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# THE FINITE VOLUME ELEMENT METHOD FOR ELLIPTIC AND PARABOLIC EQUATIONS

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ABSTRACT

An error analysis of the finite volume element method for elliptic and parabolic partial differential equations is presented. Existing results apply to discretizations of steady diffusion equations by linear finite elements. These results are extended to steady and transient advection–reaction–diffusion equations and are generalized to polynomial finite elements of arbitrary order. Optimal-order error estimates are derived in a discrete $H^1$ norm, under minimal regularity assumptions for the exact solution, the finite element triangulation, and the finite volume construction. With additional uniformity assumptions for the finite volumes, $H^1$ superconvergence results are obtained for linear finite elements.

## 1. Introduction

This paper presents an error analysis of the finite volume element method (FVE) for elliptic and parabolic partial differential equations. Full details of this analysis are given in the thesis of Trujillo[19]. As introduced by Baliga and Patankar[1] and elaborated by McCormick[15] and in the references cited therein, we can view FVE as a combination of the standard finite volume method (FV), also known as cell-centered finite differences (see Mitchell and Griffiths[13] for details), and the standard Galerkin finite element method (FE) (see Ciarlet[6] for details). Although a large body of theoretical infrastructure and of results for elliptic equations exists for both FV and FE, analysis for FVE is limited to the foundation laid by Cai and McCormick[3], Cai *et al.*[4], and culminating in Cai[5] for the numerical solution of steady diffusion equations by linear finite elements.

Starting from this foundation, we extend and generalize FVE error analysis to steady and transient advection–reaction–diffusion equations and to polynomial finite elements of arbitrary order. To construct an error analysis for FVE, we are guided by the large body of FE theory as contained in Ciarlet[6] and in references cited therein, and by modern FV error analysis based on FE-style arguments: e.g., Bank and Rose[2], Ewing *et al.*[8], Hackbusch[9], Heinrich[10], Herbin[11], Lazarov *et al.*[12], Morton and Süli[14], Samarskii *et al.*[17], Süli[18], and Weiser and Wheeler[20].

By adapting FV and FE arguments to fit the context of FVE and developing some new arguments unique to FVE, we derive optimal-order error estimates for FVE in a discrete $H^1$ norm for polynomial finite elements of arbitrary order—under minimal regularity assumptions for the exact solution, the finite element triangulation, and the finite volume construction. With additional uniformity assumptions for the finite volumes, $H^1$ superconvergence results are obtained for linear finite elements.

The remainder of the paper is organized as follows. In Section 2, we outline how FVE arises from the approximation of certain integral conservation laws that lead to elliptic partial differential equations. In Section 3, we define and discuss the computational meshes (i.e., the FE triangulation $\mathcal{T}^h$, the FVE primal mesh $\mathcal{T}^{h/k}$, and the FVE dual mesh $\mathcal{V}^{h/k}$) that are fundamental to the implementation of FVE. In Section 4, we briefly outline an FE analysis for the steady diffusion equation to set a reference point for an FVE analysis in Sections 5–10. In Section 11, we note modifications for an FVE analysis for general elliptic equations. Finally, in Section 12 we briefly outline an extension of FVE analysis to parabolic equations.

## 2. Background

The finite volume element method for elliptic equations is based on the fact that these equations arise from integral conservation laws. For definiteness and simplicity, we work in two spatial dimensions and consider the following model elliptic equation for an unknown distribution $u$:

$$\nabla \cdot (\mathbf{a}\, u - D \nabla u) + r\, u = f \quad \text{in } \Omega, \tag{1}$$

where $\Omega \subset \Re^2$ is the spatial domain, $f$ is a source term, and $\mathbf{a} = (a_1, a_2)$, $D$, and $r$ are advection, diffusion and reaction coefficients. For simplicity, we assume here that the coefficients are smooth functions of space alone to avoid difficulties caused by discontinuities and nonlinearities in our analysis, though these cases can be handled (see Section 12).

For (1), we consider Dirichlet or flux boundary conditions:

$$u = g \quad \text{or} \quad (\mathbf{a}\, u - D \nabla u) \cdot \mathbf{n} = g \quad \text{on } \partial\Omega, \tag{2}$$

where $\partial\Omega$ is the boundary of $\Omega$, $g$ is boundary data on $\partial\Omega$, and $\mathbf{n}$ is the outward unit normal on $\partial\Omega$.

Although we stated at the outset that we are investigating numerical solutions of the elliptic partial differential equation (1), we are more precisely studying integral conservation laws on subdomains $V \subseteq \Omega$ that lead to equations such as (1):

$$\int_{\partial V} (\mathbf{a}\, u - D \nabla u) \cdot \mathbf{n}\, dS + \int_V r\, u\, d\mathbf{x} = \int_V f\, d\mathbf{x}, \ \forall\, V \subseteq \Omega, \tag{3}$$

where $\partial V$ is the boundary of $V$, $\mathbf{n}$ is the outward unit normal on the boundary $\partial V$, and $dS$ is a spatial boundary measure. The conservation law (3) states that the advective-diffusive flux across volume boundary $\partial V$ is counter-balanced by reactions and sources within the volume $V$. Equation (3) is called the primitive form of the elliptic equation (1) because it contains the essence of the physical model within the least restrictive mathematical model. Finally, it is (3)—and *not* (1)—that we mimic discretely in FVE.

FVE approximates (3) using meshes described in the next section, by replacing $u$ and $f$ in (3) with finite element approximations $u^h$ and $f^h$ which are based on a finite element triangulation $\mathcal{T}^h$ that partitions $\Omega$, and by posing equation (3) on a finite subset $\mathcal{V}^{h/k}$ of volumes that partitions $\Omega$.

## 3. Computational Meshes

The most basic component in the implementation of the finite volume element method is the discretization of the domain $\Omega$ into computational meshes. The finite element triangulation $\mathcal{T}^h$ of $\Omega$ and the finite volume element volumization $\mathcal{V}^{h/k}$ of $\Omega$ are two different—yet interconnected—meshes or discretizations of $\Omega$. The two discretizations are connected by a third mesh or discretization of $\Omega$: $\mathcal{T}^{h/k}$, the $k$-fold refinement of $\mathcal{T}^h$. The nodes or vertices of $\mathcal{T}^{h/k}$ are the locations of the degrees of freedom (DOF) for $u^h \in P^k(\mathcal{T}^h)$: $u^h$ is a $C^0$ function on $\Omega$ that is a polynomial of degree $k \geq 1$ when restricted to each element $T$ of $\mathcal{T}^h$. In FVE terminology, the triangulation $\mathcal{T}^{h/k}$ is the *primal* mesh and the volumization $\mathcal{V}^{h/k}$ is the *dual* mesh. Next, we describe these three meshes and their relationships.

### *3.1. FE Triangulation*

Let $\mathcal{T}^h$ be a non-overlapping triangulation of $\overline{\Omega}$, the closure of $\Omega$, into a finite number of elements $T$. To simplify the discussion, we assume the elements of the triangulation are triangles. Other types of elements could be considered: in particular, rectangular elements for a rectangular domain $\Omega$.

For each $T$ of $\mathcal{T}^h$, we define the mesh parameters $h_T, h$, and $\rho_T$ as follows: $h_T$ is the diameter of the circumscribing circle for $T$; $h$ is the maximum value of $h_T$; and $\rho_T$ is the diameter of the inscribed circle in $T$.

We assume that the triangulation is *regular:* there exists a positive constant $\sigma$ such that

$$\frac{h_T}{\rho_T} \leq \sigma, \quad \forall T \in \mathcal{T}^h; \tag{4}$$

the family $(h_T)$ is bounded and 0 is its unique accumulation point—i.e., $h$ approaches zero. The assumption of regularity is used to simplify error estimates and to avoid degenerate triangulations (i.e., a minimum angle condition is satisfied).

### 3.2. FVE Primal Mesh

Once a nodal or base triangulation $\mathcal{T}^h$ is defined, we can define elements of $P^k(\mathcal{T}^h)$. The degrees of freedom (DOF) for $u^h \in P^k(\mathcal{T}^h)$ are located in a regular fashion in $\overline{T}$: on the vertices, along edges, and in the interior of $T$. Just as the FE triangulation $\mathcal{T}^h$ can be determined by connecting the vertices of $T$ to their nearest neighbors, an alternative triangulation $\mathcal{T}^{h/k}$ can be determined by connecting the DOF in $T$ to their nearest neighbors within $T$: let $T_k$ denote the triangular elements of $\mathcal{T}^{h/k}$. The primal mesh $\mathcal{T}^{h/k}$ can be seen as the $k$-fold refinement of $\mathcal{T}^h$: if $\text{diam}(T) = h$, then $\text{diam}(T_k) = h/k$.

### 3.3. FVE Dual Mesh

The FVE dual mesh $\mathcal{V}^{h/k}$ or volumization of $\Omega$ partitions $\overline{\Omega}$ into a finite number of non-overlapping elements $V_i$—the index $i$ refers to a one-to-one correspondence between volumes and DOF for $u^h \in P^k(\mathcal{T}^h)$. To define $\mathcal{V}^{h/k}$ from the primal mesh $\mathcal{T}^{h/k}$, we make frequent reference to $\mathcal{T}_i^{h/k}$: the union of all $T_k \in \mathcal{T}^{h/k}$ that have the location of the $i$th DOF, $\mathbf{x}_i$, as a vertex. To ensure a one-to-one correspondence between volumes and DOF, we require $V_i \subset \mathcal{T}_i^{h/k}$—further specifications for $V_i$ are outlined below.

#### 3.3.1. Volume Construction

Following Bank and Rose[2], we construct volumes as follows: (1) select a point $P \in \overline{T_k}, \quad \forall\, T_k \in \mathcal{T}_i^{h/k}$; (2) connect $P$ by straight line segments to edge midpoints of $T_k$ for the two edges of $T_k$ adjacent to the vertex $\mathbf{x}_i$, $\forall\, T_k \in \mathcal{T}_i^{h/k}$; (3) for each $T_k \in \mathcal{T}_i^{h/k}$, define a *sub-volume*, $v_i(T_k)$, as the region bounded by the line segments formed in Step 2 and line segments connecting the edge midpoints of $T_k$ with the vertex $\mathbf{x}_i$. Finally, define the volume $V_i$ as the region enclosing $\mathbf{x}_i$ in Step 2:

$$V_i \equiv \bigcup_{T_k \in \mathcal{T}_i^{h/k}} v_i(T_k). \tag{5}$$

The choice of $P$ in Step 1 is crucial in volume construction. In practice (see Cai[5]), we typically use the circumcenter volume: $P$ is the center of the circle circumscribed about $T_k$, or equivalently, the intersection of the perpendicular bisectors of the edges of $T_k$. To ensure that $P \in \overline{T_k}$, this case requires that no interior angle of $T_k$ exceed $\pi/2$—no obtuse triangles are permitted. In applications, advantages of circumcenter volumes are that they are always convex and geometrically simple, while other types of volumes (e.g., centroid, orthocenter, incenter, etc.) are usually non-convex and geometrically complex (see Bank and Rose[2] for details).

#### 3.3.2. Volume Symmetry

In the analysis to follow, superconvergence results for linear finite elements can be demonstrated for "symmetric" circumcenter volumes. To define this symmetry precisely, we need additional notation: $X_{ij}$ is the edge or line segment connecting nodes $\mathbf{x}_i$ and $\mathbf{x}_j$ of the triangulation $\mathcal{T}^{h/k}$; $\gamma_{ij}$ is the interface or volume boundary between volumes $V_i$ and $V_j$ of the volumization $\mathcal{V}^{h/k}$—i.e., $\gamma_{ij} \equiv \overline{V_i} \bigcap \overline{V_j}$. Following Cai *et al.*[4] and Cai[5], the volumization $\mathcal{V}^{h/k}$ is *symmetric* to the triangulation $\mathcal{T}^{h/k}$ if the following two symmetries hold for all volumes in $\mathcal{V}^{h/k}$: ($X$-symmetry) $\gamma_{ij}$ is a perpendicular bisector of $X_{ij}$; ($\gamma$-symmetry) $X_{ij}$ is a perpendicular bisector of $\gamma_{ij}$. We remark that for rectangular elements, only the first condition of $X$-symmetry is required for $\mathcal{V}^{h/k}$ to be symmetric to $\mathcal{T}^{h/k}$.

#### 3.3.3. Volume Regularity

Of more general significance is the notion of volume "regularity," which is assumed in all of our results. Form an auxiliary triangulation $\widetilde{\mathcal{T}^{h/k}}$ of triangular elements by connecting the endpoints of $\gamma_{ij}$ with the endpoints of $X_{ij}$ for every volume interface in $\mathcal{V}^{h/k}$. For circumcenter volumes, if this auxiliary triangulation $\widetilde{\mathcal{T}^{h/k}}$ is regular according to the definition in Section 3.1 (cf. (4)) and no interior angle of $\widetilde{T_k} \in \widetilde{\mathcal{T}^{h/k}}$ exceeds $\pi/2$, then the volumization $\mathcal{V}^{h/k}$ is *regular.*

## 4. Finite Element Analysis

Here we present the rudiments of FE analysis to establish a reference point for analogous developments to follow for FVE. For simplicity, consider a steady diffusion equation with homogeneous Dirichlet boundary condition:

$$-\nabla \cdot (D\nabla u) = f, \quad x \in \Omega, \tag{6}$$

$$u|_{\partial\Omega} = 0, \tag{7}$$

where $D$ is a continuous, bounded, non-degenerate ($0 < D_m \leq D \leq D_M < \infty$) diffusion coefficient and $\Omega \subset \Re^2$ is a domain.

A weak solution $u \in \mathcal{W} \equiv H_0^1(\Omega)$ to (6), corresponding to $f \in L^2(\Omega)$, satisfies:

$$A(u, w) = (f, w), \quad \forall\, w \in \mathcal{W}, \tag{8}$$

where

$$A(u, w) = \int_\Omega D\nabla u \cdot \nabla w \, d\mathbf{x}, \tag{9}$$

$$(f, w) = \int_\Omega f\, w\, d\mathbf{x}. \tag{10}$$

In our analysis, we assume additional regularity for $u$: i.e., $u$ lies in the (fractional-order) Sobolev space $\mathcal{W}^+ \equiv H^{S+1}(\Omega) \cap \mathcal{W}$, where $S \in (0, k]$ and $k \geq 1$. Here, we have invoked fractional-order spaces to demonstrate error estimates across the fullest extent of admissible regularities. Also, the numerical solution $u^h$ of (6) (see (11) below) will lie in $\mathcal{W}^h \equiv P_0^k(\mathcal{T}^h) \subset \mathcal{W}$—the space of $C^0$ functions that are polynomials of order $\leq k$ on each element $T$ of the triangulation $\mathcal{T}^h$ and vanish on $\partial\Omega$.

In direct analogy with (8), the (finite element) numerical solution $u^h \in \mathcal{W}^h$ to (6), corresponding to $f^h \in P^k(\mathcal{T}^h)$, satisfies:

$$A(u^h, w) = (f^h, w), \quad \forall\, w \in \mathcal{W}^h. \tag{11}$$

Here, $f^h$ is the $L^2(\Omega)$ projection of $f$ into $P^k(\mathcal{T}^h)$: i.e.,

$$(f - f^h, w) = 0, \quad \forall\, w \in P^k(\mathcal{T}^h). \tag{12}$$

The key components of finite element error analysis are ellipticity, boundedness, and approximation theory results: we introduce and discuss each below.

### *4.1. Ellipticity*

An *ellipticity* (or lower bound) condition that there exists a positive constant $\alpha$ such that

$$A(u, u) \geq \alpha\, |u|_{1,\Omega}^2, \quad \forall u \in \mathcal{W}, \tag{13}$$

is perhaps the most important component of FE error analysis: it implies that the bilinear form $A(\cdot,\cdot)$ of (9)—hereafter called the "$A$–form"—is positive definite.

### *4.2. Boundedness*

An upper bound (or *continuity*) conditions for the $A$-form (9), that there exists a positive constant $M$ such that

$$A(u, w) \leq M\, |u|_{1,\Omega} |w|_{1,\Omega}, \quad \forall\, u, w \in \mathcal{W}, \tag{14}$$

is necessary to demonstrate convergence of the numerical method as seen below.

### *4.3. Approximation Theory*

For the $H^m(\Omega)$ semi-norms ($m \in \{0, 1\}$) that we shall employ, we cite standard approximation theory results for polynomial interpolation (see Dupont and Scott[7]): there exists a constant $C$ such that

$$|\eta_k(u)|_{m,\Omega} \leq C\, h^{S+1-m} \|u\|_{S+1,\Omega}, \quad S \in (0, k], \tag{15}$$

where $\eta_k(u) \equiv \Pi_k^h u - u$ is the interpolation error for $u$ in the polynomial space $P^k(\mathcal{T}^h)$. In the remainder of the text, $C$ will represent a (generic) positive constant.

### *4.4. Error Analysis*

In the standard FE error analysis to follow, we apply the results (13) and (14)–(15); for (6), (13) and (14) hold with $\alpha = D_m$ and $M = D_M$, respectively.

#### 4.4.1. Error Splitting

Let $e \equiv u - u^h \in \mathcal{W}$ be the error in the FE approximation to (6). For theoretical purposes, define $\zeta \equiv \Pi_k^h e$ as an interpolation of the error from $\mathcal{W}$ into $\mathcal{W}^h$, so that $\zeta \equiv \Pi_k^h u - u^h \in \mathcal{W}^h$. We refer to $\zeta$ as the *representation error* in $\mathcal{W}^h$: i.e., the difference between interpolated and computed approximations in $\mathcal{W}^h$ of the exact solution $u$. Then, we employ the *error splitting* $e = \zeta - \eta_k(u)$, in which the FE error is split into its representation error and interpolation error components: $|u - u^h|_{1,\Omega} \leq |\zeta|_{1,\Omega} + |\eta_k(u)|_{1,\Omega}$. With the estimate of $|\eta_k(u)|_{1,\Omega}$ in (15), estimating $|e|_{1,\Omega}$ is reduced to estimating $|\zeta|_{1,\Omega}$. Below, we develop an estimate for $|\zeta|_{1,\Omega}$ in terms of $|\eta_k(u)|_{1,\Omega}$.

#### 4.4.2. Zero Property

For all $w \in \mathcal{W}^h$, the exact solution satisfies (8), the numerical solution satisfies (11), and the source term satisfies (12). Subtracting, we have the *zero property* or *orthogonality* of the error,

$$A(u - u^h, w) = 0, \quad \forall\, w \in \mathcal{W}^h, \tag{16}$$

which in our context is more appropriately expressed as

$$A(\zeta, w) = A(\eta_k(u), w), \quad \forall\, w \in \mathcal{W}^h. \tag{17}$$

#### 4.4.3. Error Estimate

Since the representation error, $\zeta$, is in $\mathcal{W}^h$, we take $w = \zeta$ in (17) to obtain

$$\begin{aligned} \alpha\,|\zeta|_{1,\Omega}^2 &\leq A(\zeta, \zeta) \\ &= A(\eta_k(u), \zeta) \\ &\leq M\,|\eta_k(u)|_{1,\Omega}|\zeta|_{1,\Omega}. \end{aligned} \tag{18}$$

The three lines of (18) use (13), (17), and (14), respectively. By (15), (18) implies the following optimal-order *error estimate* for the finite element method:

$$|u - u^h|_{1,\Omega} \leq C\,h^S\,\|u\|_{S+1,\Omega}, \quad S \in (0, k]. \tag{19}$$

In the FVE analysis to follow, we obtain error estimates that match or surpass the $k$th-order estimate (19).

## 5. Preliminaries

We employ the same continuous and discrete space notations, definitions, and structures as in the previous section for the FVE continuous and discrete trial spaces that represent the solution to (6): $u \in \mathcal{W} \equiv H_0^1(\Omega)$ and $u^h \in \mathcal{W}^h \equiv P_0^k(\mathcal{T}^h)$, for $k \geq 1$. Similarly in FVE analysis, we assume additional regularity for $u$: i.e., $u$ lies in the (fractional-order) Sobolev space $\mathcal{W}^+ \equiv H^{S+1}(\Omega) \cap \mathcal{W}$, where $S \in (1/2, k]$. The $1/2$ lower limit on $S$ (instead of 0 as in FE) ensures a bounded diffusive flux, due to the Sobolev trace theorem (as discussed below).

In addition, let $\mathcal{X}^h$ denote the FVE test space $P_0^0(\mathcal{V}^{h/k})$—the polynomials that are constant on each volume $V$ of $\mathcal{V}^{h/k}$ and vanish on $\partial\Omega$. Thus, test functions corresponding to volumes that intersect $\partial\Omega$ are identically zero. Also, let $\chi_V$ denote the characteristic function associated with volume $V \subseteq \Omega$ and let $\mathcal{V}$ denote the collection of all such $V$.

Recalling the integral equation (3), (in the absence of elliptic regularity) a weak solution $u$ of (6) satisfies: given $f \in L^2(\Omega)$, find $u \in \mathcal{W}^+$ such that

$$B(u, \chi_V) = (f, \chi_V), \quad \forall\, V \in \mathcal{V}, \tag{20}$$

where

$$B(u, \chi_V) = -\int_{\partial V} D\nabla u \cdot \mathbf{n}\, dS, \tag{21}$$

$$(f, \chi_V) = \int_V f\, d\mathbf{x}. \tag{22}$$

FVE poses the weak form (20) on the volumization $\mathcal{V}^{h/k}$—a finite subset of $\mathcal{V}$ with non-overlapping volumes—and employs standard FE representations of $u$ and $f$. Thus, FVE replaces (20) with: given $f^h \in P^k(\mathcal{T}^h)$ find $u^h \in \mathcal{W}^h$ such that

$$B(u^h, \chi_V) = (f^h, \chi_V), \quad \forall\, V \in \mathcal{V}^{h/k}. \tag{23}$$

Here, $f^h$ satisfies a non-standard $L^2(\Omega)$ projection relationship with $f$:

$$(f - f^h, \chi_V) = 0, \quad \forall\, V \in \mathcal{V}^{h/k}, \tag{24}$$

which ensures that the numerical source term conserves mass on each volume.

Since $\mathcal{V}^{h/k}$ partitions $\Omega$ and the volumes in $\mathcal{V}^{h/k}$ are related (through $\mathcal{T}^{h/k}$) to the triangulation $\mathcal{T}^h$ as discussed in Section 3.3, (23) is more amenable than (20) to an FE-style analysis. Below, we point out differences between FVE and FE from the perspective of an FE-style framework for analysis.

## 6. Motivation

To motivate our FVE analysis and its relation to previous FE work, reconsider (6)–(7) and the FE analysis of Section 4. We saw that the crucial ellipticity result (13) for the $A$-form and boundedness result (14) are stated in terms of the continuum space $\mathcal{W}$, but the results also held for the discrete space $\mathcal{W}^h$ (i.e., $\mathcal{W}^h \subset \mathcal{W}$). It could appear that the focus was on the continuum problem (8) rather than on the FE discrete problem (11). However, in Section 4.4, the error analysis emphasized results for the discrete FE problem (11) and the associated discrete spaces. Recall that the exact weak solution $u$ was invoked only in the zero property (17); furthermore, recall that the key observation in (17) was that $u$ conformed to the FE discrete equation (11). Therefore, to begin an FE-style analysis for FVE, we need to generate results analogous to those of Sections 4.1, 4.2, and 4.3 for the FVE discrete problem (23) rather than for (20).

The continuum problems (8) and (20) are different in physical character (minimization vs. conservation) as well as in mathematical content, discouraging a single, unifying framework for their analysis. For the discrete problems (11) and (23), the situation is more favorable for this. We now manipulate (23) to be more reminiscent of (11).

If $w \in \mathcal{X}^h$, put

$$w \equiv \sum_{i \in I} w_i \chi_i, \tag{25}$$

where $\chi_i = \chi_{V_i}$ and $\mathcal{V}^{h/k} = \{V_i : i \in I\}$. Now the FVE solution $u^h \in \mathcal{W}^h$ of (23) also satisfies:

$$B(u^h, w) = (f^h, w), \quad \forall\, w \in \mathcal{X}^h, \tag{26}$$

where

$$B(u^h, w) = \sum_{i \in I} B(u^h, w_i \chi_i), \tag{27}$$

$$(f^h, w) = \sum_{i \in I} (f^h, w_i \chi_i). \tag{28}$$

That is, if $u^h$ satisfies the each of the local equations of (23), then, by (26), $u^h$ also satisfies any linear combination of those equations. Notice that (27) and (28) are now global in extent, like their FE counterparts (9) and (10), and (26) with $w \equiv 1$ represents a global conservation law on the union or aggregate of all volumes in $\mathcal{V}^{h/k}$. Like the FE $A$-form, we refer to (27) as the $B$-form.

Similarly, the $L^2(\Omega)$ projection relationship (24) for $f \in P^k(\mathcal{T}^h)$ is transformed into

$$(f - f^h, w) = 0, \quad \forall\, w \in P^0(\mathcal{V}^{h/k}). \tag{29}$$

As in FE error analysis, the key components of FVE analysis are ellipticity, boundedness, and approximation theory results: we introduce and discuss each result below. All follow the basic framework introduced in the FE section after appropriate modifications for FVE. The proofs of these results are in the thesis of Trujillo[19]; the intent here is to summarize them and to compare and contrast FE and FVE analysis.

## 7. Ellipticity

In Section 4, the FE ellipticity result required that the trial space representation error be used as a test function. In FE, test and trial spaces coincide so this was straightforward. In FVE, test and trial spaces differ, and we must describe the test function representation of a trial function.

Let $u \in \mathcal{W}^h$ with nodal representation $u \equiv \sum_{i\in I} u_i\, \phi_i$, where $\{\phi_i\}_{i\in I}$ is the DOF basis for $\mathcal{W}^h$. Then $u$ also has a corresponding *lumped* or test space representation, $\overline{u}$, in $\mathcal{X}^h$:

$$\overline{u} \equiv \sum_{i\in I} u_i\, \chi_i. \tag{30}$$

In other words, if $u \in P^k(\mathcal{T}^h)$, $\overline{u} \in P^0(\mathcal{V}^{h/k})$ is a volume-wise constant interpolant of $u$. Our ellipticity result for FVE will use the lumped represention defined by (30) to characterize test functions.

As in FE, a discrete ellipticity result (for $h$ sufficiently small when $k \geq 2$),

$$B(u,\overline{u}) \geq \alpha\, |u|^2_{1,\omega}, \quad \forall\, u \in \mathcal{W}^h, \tag{31}$$

is perhaps the most important component of FVE analysis. On the right-hand side of (31), $|\cdot|_{1,\omega}$ is a discrete $H^1_0(\Omega)$ norm; this can be understood as a restriction of the continuous $H^1_0(\Omega)$ norm to $P^1_0(\mathcal{T}^{h/k})$: i.e., $|u|_{1,\omega} \equiv |\Pi_1^{h/k} u|_{1,\Omega}$. Therefore, (31) implies that the $B$-form is positive definite.

In addition, we see that $B(u,\overline{u}) \equiv \mathbf{u}^T\mathbf{B}\mathbf{u} \equiv Q_{\mathbf{B}}(\mathbf{u}), \quad \forall\, u \in \mathcal{W}^h$, where $\mathbf{u}$ is the vector of nodal values for $u \in \mathcal{W}^h$, $\mathbf{B}$ is the FVE matrix generated by the $B$-form, and $Q_{\mathbf{B}}(\cdot)$ is the associated quadratic form. Therefore, (31) simply states that $\mathbf{B}$ is positive definite: i.e.,

$$Q_{\mathbf{B}}(\mathbf{u}) \geq 0, \quad \forall \mathbf{u}, \tag{32}$$

with equality holding in (32) only when $\mathbf{u}$ (i.e., $u \in \mathcal{W}^h$) is identically zero. Hence, (31) guarantees the existence and uniqueness of the numerical solution $u^h \in \mathcal{W}^h$.

The FVE ellipticity result (31) clearly has the same form as the corresponding FE result (13) (when restricted to $\mathcal{W}^h$). In fact, the two results are identical for (6) when $D$ is constant and $\mathcal{W}^h = P^1_0(\mathcal{T}^h)$; that is, the FVE (and FV) matrix $\mathbf{B}$ and the FE matrix $\mathbf{A}$ are identical for piecewise linear polynomials on a general triangulation as demonstrated by Bank and Rose.[2] This equivalence between FV and FE for a piecewise linear trial space is the key observation that drives much of the FV analysis in the references cited in Section 1. Such arguments may seem to imply dependence of FV (or FVE) analysis on the corresponding FE formulation of the same problem. As we will see in the following sections, this is not the case: FVE (or FV) operates independently of FE and its analysis need not refer to FE or be confined to the cases when FVE and FE are equivalent in some sense. Our FVE analysis is similar to FE in style but not in substance.

## 8. Boundedness

In Section 4.2, the upper bound (14) for the $A$-form is given in terms of products of certain bounded functionals or norms (to be specified according to the context of a problem) of a continuous trial function and a discrete test function. In the case of FE, these functionals are standard integer-order Sobolev norms (or sub-linear functionals). Since these bounds are to be used in conjunction with the ellipticity result previously discussed, the only theoretical constraint on the bound is that the functional or norm applied to the test function must match (or can be made to match via an auxiliary result like the Poincaré inequality) the functional or norm in the ellipticity result—i.e., $|\cdot|_{1,\Omega}$ for FE and $|\cdot|_{1,\omega}$ for FVE.

After this requirement is satisfied, there is great freedom of choice in the functional applied to the continuous trial function: this can be problem-dependent and can adapt to any additional constraints or exploit any additional information in a problem. Practically speaking, we want this functional to admit optimal-order or even superconvergent approximation theorems, since the continuous trial function in our analysis is the interpolation error for the exact solution.

In FE analysis, the $H_0^1(\Omega)$ norm constraint on the test function leads to the choice of the $H_0^1(\Omega)$ norm for the trial functions. However, in FVE analysis, the $H_0^1(\omega)$ constraint on the test function leads to more general choices of problem-dependent bounded linear functionals for the trial functions; these functionals correspond to the diffusion term and to the source term (and in Section 11.2 to reaction and advection terms).

The boundedness result for the $B$-form (27) is (cf. (14) of Section 4.2):

$$B(u,w) \le \mathcal{D}(u)|w|_{1,\omega}, \quad \forall\, u \in \mathcal{W}^+, w \in \mathcal{X}^h. \tag{33}$$

Here $\mathcal{D}(\cdot)$ is a bounded functional defined as

$$\mathcal{D}(u) \equiv \Big( \sum_{\{i,j\}} \mathcal{D}_{ij}^2(u) \frac{|X_{ij}|}{|\gamma_{ij}|} \Big)^{1/2}, \tag{34}$$

where $|\cdot|$ is the (local) Lebesgue measure, $X_{ij}$ and $\gamma_{ij}$ are defined in Section 3.3.2, and $\mathcal{D}_{ij}(\cdot)$ is a bounded linear functional involving the FVE diffusion term,

$$\mathcal{D}_{ij}(u) \equiv -\int_{\gamma_{ij}} D\nabla u \cdot \mathbf{n}_{ij}\, dS, \tag{35}$$

and $\mathbf{n}_{ij}$ is a normal pointing outward from $V_i$ into $V_j$. By the Sobolev trace theorem, $u \in H^{S+1}(\Omega)$ for $S > 1/2$ ensures that (35) is bounded.

With an upper bound for the $B$-form in terms of the non-standard, even problem-dependent, bounded linear functional (35), we need an approximation theorem: the result is summarized and discussed below.

## 9. Approximation Theory

By applying the linear and bilinear Bramble-Hilbert lemmas[6] to the bounded linear functional (35) we can demonstrate an approximation theorem for the diffusion functional $\mathcal{D}(\cdot)$. Our criterion is that the Bramble-Hilbert result be equivalent to the standard optimal-order estimates for polynomial interpolation (in terms of the integer-order Sobolev norms) as demonstrated by (15) of Section 4.3. Furthermore, in the case of the diffusion functional, we can prove superconvergence (see Cai[5]) that actually surpasses the standard results.

For $\mathcal{D}(\cdot)$, we have an ($H^1$-equivalent) optimal-order result for interpolation into $P^k(\mathcal{T}^h)$ when the triangulation $\mathcal{T}^h$ and volumization $\mathcal{V}^{h/k}$ are regular (defined in Sections 3.1 and 3.3.3):

$$\mathcal{D}(\eta_k(u)) \leq C\, h^S \|u\|_{S+1,\Omega}, \quad S \in (1/2, k]. \tag{36}$$

However, for interpolation into $P_0^1(\mathcal{T}^h)$, we have a choice between an ($H^1$-equivalent) optimal-order result and a superconvergence result that depends on whether the volumization $\mathcal{V}^{h/k}$ is symmetric to the triangulation $\mathcal{T}^{h/k}$ (see Section 3.3.2):

$$\mathcal{D}(\eta_1(u)) \leq C\, h^S \|u\|_{S+1,\Omega}, \quad S \in (1/2, K], \tag{37}$$

where $K = 1$ (optimal-order) or 2 (superconvergence) in the absence or presence of volume symmetry, respectively. Notice (cf. (36)) that the superconvergence is consistent with the use of a quadratic, rather than linear, trial space; indeed, the linear FVE on a symmetric volumization is identical to a quadratic FVE on a special "degenerate" volumization (see Trujillo[19] for details).

Unlike standard FE superconvergence results that are usually local and one-dimensional, the FVE superconvergence result (37) is always global and multidimensional. Since superconvergence is of great importance in both theory and practice, the focus of FVE analysis here is on establishing superconvergence results for the linear trial space before moving on to optimal-order results for higher-order trial spaces.

## 10. Error Analysis

With the components developed in the preceding sections, expressed in (31), (33), and (36)–(37), we can outline the standard FVE analysis. To combine optimal-order and superconvergence results, define

$$\kappa \equiv \max\{k, K\}, \tag{38}$$

where $k$ and $K$ are the optimal-order and superconvergence parameters employed in the preceding section. The pathway to our error estimate is detailed below.

### *10.1. Error Splitting*

As in the FE analysis of Section 4.4, we define the error in the FVE approximation to (6) as $e \equiv u - u^h \in \mathcal{W}$ and write $e = \zeta - \eta_k(u)$, so that the FVE error is split into its representation error ($\zeta \equiv \Pi_k^h u - u^h \in \mathcal{W}^h$) and interpolation error ($\eta_k(u) \equiv \Pi_k^h u - u \in \mathcal{W}$) components. Since $|u - u^h|_{1,\omega} \equiv |\zeta|_{1,\omega}$, estimating $|e|_{1,\omega}$ is reduced to estimating $|\zeta|_{1,\omega}$. The latter estimate can be developed as below in terms of estimates (36)–(37) for the interpolation error $\eta_k(u)$.

### *10.2. Zero Property*

Knowing that (for all $w \in \mathcal{X}^h$) the exact solution of (20) satisfies $B(u, w) = (f, w)$, the numerical solution satisfies (26), and the source term satisfies (29), we have the *zero property*,

$$B(u - u^h, w) = 0, \quad \forall\, w \in \mathcal{X}^h, \tag{39}$$

which in our context is more appropriately expressed as

$$B(\zeta, w) = B(\eta_k(u), w), \quad \forall\, w \in \mathcal{X}^h. \tag{40}$$

### *10.3. Error Estimate*

Since $\overline{\zeta}$, the lumped representation (or $P^0(\mathcal{V}^{h/k})$-interpolant) of $\zeta$, is in $\mathcal{X}^h$, take $w = \overline{\zeta}$ in (40) and obtain

$$\begin{aligned} \alpha\, |\zeta|_{1,\omega}^2 &\leq B(\zeta, \overline{\zeta}) \\ &= B(\eta_k(u), \overline{\zeta}) \\ &\leq \mathcal{D}(\eta_k(u)) |\zeta|_{1,\omega}. \end{aligned} \tag{41}$$

The three lines of (41) use (31), (40), and (33), respectively. Finally, by (36)–(37) we have the following combined optimal-order and superconvergent error estimate for the finite volume element method:

$$|u - u^h|_{1,\omega} \leq C\, h^S\, \|u\|_{S+1,\Omega}, \quad S \in (1/2, \kappa], \tag{42}$$

where $\kappa$ is defined in (38). Note that (42) matches or surpasses the corresponding $k$th-order finite element result (19).

This concludes our presentation for diffusion equations. In the next section, we outline the necessary additions and modifications for the inclusion of reaction and advection terms, as in the general elliptic equation (1) that arises from the general integral conservation law (3) of Section 2.

## 11. General Elliptic Equations

In general, the FVE $B$-form is given by

$$B(u,w) \equiv \mathcal{A}(u,w) + \mathcal{D}(u,w) + \mathcal{R}(u,w), \quad u \in \mathcal{W}, w \in \mathcal{X}^h, \tag{43}$$

where the bilinear forms corresponding to advection, diffusion, and reaction are

$$\mathcal{A}(u,w) \equiv \sum_{i \in I} \int_{\partial V_i} \mathbf{a}u \cdot \mathbf{n}\, w_i\, dS, \tag{44}$$

$$\mathcal{D}(u,w) \equiv \sum_{i \in I} - \int_{\partial V_i} D\nabla u \cdot \mathbf{n}\, w_i\, dS, \tag{45}$$

$$\mathcal{R}(u,w) \equiv \sum_{i \in I} \int_{V_i} ru\, w_i\, d\mathbf{x}. \tag{46}$$

Then (44)–(46) correspond to the left-hand side of the general integral equation (3), so that (43) is the starting point for an FVE analysis for the general elliptic equation (1). Sections 7–9 analyzed the diffusion bilinear form (45); here we outline the corresponding results for the advection and reaction forms (44) and (46).

### *11.1. Ellipticity*

For reaction problems, we assume continuous and bounded ($0 < r_m \leq r \leq r_M < \infty$) reaction coefficients. If $u$ in (46) is replaced by the lumped representation $\overline{u} \in \mathcal{X}^h$ (recall (30)), a discrete ellipticity condition analogous to (31) is:

$$\mathcal{R}(\overline{u},\overline{u}) \geq r_m\, |u|^2_{0,\omega}, \quad \forall\, u \in \mathcal{W}^h. \tag{47}$$

Using (31) and (47), we find that for the general reaction term (46): there exists a positive constant $\beta = r_m(1-\epsilon)$ for $\epsilon$ in $(0,1)$ such that

$$\mathcal{R}(u,\overline{u}) \geq \beta\, |u|^2_{0,\omega} - \alpha\, C\, h^2 |u|^2_{1,\omega}, \quad \forall\, u \in \mathcal{W}^h. \tag{48}$$

Then (48) and (31) imply discrete ellipticity for a reaction-diffusion problem when $h$ is sufficiently small (e.g., $h$ satisfies $C\, h^2 \leq \epsilon/2 < 1/2$, for a given $\epsilon$).

For advection problems, we assume continuously differentiable and bounded ($||\mathbf{a}||$, $||\nabla \cdot \mathbf{a}|| \leq M < \infty$) velocity vectors. After some algebra to split (44) into $L^2$ and $H^1$-equivalent components, we find for $\epsilon < \min\{1, M\}/4$ that

$$\mathcal{A}(u,\overline{u}) \geq -\frac{M^2}{\alpha\epsilon}\, |u|^2_{0,\omega} - \alpha\, \frac{\epsilon}{2}\, |u|^2_{1,\omega}, \quad \forall\, u \in \mathcal{W}^h. \tag{49}$$

Then (49), with (31) and (48), implies discrete ellipticity for (43) when $h = h(\epsilon)$ is sufficiently small and $r_m = r_m(\epsilon, \alpha, M)$ is sufficiently large.

### *11.2. Boundedness*

For the advection bilinear form (44), similar to the diffusion bound (33) we have

$$\mathcal{A}(u, w) \leq \mathcal{A}(u)|w|_{1,\omega}, \quad \forall\, u \in \mathcal{W}, w \in \mathcal{X}^h. \tag{50}$$

Here $\mathcal{A}(\cdot)$ is a bounded functional defined as

$$\mathcal{A}(u) \equiv \Big( \sum_{\{i,j\}} \mathcal{A}_{ij}^2(u)\, \frac{|X_{ij}|}{|\gamma_{ij}|} \Big)^{1/2}, \tag{51}$$

where $\mathcal{A}_{ij}(\cdot)$ is a bounded linear functional relating to the FVE advection term:

$$\mathcal{A}_{ij}(u) \equiv \int_{\gamma_{ij}} \mathbf{a}u \cdot \mathbf{n}_{ij}\, dS. \tag{52}$$

For the reaction bilinear form (46), we derive

$$\mathcal{R}(u, w) \leq \mathcal{R}(u)|w|_{0,\omega}, \quad \forall\, u \in \mathcal{W}, w \in \mathcal{X}^h. \tag{53}$$

On the right-hand side of (53), $\mathcal{R}(\cdot)$ is a bounded functional defined as

$$\mathcal{R}(u) \equiv \Big( \sum_{i \in I} \frac{1}{|V_i|}\, \mathcal{R}_i^2(u) \Big)^{1/2}, \tag{54}$$

where $\mathcal{R}_i(\cdot)$ is a bounded linear functional relating to the FVE reaction term:

$$\mathcal{R}_i(u) = \int_{V_i} ru\, d\mathbf{x}; \tag{55}$$

also, $|\cdot|_{0,\omega}$ is a discrete $L^2(\Omega)$ norm formed by restricting the continuous $L^2(\Omega)$ norm to $P^0(\mathcal{V}^{h/k})$: i.e., $|u|_{0,\omega} \equiv |\Pi_0^{h/k} u|_{0,\Omega}$.

### *11.3. Approximation Theory*

For the advection functional (51), we apply the Bramble-Hilbert lemmas to the bounded linear functional (52) to obtain the analogue of (36):

$$\mathcal{A}(\eta_k(u)) \leq C\, h^{S+1} \|u\|_{S+1,\Omega}, \quad S \in (1/2, k], \tag{56}$$

which is an ($L^2$-equivalent) optimal-order result for interpolation into $P^k(\mathcal{T}^h)$ when the triangulation $\mathcal{T}^h$ and volumization $\mathcal{V}^{h/k}$ are regular. Most importantly, we see that (56) preserves the superconvergence result (37) for linear finite elements.

For the reaction functional (54), Bramble-Hilbert arguments applied to (55) show that (54) shares the same estimate as (56).

### *11.4. Error Analysis*

With the results of the previous subsections, the basic structure of the error analysis (41) presented in Section 10 is relatively unchanged. Combining the ellipticity results of Sections 7 and 11.1, we find by a discrete Poincaré inequality that the first line of (18) is modified to read

$$\alpha_0 \, |\zeta|^2_{1,\omega} \leq \mathcal{A}(\zeta, \overline{\zeta}) + \mathcal{D}(\zeta, \overline{\zeta}) + \mathcal{R}(\zeta, \overline{\zeta}), \tag{57}$$

where $\alpha_0$ is a positive constant formed after combining (31), (48), and (49) for $h$ sufficiently small and $r_m$ sufficiently large. Combining the upper bound results of Sections 8 and 11.2, the third line of (41) becomes

$$B(\eta_k(u), \overline{\zeta}) \leq (\mathcal{A}(\eta_k(u)) + \mathcal{D}(\eta_k(u)))|\zeta|_{1,\omega} + \mathcal{R}(\eta_k(u))|\zeta|_{0,\omega}. \tag{58}$$

With these modifications and the approximation results of Sections 9 and 11.3, the optimal-order and superconvergence result of (42) is maintained.

## 12. Conclusions

A summary of FVE error analysis for elliptic partial differential equations with smooth coefficients has been presented: $H^1$-equivalent optimal-order and superconvergence results have been obtained. These FVE results can be extended to problems with discontinuous and nonlinear coefficients: the modifications for discontinuous coefficients are analogous to those of Samarskii *et al.*[17] for FV; the modifications for nonlinear coefficients are analogous to those of Russell[16] for FE.

Furthermore, by an FVE variant of the elliptic projection argument of Wheeler[21], the FVE results presented here can be extended to parabolic equations in a straightforward manner. That is, an analysis of a method that uses finite differences in time and FVE in space yields error estimates of the form

$$\begin{aligned} \max_{0 \leq N \leq NT} &|(u - u^h)(\cdot, t^N)|_{1,\omega} \\ &\leq C \left( \Delta t^R ||\partial_t^{R+1} u||_{L^2(L^2)} + h^S \left( ||\partial_t u||_{L^2(H^S)} + ||u||_{L^\infty(H^{S+1})} \right) \right), \end{aligned} \tag{59}$$

where $S \in (1/2, \kappa]$ and $R = 1$ or 2 depending on whether backward Euler ($R = 1$) or Crank-Nicolson ($R = 2$) time differencing is used.

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