Unlocking with Residual-Free Bubbles

by

Leopoldo P. Franca
Department of Mathematics
University of Colorado at Denver
P.O.Box 173364, Campus Box 170
Denver, CO 80217-3364

Alessandro Russo
Istituto di Analisi Numerica del CNR
Via Abbiategrasso, 209
27100 Pavia, Italy

Abstract

Residual-free bubbles are derived for the Timoshenko beam problem. Eliminating these bubbles the resulting formulation is form-identical to using the following tricks to the standard variational formulation: i) one-point reduced integration on the shear energy term; ii) replace its coefficient $1/\epsilon^2$ by $1/(\epsilon^2 + (h_K^2/12))$ in each element; iii) modify consistently the right-hand-side. This final formulation is 'legally' obtained in that the Galerkin method enriched with residual-free bubbles is developed using full integration throughout. Furthermore this method is nodally exact by construction.

Submitted to:
Computer Methods in Applied Mechanics and Engineering

Preprint
July 1995

1. INTRODUCTION

The deflection of a beam taking into account bending and shear deformations is described by the Timoshenko model. Standard Galerkin finite element method using equal-order piecewise linear approximations for the unknown dependent variables rotation (θ) and displacement (w) yields "locking" and spurious oscilations for the shear forces. Selective reduced integration has been suggested to cure some of these pathologies and has been justified resorting to an equivalent mixed variational formulation [1,8].

In this work we enrich the standard piecewise linears with residual-free bubbles [2-4] and show that the Galerkin method without tricks (using full integration) produces selective reduced integration with a coefficient for the shear term that is form-identical to the "residual bending flexibility" coefficient suggested by MacNeal [7]. The right-hand-side load terms get some correction as well. The final formulation is nodally exact, a result which has been achieved before with an unrelated idea [6].

2. RESIDUAL FREE BUBBLE FORMULATION

The Timoshenko beam model is governed by the following differential equations (after non-dimensionalization – e.g., see [1,6]):

$$\begin{aligned} -\theta'' - \frac{1}{\epsilon^2}(w' - \theta) &= 0 \qquad \text{in } \Omega \\ -\frac{1}{\epsilon^2}(w'' - \theta') &= f \qquad \text{in } \Omega \end{aligned} \tag{1}$$

where prime denotes differentiation with respect to $x \in \Omega = (0,1)$, θ and w are the rotation and displacement variables, f is the load and ϵ is a non-dimensional parameter proportional to the beam thickness.

To (1) we append the following clamped boundary conditions (other boundary conditions may be used without major changes in what follows):

$$\begin{aligned} w(0) = w(1) = 0 \\ \theta(0) = \theta(1) = 0\,. \end{aligned} \tag{2}$$

The variational formulation corresponding to (1)-(2) is given by: Find $\{\theta, w\} \in H_0^1(\Omega)^2$ such that

$$(\theta', \psi') + \frac{1}{\epsilon^2}(w' - \theta, v' - \psi) = (f, v) \qquad \forall\{\psi, v\} \in H_0^1(\Omega)^2 \tag{3}$$

where $H_0^1(\Omega)$ is the Hilbert space of functions with square-integrable value and derivative in Ω satisfying (2) and we use the notation $(f, g) = \int_\Omega fg\, d\Omega$.

Consider a partition of Ω into non-overlapping elements in the usual way. Then the exact solution of our problem can be decomposed into:

$$\begin{aligned} \theta = \theta_1 + \theta_b \\ w = w_1 + w_b \end{aligned} \tag{4}$$

where θ_1 and w_1 are spanned by the standard continuous piecewise linears of finite element methods, and θ_b and w_b are assumed to satisfy the following differential equations in each element K:

$$\begin{aligned} -\theta_b'' - \frac{1}{\epsilon^2}(w_b' - \theta_b) &= -\big(-\theta_1'' - \frac{1}{\epsilon^2}(w_1' - \theta_1)\big) \\ -\frac{1}{\epsilon^2}(w_b'' - \theta_b') &= -\big(-\frac{1}{\epsilon^2}(w_1'' - \theta_1') - f\big) \end{aligned} \tag{5}$$

and subjected to the boundary conditions:

$$\theta_b = w_b = 0 \qquad \text{on } \partial K\,. \tag{6}$$

Equations (5) can be rewritten as (note that $\theta_1'' = w_1'' = 0$ in K):

$$\begin{aligned} -\epsilon^2\theta_b'' + \theta_b - w_b' &= w_1' - \theta_1 \\ \theta_b' - w_b'' &= -\theta_1' + \epsilon^2 f \,. \end{aligned} \tag{7}$$

From $(7)_1$

$$\theta_b - w_b' = w_1' - \theta_1 + \epsilon^2\theta_b''$$

and combining with $(7)_2$ we get

$$\theta_b''' = f \qquad \text{in } K \,. \tag{8}$$

Integrating three times (with respect to the local variable in the element, $\xi \in [0, h_K], h_K = x_{i+1} - x_i, \xi = x - x_i$) and assuming piecewise constant load f, and for notation's sake dropping the subscripts for h and f (nowhere we need to assume that h_K is constant in what follows) we get:

$$\theta_b(\xi) = \frac{\xi^3}{6} f + c_1\frac{\xi^2}{2} + c_2\xi + c_3 \,. \tag{9}$$

Applying the boundary conditions $\theta_b(0) = \theta_b(h) = 0$ above gives:

$$\theta_b(\xi) = \frac{\xi}{6} f(\xi^2 - h^2) + c_1\frac{\xi}{2}(\xi - h)\,. \tag{10}$$

Using this expression into the first equation of (7) after one integration we get:

$$\begin{aligned} w_b(\xi) = \int_0^\xi \theta_1(t)\,dt - w_1(\xi) - \epsilon^2\left[\frac{f}{6}(3\xi^2 - h^2) + \frac{c_1}{2}(2\xi - h)\right] + \frac{f}{6}\left[\frac{\xi^4}{4} - \frac{\xi^2}{2}h^2\right] \\ - \frac{c_1}{12}\xi^2(3h - 2\xi) + c_4 \,. \end{aligned} \tag{11}$$

Applying the boundary conditions $w_b(0) = w_b(h) = 0$ in (11) we get expressions for the remaining constants c_1 and c_4 and the expressions for the residual-free bubble functions are then given by:

$$\begin{aligned} \theta_b(\xi) = f\left\{\frac{\xi}{6}(\xi^2 - h^2) + \frac{h\xi}{4}(h - \xi)\right\} \\ + \frac{1}{\epsilon^2 + \frac{h^2}{12}}\frac{\xi(\xi - h)}{2}\left\{\theta_1(\frac{h}{2}) - \frac{w_1(h) - w_1(0)}{h}\right\} \,, \end{aligned} \tag{12}$$

$$
\begin{aligned}
w_b(\xi) = {} & \xi\left(1-\frac{\xi}{2h}\right)\theta_1(0)+\frac{\xi^2}{2h}\theta_1(h)+\frac{\xi}{h}[w_1(0)-w_1(h)] \\
& -\xi\left[\epsilon^2-\frac{\xi^2}{6}+\frac{h\xi}{4}\right]\left\{\frac{1}{\epsilon^2+\frac{h^2}{12}}\left[\theta_1(\frac{h}{2})-\frac{w_1(h)-w_1(0)}{h}\right]-\frac{hf}{2}\right\} \\
& +\frac{f\xi^2}{2}\left[-\epsilon^2+\frac{\xi^2}{12}-\frac{h^2}{6}\right].
\end{aligned}
\tag{13}
$$

If we take the test functions $\psi=\psi_1$ and $v=v_1$, where ψ_1 and v_1 are spanned by continuous piecewise linears, then using decomposition (4) the variational formulation (3) can be rewritten as

$$
(\theta_1',\psi_1')+\frac{1}{\epsilon^2}(w_1'-\theta_1,v_1'-\psi_1)-(f,v_1)+\frac{1}{\epsilon^2}(w_b'-\theta_b,v_1'-\psi_1)=0 \tag{14}
$$

where, by integration-by-parts, we used that:

$$
(\theta_b',\psi_1')=\sum_K(\theta_b',\psi_1')_K=\sum_K\left[(\theta_b,\psi_1')_{\partial K}-(\theta_b,\psi_1'')_K\right]=0\,.
$$

Note that (14) consists of the Galerkin method for equal-order piecewise linear approximations for θ and w (without tricks, using full integration) plus a "perturbation term" that we need to compute based on the bubble functions given by (12) and (13). First by (12) and (13) we compute:

$$
\begin{aligned}
w_b'-\theta_b = {} & \theta_1(0)+\frac{\xi}{h}[\theta_1(h)-\theta_1(0)]-\frac{w_1(h)-w_1(0)}{h} \\
& -\frac{\epsilon^2}{\epsilon^2+\frac{h^2}{12}}\left[\theta_1(\frac{h}{2})-\frac{w_1(h)-w_1(0)}{h}\right]+\epsilon^2 f\left(\frac{h}{2}-\xi\right).
\end{aligned}
\tag{15}
$$

Note also that

$$
w_1'-\theta_1=\frac{w_1(h)-w_1(0)}{h}-\left(1-\frac{\xi}{h}\right)\theta_1(0)-\frac{\xi}{h}\theta_1(h)\,. \tag{16}
$$

Thus summing (15) to (16)

$$
w_1'-\theta_1+w_b'-\theta_b=\epsilon^2 f\left(\frac{h}{2}-\xi\right)-\frac{\epsilon^2}{\epsilon^2+\frac{h^2}{12}}\left[\theta_1(\frac{h}{2})-\frac{w_1(h)-w_1(0)}{h}\right]. \tag{17}
$$

Therefore, using (17), the variational formulation given by (14) reduces to

$$(\theta_1', \psi_1') + \sum_K \frac{1}{\epsilon^2 + \frac{h_K^2}{12}} \left(\frac{w_1(h_K) - w_1(0)}{h_K} - \theta_1(\frac{h_K}{2}), v_1' - \psi_1 \right)_K = (f, v_1) + \sum_K f_K(\xi - \frac{h_K}{2}, v_1' - \psi_1)_K \quad (18)$$

where we reintroduced the subscripts for h and the piecewise constant load f. This can also be rewritten as

$$(\theta_1', \psi_1') + \sum_K \frac{1}{\epsilon^2 + \frac{h_K^2}{12}} (w_1' - R\theta_1, v_1' - \psi_1)_K = (f, v_1) + \sum_K f_K(\xi - \frac{h_K}{2}, v_1' - \psi_1)_K \quad (19)$$

where R stands for a reduced integration operator.

Formulation (19) was *derived* using full integration throughout and by construction its solution is nodally exact. The final form is identical to applying the following tricks to the standard variational formulation:

i) Use one-point reduced integration on the shear energy term;

ii) Replace its coefficient $1/\epsilon^2$ by $1/(\epsilon^2 + (h_K^2/12))$ in each element;

iii) Correct the right-hand-side as in equation (19) for piecewise-constant loads.

To emerge with these collection of “tricks” requires ingenuity and for the first two tricks different arguments have been given before by several authors (see references in [5,7,9]). We wish to point out that the residual-free bubbles point-of-view provides us with a systematic approach to construct discretization procedures that may shed some light on existing schemes and possibly improve them.

ACKNOWLEDGMENT

During the course of this work Leopoldo P. Franca has been partially supported by the Istituto di Analisi Numerica del CNR, Pavia, Italy.

REFERENCES

[1] D.N. Arnold, "Discretization by finite elements of a model parameter dependent problem," *Numerische Mathematik* Vol.**37** (1981) 405-421.

[2] F. Brezzi and A. Russo, "Choosing bubbles for advection-diffusion problems," *Math. Models Meths. Appl. Sci.* Vol.**4** (1994) 571-587.

[3] L.P. Franca and A. Russo, "Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles," *Applied Mathematics Letters*, to appear.

[4] L.P. Franca and A. Russo, "Mass lumping emanating from residual-free bubbles," *Computer Methods in Applied Mechanics and Engineering*, submitted.

[5] T.J.R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

[6] A.F.D. Loula, T.J.R. Hughes and L.P. Franca, "Petrov-Galerkin formulations of the Timoshenko beam problem," *Computer Methods in Applied Mechanics and Engineering* 1987.

[7] R.H. MacNeal, *Finite Elements: Their Design and Performance*, Marcel Dekker Inc., New York, 1994.

[8] D.S. Malkus and T.J.R. Hughes, "Mixed finite element methods — reduced and selective integration techniques: a unification of concepts," *Computer Methods in Applied Mechanics and Engineering*, Vol.**15** (1978) 63-81.

[9] O.C. Zienkiewicz and R.L. Taylor, *The Finite Element Method*, 4th ed., McGraw-Hill, London, New York, 1989.