution of (2.3) in the local coordinate ξ can be written as

$$\text{(2.9)} \qquad \phi_f(\xi) = \frac{f}{\sigma}\left[1 - \frac{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,(h_K - \xi)\right) + \sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,\xi\right)}{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h_K\right)}\right]$$

$$= \frac{f}{\sigma}\,[\,1 - (\,\lambda_1(\xi) + \lambda_2(\xi)\,)\,] = -\frac{f}{\sigma}\,[\,\phi_1(\xi) + \phi_2(\xi)\,]\,.$$

.

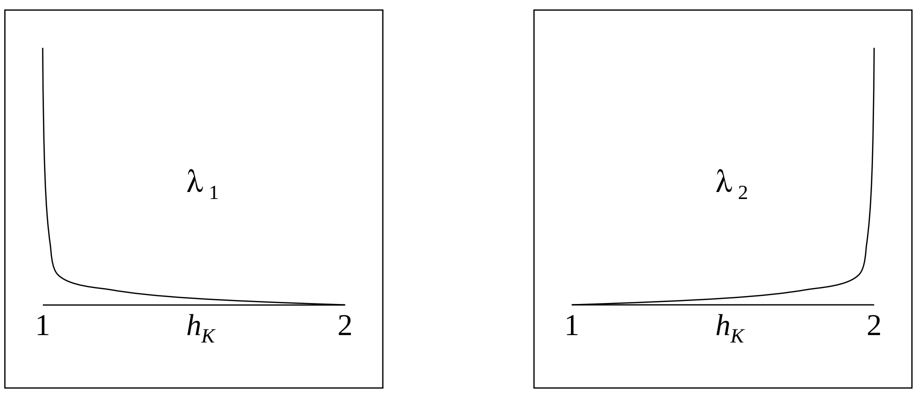

Figure 3: The layer basis functions λ_1 and λ_2 for $\kappa \ll \sigma$

The Galerkin method given by eq. (1.18) for this model problem is

$$\text{(2.10)} \qquad (\sigma u_1, v_1) + (\kappa u_1', v_1') + \sum_{K \in \mathcal{P}_h} (\sigma u_{b,K}, v_1)_K = (f, v_1)$$

or rewriting the first two terms as a sum over all elements $K \in \mathcal{P}_h$ and substituting u_1 and $u_{b,K}$ by their linear combination with respect to their basis functions (cf. eq.

(1.15)):

$$\sum_{K\in\mathcal{P}_h}\sum_{i=1}^{n_{\rm en}} c_{i,K}\left\{(\sigma\psi_{i,K},v_1)_K+(\kappa\psi'_{i,K},v'_1)_K+(\sigma\phi_{i,K},v_1)_K\right\} = (f,v)-\sum_{K\in\mathcal{P}_h}(\sigma\phi_{f,K},v_1)_K. \tag{2.11}$$

Substituting $\phi_{i,K}=\lambda_{i,K}-\psi_{i,K}$ (cf. eq. (2.4)) we have:

$$\sum_{K\in\mathcal{P}_h}\sum_{i=1}^{n_{\rm en}} c_{i,K}\left\{(\kappa\psi'_{i,K},v'_1)_K+(\sigma\lambda_{i,K},v_1)_K\right\}=(f,v)-\sum_{K\in\mathcal{P}_h}(\sigma\phi_{f,K},v_1)_K. \tag{2.12}$$

Equation (2.12) gives rise to the matrix formulation by selecting $v_1=\psi_I$, the global piecewise linear basis function (see Figure 4), for $I=1\ldots,N$ where N is the number of interior points in the mesh. To illustrate the implication of this formulation, let us compute the typical interior node equation in the case of globally constant σ, κ, f and of a uniform mesh. Thus, take $v_1=\psi_I$ in (2.12) to obtain

$$\begin{aligned}\kappa\left((\psi'_{I-1},\psi'_I)\,c_{I-1}+(\psi'_I,\psi'_I)\,c_I+(\psi'_{I+1},\psi'_I)\,c_{I+1}\right)+\\ \sigma\left((\lambda_{I-1},\psi_I)\,c_{I-1}+(\lambda_I,\psi_I)\,c_I+(\lambda_{I+1},\psi_I)\,c_{I+1}\right)=(f,\psi_I)-\sigma(\phi_{f,I},\psi_I)\end{aligned} \tag{2.13}$$

where λ_I is defined by gluing together λ_1 and λ_2 as in Figure 4 and

$$\phi_{f,I}=\frac{f}{\sigma}\left(1-(\lambda_{I-1}+\lambda_I+\lambda_{I+1})\right). \tag{2.14}$$

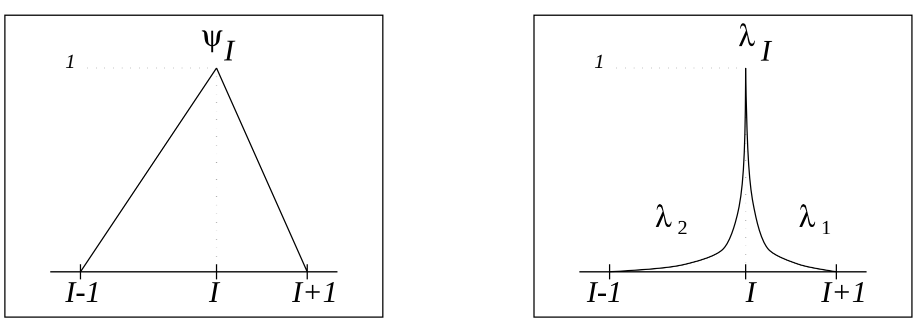

Figure 4: The global piecewise linear basis function ψ_I and the function λ_I for $\kappa\ll\sigma$

Inserting (2.14) in (2.13), we have the final scheme:

$$\kappa\left((\psi'_{I-1},\psi'_I)\ c_{I-1} + (\psi'_I,\psi'_I)\ c_I + (\psi'_{I+1},\psi'_I)\ c_{I+1}\right) + \sigma\left((\lambda_{I-1},\psi_I)\left(c_{I-1}-\frac{f}{\sigma}\right) + (\lambda_I,\psi_I)\left(c_I-\frac{f}{\sigma}\right) + (\lambda_{I+1},\psi_I)\left(c_{I+1}-\frac{f}{\sigma}\right)\right) = 0 \tag{2.15}$$

Computing exactly the integrals in (2.15) we get

$$\frac{\kappa}{h}\left[-c_{I-1}+2c_I-c_{I+1}\right] + \frac{\sigma}{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right)}\left[2\left(\frac{\cosh\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right)}{\sqrt{\frac{\sigma}{\kappa}}} - \frac{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right)}{\frac{\sigma}{\kappa}\,h}\right)\left(c_I-\frac{f}{\sigma}\right) + \left(\frac{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right)}{\frac{\sigma}{\kappa}\,h} - \frac{1}{\sqrt{\frac{\sigma}{\kappa}}}\right)\left(c_{I-1}+c_{I+1}-2\frac{f}{\sigma}\right)\right] = 0. \tag{2.16}$$

Note that the terms in the first square brackets cancel with some of the terms in the second square brackets, yielding:

$$\frac{\sqrt{\sigma\kappa}}{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right)}\left[2\left(c_I-\frac{f}{\sigma}\right)\cosh\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right) - c_{I-1} - c_{I+1} + 2\,\frac{f}{\sigma}\right] = 0. \tag{2.17}$$

The last formula can be rearranged in a more illuminating way as follows:

$$\mathcal{A}_h\left[\frac{-c_{I-1}+2c_I-c_{I+1}}{h}\right] + \mathcal{B}_h c_I = \mathcal{C}_h f \tag{2.18}$$

where

$$\mathcal{A}_h = \kappa\,\frac{\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right)}{\sinh\left(\sqrt{\frac{\sigma}{\kappa}}\,h\right)},\quad \mathcal{B}_h = 2\sqrt{\sigma\kappa}\,\tanh\left(\frac{1}{2}\sqrt{\frac{\sigma}{\kappa}}\,h\right),\quad \mathcal{C}_h = \frac{1}{\sigma}\,\mathcal{B}_h. \tag{2.19}$$

This is the method implied by the residual-free bubbles viewpoint. Equation (2.18) gives the exact solution for all values of κ, σ, f and h. For small $\sqrt{\frac{\sigma}{\kappa}}\,h$ (when h is sufficiently small) we have

$$\mathcal{A}_h \approx \kappa,\quad \mathcal{B}_h \approx \sigma h,\quad \mathcal{C}_h \approx h \tag{2.20}$$

which simplifies (2.18) to

$$\kappa\left[\frac{-c_{I-1}+2c_I-c_{I+1}}{h}\right]+\sigma h\,c_I=hf \tag{2.21}$$

which is form-identical to the method obtained from the classical variational formulation using full integration on the second order term and mass lumping on the zeroth order term. On the other hand, if we keep h fixed and we let κ going to zero, then from (2.19) we see that the resulting equation is

$$2\sqrt{\sigma\kappa}\,c_I \approx 2\sqrt{\frac{\kappa}{\sigma}}f, \tag{2.22}$$

which simplifies to

$$\sigma c_I \approx f. \tag{2.23}$$

We then have that as κ goes to zero, the solution of (2.19) and the solution obtained with mass lumping tend to the same limit.

Remarks

1) By construction, the method derived by the residual-free bubbles is nodally exact. The reason is that we are able to solve exactly for the bubble shape functions given by eqs. (2.2) and (2.3) and therefore the decomposition given by (1.16) reproduces the exact solution everywhere, in particular, it gives u at each node which equals to the nodal value of u_1.
2) We wish to emphasize that mass lumping is only nodally exact in the limit as $h \to 0$, otherwise one should employ the exact scheme (or "legal mass lumping") emanating from residual-free bubbles.
3) Using static condensation Franca and Farhat [4] have devised an unusual stabilized method that does not employ "mass lumping" in the zeroth order term. The numerical results using that scheme are more diffusive than simply using mass lumping.
4) The residual-free bubbles approach for the Timoshenko beam problem is discussed in [6]. Further application of this technique will be the object of future work.

3. A numerical experiment.

We show in this Section a numerical experiment which demonstrates the difference between the schemes previously discussed. Referring to eq. (2.1), we take $L = 1$, $\kappa = 10^{-3}$, $f = 1$, a uniform partition with $h = 0.1$, and compare in Fig. 5 the nodally

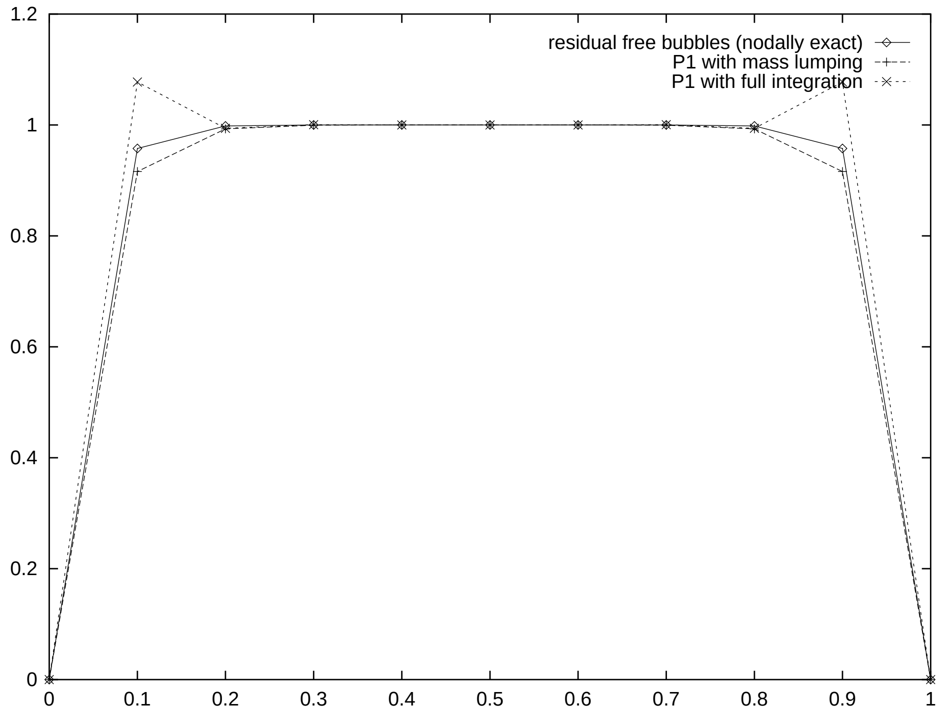

Figure 5: A test case with $L = 1$, $\sigma = 1$, $\kappa = 10^{-3}$ *and* $h = 0.1$

exact residual-free bubbles solution (continuous line) with the Galerkin method with piecewise linears and mass lumping (dashed), and full integration (dotted).

References

[**1**] C. Baiocchi, F. Brezzi and L. P. Franca, *Virtual bubbles and the Galerkin-least-squares method*, Comput. Meths. Appl. Mech. Engrg. **105** (1993), 125-141.

[**2**] F. Brezzi, M. O. Bristeau, L. P. Franca, M. Mallet and G. Roge, *A relationship between stabilized finite element methods and the Galerkin method with bubble functions*, Comput. Meths. Appl. Mech. Engrg. **96** (1992), 117-129.

[**3**] F. Brezzi and A. Russo, *Choosing bubbles for advection-diffusion problems*, Math. Models Meths. Appl. Sci. **4** (1994), 571-587.

[**4**] L. P. Franca and C. Farhat, *Bubble functions prompt unusual stabilized finite element methods*, Comput. Meths. Appl. Mech. Engrg. **123** (1995), 299-308.

[**5**] L. P. Franca and A. Russo, *Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles*, to appear in Appl. Math. Lett., Tech. Rep. 975, Istituto di Analisi Numerica del CNR, Pavia, 1995.

[**6**] L. P. Franca and A. Russo, *Unlocking with residual-free bubbles*, submitted to Comput. Meths. Appl. Mech. Engrg., Tech. Rep. 977, Istituto di Analisi Numerica del CNR, Pavia, 1995.

[**7**] A. Russo, *Bubble stabilization of finite element methods for the linearized incompressible Navier-Stokes equations*, to appear in Comput. Meths. Appl. Mech. Engrg., Tech. Rep. 930, Istituto di Analisi Numerica del CNR, Pavia, 1994.

[**8**] A. Russo, *Residual-free bubbles and stabilized methods*, to appear in Proceedings of the IX International Conference on Finite Element in Fluids, Venice 1995.