Building a Cyclic q-Clan

*S. E. Payne** *Tim Penttila* *Gordon F. Royle*

February 1996

Abstract

For $q = 2^e$ let $\mathcal{C}$ be a q-clan whose associated generalized quadrangle $GQ(\mathcal{C})$ of order (q^2, q) admits a collineation θ that acts as a cycle of length $q+1$ on the lines through the special point (∞). With some natural additional assumptions on the action of θ on certain subquadrangles of order q i.e., on the associated ovals, the q-clan $\mathcal{C}$ can be constructed from one carefully chosen matrix in $\mathcal{C}$ and the action of θ. With the help of computer computations this approach is used to construct new examples with $q = 4^k$ for $k = 5$, $k = 6$, $k = 7$ and $k = 8$. We conjecture that these examples belong to a new infinite family.

1 Introduction and Review

This essay is a sequel to [4], but we recall sufficiently many definitions and results to render this work intelligible to a reader already familiar with the construction of a generalized quadrangle $GQ(\mathcal{C})$ of order (q^2, q) from a q-clan $\mathcal{C}$ (cf. [3, 2]).

Let $q = 2^e$, $F = GF(q)$, $\tilde{F} = F \cup \{\infty\}$. We assume that each q - clan $\mathcal{C} = \{A_t : t \in F\}$ is normalized so that
$A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and for $t \in F$, $A_t = \begin{pmatrix} x_t & t^{1/2} \\ 0 & z_t \end{pmatrix}$. As in [4] we also write $A_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.
Let $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and for $\alpha, \beta \in F^2$ put $\alpha \circ \beta = \alpha P \beta^T$. Then the elation group $G^\otimes$ of $GQ(\mathcal{C})$ is defined as follows:

$$G^\otimes = \{((\alpha, \beta), c) \in F^2 \times F^2 \times F : \alpha, \beta \in F^2, c \in F\}$$

*The research for this paper was performed while the first author enjoyed the warm hospitality of the Combinatorial Computing Research Group of the University of Western Australia and the financial support of the Australian Research Council

with binary operation given by

$$((\alpha,\beta),c)\circ((\alpha',\beta'),c') = ((\alpha+\alpha',\beta+\beta'),c+c'+\beta\circ\alpha'). \tag{1}$$

There are two families of subgroups of $G^{\otimes}$, each subgroup having order q^3 and being elementary abelian, that play important roles in q-clan geometry.

$$\text{For } \vec{0}\neq\gamma\in F^2,\quad \mathcal{L}_\gamma = \{(\gamma\otimes\alpha,c)\in G^{\otimes} : \alpha\in F^2, c\in F\}. \tag{2}$$

$$\text{For } \vec{0}\neq\alpha\in F^2,\quad \mathcal{R}_\alpha = \{(\gamma\otimes\alpha,c)\in G^{\otimes} : \gamma\in F^2, c\in F\}. \tag{3}$$

For nonzero $\alpha,\gamma\in F^2$, $\mathcal{L}_\gamma=\mathcal{L}_\alpha$ (resp., $\mathcal{R}_\gamma=\mathcal{R}_\alpha$) if and only if $\{\alpha,\gamma\}$ is F-dependent. Hence we may think of the $\mathcal{L}_\gamma$ (resp., $\mathcal{R}_\alpha$) as being indexed by the points of $PG(1,q)$. Identify the elements of $\tilde{F}$ with the points of $PG(1,q)$ in the following way. For $t\in F$, write $\gamma_t=(1,t)$, and $\gamma_\infty=(0,1)$. So for $\vec{0}\neq\gamma=(a,b)$, $\gamma\equiv\gamma_{b/a}$. For $t\in\tilde{F}$ we then write $\mathcal{L}_t=\mathcal{L}_{\gamma_t}$ and $\mathcal{R}_t=\mathcal{R}_{\gamma_t}$.

Now let $\mathcal{C}=\{A_t : t\in\tilde{F}\}$ be a normalized q-clan, so for distinct $t,s\in F$, A_t-A_s is anisotropic. For $t\in\tilde{F}$ and $\alpha\in F^2$, write $g_t(\alpha)=\alpha A_t\alpha^T$. Then we may use $\mathcal{C}$ to define subgroups of $G^{\otimes}$ as follows.

$$\text{For } t\in\tilde{F},\quad A(t)=\{(\gamma_{t^{\frac{1}{2}}}\otimes\alpha, g_t(\alpha))\in G^{\otimes} : \alpha\in F^2\}. \tag{4}$$

It follows that for each $t\in\tilde{F}$, $A(t)$ is a subgroup of $\mathcal{L}_{t^{\frac{1}{2}}}$ where $\infty^{\frac{1}{2}}=\infty$ by convention. And then $\mathcal{J}(\mathcal{C})=\{A(t) : t\in\tilde{F}\}$ is a 4-gonal family for $G^{\otimes}$, and $\mathcal{L}_{t^{\frac{1}{2}}}$ is the tangent space with respect to $\mathcal{J}(\mathcal{C})$ at $A(t)$. This notation for $G^{\otimes}$ and $\mathcal{J}(\mathcal{C})$was introduced in [4]. By the Fundamental Theorem of q-clan geometry (cf. [5, 1]) and its corollaries each collineation of $GQ(\mathcal{C})$ that fixes the two points (∞) and $((\vec{0},\vec{0}),0)$ is induced by an automorphism θ of the elation group $G^{\otimes}$. Moreover, each such automorphism is defined in terms of a field automorphism $\sigma\in Aut(F)$ and two 4×4 matrices over F, H and S, with H invertible. From Eq.(10) of [4] and the material that follows, if we write $\theta=\theta(\sigma,H,S)$, then we have

$$\theta(\sigma_1,H_1,S_1)\cdot\theta(\sigma_2,H_2,S_2)=\theta(\sigma_1\cdot\sigma_2, {H_1}^{\sigma_2}H_2, [det(H_2)]^{\frac{1}{2}}{S_1}^{\sigma_2}+{H_1}^{\sigma_2}S_2({H_1}^{\sigma_2})^T). \tag{5}$$

By putting $\tau=2$ in Theorem 3.1 of [4] (since $y_t=t^{\frac{1}{2}}$ here), we see that if $\theta(\sigma,H,S)$ stabilizes the 4-gonal family $\mathcal{J}(\mathcal{C})$, then $H=A\otimes B$, where $A,B\in GL(2,q)$. We may assume $\Delta=det(B)=1$ and write $A=\begin{pmatrix} a_4 & a_2\\ a_3 & a_1\end{pmatrix}$. Then

$$S=(I\otimes B)\begin{pmatrix} {a_4}^2A_{(a_2/a_4)^2} & a_2a_3P\\ 0 & {a_3}^2A_{(a_1/a_3)^2}\end{pmatrix}(I\otimes B)^T. \tag{6}$$

And it follows (cf. part (ii) of Theorem 3.1 of []) that if $\mu = det(A), \quad 1 = det(B)$, then

$$\theta = \theta(\sigma, A \otimes B, S) : ((\alpha, \beta), c) \mapsto$$

$$((\alpha^\sigma, \beta^\sigma)(A \otimes B), \quad \mu c^\sigma + {a_4}^2 \alpha^\sigma (B A_{(a_2/a_4)^2} B^T)(\alpha^\sigma)^T + \tag{7}$$

$$a_2 a_3 (\alpha^\sigma \circ \beta^\sigma) + {a_3}^2 \beta^\sigma (B A_{(a_1/a_3)^2} B^T)(\beta^\sigma)^T).$$

Until further notice we assume that $\theta(\sigma, H, S)$ is *linear*, i.e., $\sigma = id$. And since S is determined by $A \otimes B$, we may write $\theta(id, A \otimes B, S) = \theta(A \otimes B)$. With A as above and $B = \begin{pmatrix} b_4 & b_2 \\ b_3 & b_1 \end{pmatrix}$, we now have (cf. Eqs. (22) through (25) of [])

$$\begin{array}{ll} (i) & \theta(A \otimes B) : \mathcal{L}_\gamma \mapsto \mathcal{L}_{\gamma A}, \qquad \text{so that } \mathcal{L}_t \mapsto \mathcal{L}_{\bar{t}}, \\ & \text{where } \bar{t} = \frac{a_1 t + a_2}{a_3 t + a_4}, \quad t \in \tilde{F}. \\ & \\ (ii) & \theta(A \otimes B) : \mathcal{R}_\alpha \mapsto \mathcal{R}_{\alpha B}, \qquad \text{so that } \mathcal{R}_t \mapsto \mathcal{R}_{\bar{t}}, \\ & \text{where } \bar{t} = \frac{b_1 t + b_2}{b_3 t + b_4}, \quad t \in \tilde{F}. \end{array} \tag{8}$$

From [] we recall that for $\alpha \in PG(1,q)$, $A(t) \cap \mathcal{R}_\alpha = \{d(\gamma_{t^{\frac{1}{2}}} \otimes \alpha, g_t(\alpha)) = (\gamma_{t^{\frac{1}{2}}} \otimes d\alpha, g_t(d\alpha)) : d \in F\}$. So $A(t) \cap \mathcal{R}_\alpha$ is the point $\bar{p}_\alpha(t) = (\gamma_{t^{\frac{1}{2}}} \otimes \alpha, g_t(\alpha))$ in the projective plane $\bar{\mathcal{R}}_\alpha$. And $\bar{\mathcal{O}}_\alpha = \{\bar{p}_\alpha(t) : t \in \tilde{F}\}$ is an oval in $\bar{\mathcal{R}}_\alpha$. The map (use the notation $(a,b)^{(2)} = (a^2, b^2)$)

$$\pi_\alpha : \bar{\mathcal{R}}_\alpha \to PG(2,q) : (\gamma \otimes \alpha, c) \mapsto (\gamma^{(2)}, c) \tag{9}$$

is an isomorphism mapping $\bar{\mathcal{O}}_\alpha$ to the oval

$$\mathcal{O}_\alpha = \{p_\alpha(t) = (\gamma_t, g_t(\alpha)) : t \in \tilde{F}\} \text{ in } PG(2,q). \tag{10}$$

It follows that in $\theta = \theta(A \otimes B)$ as given above, the matrix A determines the action of θ on the subgroups $\mathcal{L}_t$, and hence on the lines of $GQ(\mathcal{C})$ through the point (∞). Similarly, the matrix B determines the action of θ on the subgroups $\mathcal{R}_\alpha$, and hence on the ovals $\mathcal{O}_\alpha$. In [6] it was shown that for the Subiaco q-clans $\mathcal{C}$ there was a certain collineation $\theta = \theta(A \otimes B)$, where $A = M$ and $B = M^{-5}$ for a carefully chosen M, and where θ permutes the lines through (∞) in one cycle oflength $q+1$.

Motivated by the computer-generated examples of q-clans for $q = 64$ and $q = 256$, we wish to mimic this situation and make the assumption that a collineation of the form $\theta(M \otimes M^{-p})$ exists for some other value of p. The general theory is worked out in Section 2, where it is shown how to use $\theta(M \otimes M^{-p})$ to generate all the matrices of $\mathcal{C}$ starting with a special member. Then in Section 3, again motivated by the existing examples, we consider an additional assumption that the generalized quadrangle $GQ(\mathcal{C})$ admit a certain involution. In Section 4 we report on some computer results based on these ideas, whereby a search loops over all the possibilities for p and a starting matrix and attempts to generate a complete q-clan. This search succeeds in producing a q-clan for $q=1024$, $q=4096$, $q=16384$ and $q=65536$. In each case the value of p is $(q-1)/3$, and hence we conjecture the existence of an infinite family of this form.

2 A Cyclic q-Clan

Let $F \subset E = GF(q^2)$, and let ζ be a primitive element for E. Put $\lambda = \zeta^{q-1}$ and $\delta = \lambda + \lambda^{-1}$. For any rational number a with denominator prime to $q+1$, put $[a] = \lambda^a + \lambda^{-a} \in F$. Then we have the following basic arithmetic properties for the symbol $[a]$:

$$\begin{array}{ll} (i) & [a] = \lambda^a + \lambda^{-a}; \quad [1] = \delta; \quad [0] = 0. \\ (ii) & [a] = [b] \text{ if and only if } a \equiv \pm b \pmod{q+1}. \\ (iii) & [a][b] = [a+b] + [a-b]. \\ (iv) & \text{For } \sigma = 2^i \in Aut(F), \quad [a]^\sigma = [a\sigma]. \\ (v) & [\frac{a+c}{2}][\frac{a}{2}][\frac{c}{2}] = [a+c] + [a] + [c]. \end{array} \tag{11}$$

Put $M = \begin{pmatrix} 0 & 1 \\ 1 & \delta^{\frac{1}{2}} \end{pmatrix}$. Then M has eigenvalues $\lambda^{\frac{1}{2}}$ and $\lambda^{\frac{-1}{2}}$. By diagonalizing M over E, with a bit of routine computation we may compute the powers of M:

$$M^j = \frac{1}{\delta^{\frac{1}{2}}} \begin{pmatrix} \left[\frac{j-1}{2}\right] & \left[\frac{j}{2}\right] \\ \left[\frac{j}{2}\right] & \left[\frac{j+1}{2}\right] \end{pmatrix}. \tag{12}$$

We note that M is symmetric with determinant 1 and that M has multiplicative order $q+1$. Also observe that $MPM = P$ and hence $M^jPM^j = P$ for any j.

We are now ready to make the basic assumption of this work.

Basic Assumption: There is a normalized q-clan $\mathcal{C} = \{A_t : t \in F\}$ with $A_\delta = \begin{pmatrix} x_\delta & \delta^{\frac{1}{2}} \\ 0 & z_\delta \end{pmatrix}$ given, and there is an integer p modulo $q+1$ for which $\theta = \theta(M \otimes M^{-p})$ is an automorphism of $G^\otimes$ leaving $\mathcal{J}(\mathcal{C})$ invariant.

Since M has eigenvalues $\lambda^{\pm\frac{1}{2}}$, $\theta(M \otimes M^{-p})$ will permute the members of $\mathcal{J}(\mathcal{C})$ (and hence the members of $\mathcal{C}$) in a cycle of length $q+1$.

Basic Goal: Starting with $A_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = A_\infty$, and $A_\delta = \begin{pmatrix} x_\delta & \delta^{\frac{1}{2}} \\ 0 & z_\delta \end{pmatrix}$, and the collineation $\theta(M \otimes M^{-p})$, we want to reconstruct the entire q-clan $\mathcal{C}$.

To achieve this goal we will make use of equations 5, 6 and 12. For linear automorphisms we have

$$\theta(H_1, S_1)\theta(H_2, S_2) = \theta(H_1 H_2, S_3),$$

where we can compute S_3 either by using Equation 5 or by using Equation 6 to go directly from $H_1 H_2$ to S_3.

We illustrate the procedure and the computational techniques by computing θ^2 before proceeding to the general case.

By using Equation 6 on θ and on θ^2 we see that

$$\theta = \theta(M \otimes M^{-p}) = \theta\left(M \otimes M^{-p}, \begin{pmatrix} 0 & P \\ 0 & M^{-p} A_\delta M^{-p} \end{pmatrix}\right), \tag{13}$$

and that

$$\theta^2 = \theta(M^2 \otimes M^{-2p}) = \theta\left(M^2 \otimes M^{-2p}, \begin{pmatrix} M^{-2p} A_\delta M^{-2p} & \delta P \\ 0 & M^{-2p}\left(\delta A_{\left(\frac{[3]}{[2]}\right)}\right) M^{-2p} \end{pmatrix}\right). \tag{14}$$

Now we want to use Eq. 5 to compute θ^2 as $\theta \cdot \theta$. So

$$\begin{aligned} \theta \cdot \theta \;=\; & \theta\left(M^2 \otimes M^{-2p}, \; \begin{pmatrix} 0 & P \\ 0 & M^{-p} A_\delta M^{-p} \end{pmatrix} + \right. \\ & \left. \left(\begin{pmatrix} 0 & 1 \\ 1 & \delta^{\frac{1}{2}} \end{pmatrix} \otimes M^{-p}\right) \begin{pmatrix} 0 & P \\ 0 & M^{-p} A_\delta M^{-p} \end{pmatrix} \left(\begin{pmatrix} 0 & 1 \\ 1 & \delta^{\frac{1}{2}} \end{pmatrix} \otimes M^{-p}\right)\right). \end{aligned} \tag{15}$$

To simplify the bottom line of Eq. 15 we observe that

$$\begin{pmatrix} 0 & M^{-p} \\ M^{-p} & \delta^{\frac{1}{2}} M^{-p} \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & M^{-p} A_\delta M^{-p} \end{pmatrix} \begin{pmatrix} 0 & M^{-p} \\ M^{-p} & \delta^{\frac{1}{2}} M^{-p} \end{pmatrix} =$$

$$= \begin{pmatrix} M^{-2p} A_\delta M^{-2p} & \delta^{\frac{1}{2}} M^{-2p} A_\delta M^{-2p} \\ M^{-p} P M^{-p} + \delta^{\frac{1}{2}} M^{-2p} A_\delta M^{-2p} & \delta^{\frac{1}{2}} M^{-p} P M^{-p} + \delta M^{-2p} A_\delta M^{-2p} \end{pmatrix}.$$

We now mention two computational simplifications that will be used without special mention from now on. First recall that if $A = \begin{pmatrix} x & y \\ w & z \end{pmatrix}$ and $A' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A \equiv A'$ means that $x = a, x = d$, and $y + w = b + c$.

For $S = \begin{pmatrix} C & D_1 \\ D_2 & E \end{pmatrix}$, $\alpha S \beta^T = \alpha C \alpha^T + \alpha(D_1 + {D_2}^T)\beta^T + \beta E \beta^T$. Hence in $\theta(H, S)$, we may replace C with any $C' \equiv C$, and E with any $E' \equiv E$, where we usually choose C' and E' to be upper triangular. And we may replace D_1 with $D_1 + {D_2}^T$ and D_2 with 0.

Also, $M^jP(M^j)^T = M^jPM^j = P$. Then in the lower right hand block of the last displayed equation above, $M^{-p}PM^{-p} \equiv 0$. And after the indicated simplifications, the upper right hand block is $M^{-p}PM^{-p} + \delta^{\frac{1}{2}}M^{-2p}(A_\delta + A_\delta{}^T)M^{-2p} \equiv P + \delta M^{-2p}PM^{-2p} \equiv (1+\delta)P$. Now we may complete the computation of $\theta \cdot \theta$.

$$\theta^2 = \theta \cdot \theta = \theta\left(M^2 \otimes M^{-2p}, \begin{pmatrix} M^{-2p}A_\delta M^{-2p} & \delta P \\ 0 & M^{-p}A_\delta M^{-p} + \delta M^{-2p}A_\delta M^{-2p} \end{pmatrix}\right). \quad (16)$$

Comparing the last two equations for θ^2 we find

$$\delta A_{\left(\frac{[3]}{[2]}\right)} \equiv \frac{[2]}{\delta} A_{\left(\frac{[3]}{[2]}\right)} \equiv M^p A_\delta M^p + \delta A_\delta. \quad (17)$$

Now we shall perform a similar computation, but in the general case. Since $\theta^j = \theta(M^j \otimes M^{-pj}, S_j)$ for some S_j, we may use Eqs. 6 and 12 to write out the following:

$$\theta^j = \theta\left(M^j \otimes M^{-pj}, \begin{pmatrix} M^{-pj}C_jM^{-pj} & D_j \\ 0 & M^{-pj}E_jM^{-pj} \end{pmatrix}\right), \quad \text{where}$$

$$(i) \quad C_j = \tfrac{[j-1]}{\delta} A_{\left(\frac{[j]}{[j-1]}\right)}, \quad (18)$$

$$(ii) \quad D_j = \tfrac{[j]}{\delta} P,$$

$$(iii) \quad E_j = \tfrac{[j]}{\delta} A_{\left(\frac{[j+1]}{[j]}\right)} = C_{j+1}.$$

Remark 2.1 $\{\infty = \frac{[1]}{[0]},\ \delta = \frac{[2]}{[1]},\ \ldots,\ \frac{[q+1]}{[q]} = 0\} \ = \ \tilde{F}.$

Proof: We have $\frac{[j+1]}{[j]} = \frac{[k+1]}{[k]}$ iff $[j+1][k] = [j][k+1]$ iff $[j+1+k] + [j+1-k] = [j+k+1] + [j-k-1]$ iff $j-k+1 \equiv \pm(j-k-1) \pmod{q+1}$ iff $2(j-k) \equiv 0 \pmod{q+1}$. ∎

Theorem 2.2 *There is a recurrence formula for* A_t*:*

$$(i) \qquad A_{\left(\frac{[j+1]}{[j]}\right)} \equiv A_\delta + \tfrac{[j-1]}{[j]} M^p A_{\left(\frac{[j]}{[j-1]}\right)} M^p,$$

which is equivalent to

$$(ii) \quad E_j \equiv \tfrac{[j]}{\delta} A_\delta + M^p E_{j-1} M^p. \quad (\textit{For } E_j \textit{ as in Eq. 18}).$$

Proof: The idea is to use the fact that θ^{j+1} as given by Eq. 18 must be the same as that computed according to $\theta^{j+1} = \theta^j \cdot \theta$. So $\theta^j \cdot \theta =$

$$\theta\left(M^j \otimes M^{-pj},\ \begin{pmatrix} M^{-pj}C_jM^{-pj} & \frac{[j]}{\delta}P \\ 0 & M^{-pj}E_jM^{-pj} \end{pmatrix}\right) \cdot \theta\left(M \otimes M^{-p},\ \begin{pmatrix} 0 & P \\ 0 & M^{-p}A_\delta M^{-p} \end{pmatrix}\right)$$

$$= \theta\left(M^{j+1} \otimes M^{-p(j+1)},\ \begin{pmatrix} M^{-pj}C_jM^{-pj} & \frac{[j]}{\delta}P \\ 0 & M^{-pj}E_jM^{-pj} \end{pmatrix} + \right.$$

$$\left. \frac{1}{\delta}\begin{pmatrix} \left[\frac{j-1}{2}\right]M^{-pj} & \left[\frac{j}{2}\right]M^{-pj} \\ \left[\frac{j}{2}\right]M^{-pj} & \left[\frac{j+1}{2}\right]M^{-pj} \end{pmatrix} \begin{pmatrix} 0 & P \\ 0 & M^{-p}A_\delta M^{-p} \end{pmatrix} \begin{pmatrix} \left[\frac{j-1}{2}\right]M^{-pj} & \left[\frac{j}{2}\right]M^{-pj} \\ \left[\frac{j}{2}\right]M^{-pj} & \left[\frac{j+1}{2}\right]M^{-pj} \end{pmatrix}\right)$$

$= \theta\left(M^{j+1} \otimes M^{-pj},\ S\right)$, where we need to work a bit to find S.

$$S = \begin{pmatrix} M^{-pj}C_jM^{-pj} & \frac{[j]}{\delta}P \\ 0 & M^{-pj}E_jM^{-pj} \end{pmatrix} +$$

$$\frac{1}{\delta}\begin{pmatrix} 0 & \left[\frac{j-1}{2}\right]M^{-pj}P + \left[\frac{j}{2}\right]M^{-p(j+1)}A_\delta M^{-p} \\ 0 & \left[\frac{j}{2}\right]M^{-pj}P + \left[\frac{j+1}{2}\right]M^{-p(j+1)}A_\delta M^{-p} \end{pmatrix} \begin{pmatrix} \left[\frac{j-1}{2}\right]M^{-pj} & \left[\frac{j}{2}\right]M^{-pj} \\ \left[\frac{j}{2}\right]M^{-pj} & \left[\frac{j+1}{2}\right]M^{-pj} \end{pmatrix}$$

$$\equiv \begin{pmatrix} M^{-pj}C_jM^{-pj} & \frac{[j]}{\delta}P \\ 0 & M^{-pj}E_jM^{-pj} \end{pmatrix} +$$

$$\frac{1}{\delta}\begin{pmatrix} [j]M^{-p(j+1)}A_\delta M^{-p(j+1)} & \left[\frac{j-1}{2}\right]\left[\frac{j+1}{2}\right]P + \left[\frac{j}{2}\right]\left[\frac{j+1}{2}\right]M^{-p(j+1)}A_\delta M^{-p(j+1)} \\ [j]P + \left[\frac{j+1}{2}\right]\left[\frac{j}{2}\right]M^{-p(j+1)}A_\delta M^{-p(j+1)} & [j+1]M^{-p(j+1)}A_\delta M^{-p(j+1)} \end{pmatrix}$$

$$\equiv \begin{pmatrix} M^{-pj}C_jM^{-pj} + \frac{[j]}{\delta}M^{-p(j+1)}A_\delta M^{-p(j+1)} & \frac{\left[\frac{j-1}{2}\right]\left[\frac{j+1}{2}\right]}{\delta}P + \frac{\left[\frac{j+1}{2}\right]\left[\frac{j}{2}\right]}{\delta}M^{-p(j+1)}(\left[\frac{1}{2}\right]P)M^{-p(j+1)} \\ 0 & \frac{[j+1]}{\delta}M^{-p(j+1)}A_\delta M^{-p(j+1)} + M^{-pj}E_jM^{-pj} \end{pmatrix}$$

$$\equiv \begin{pmatrix} M^{-pj}C_jM^{-pj} + \frac{[j]}{\delta}M^{-p(j+1)}A_\delta M^{-p(j+1)} & \frac{[j+1]}{\delta}P \\ 0 & M^{-pj}E_jM^{-pj} + \frac{[j+1]}{\delta}M^{-p(j+1)}A_\delta M^{-p(j+1)} \end{pmatrix}.$$

If we now use Eq. 18 to see what the above matrix should be, both the upper left and lower right blocks yield the desired recurrence. ■

Corollary 2.3 *The q-clan being generated by powers of θ is automatically indexed so that if $A_t \equiv \begin{pmatrix} x_t & y_t \\ 0 & z_t \end{pmatrix}$, then $y_t = t^{\frac{1}{2}}$.*

Proof: The "y" entry of $A_{\left(\frac{[j+1]}{[j]}\right)}$ should be $\frac{\left[\frac{j+1}{2}\right]}{\left[\frac{j}{2}\right]}$. By Theorem 2.2 (since M^p is symmetric with determinant 1) it is $\left[\frac{1}{2}\right] + \frac{[j-1]\left[\frac{j}{2}\right]}{[j]\left[\frac{j-1}{2}\right]} = \left[\frac{1}{2}\right] + \frac{\left[\frac{j-1}{2}\right]}{\left[\frac{j}{2}\right]} = \frac{\left[\frac{j+1}{2}\right]}{\left[\frac{j}{2}\right]}$. ■

Corollary 2.4 $A_{\left(\frac{[j+1]}{[j]}\right)} \equiv \sum_{i=0}^{j-1} \frac{[i+1]}{[j]} M^{p(j-1-i)} A_\delta M^{p(j-1-i)} \equiv \sum_{k=0}^{j-1} \frac{[j-k]}{[j]} M^{pk} A_\delta M^{pk}, j \geq 1.$

Proof: Straightforward induction starting with part (i) of Theorem 2.2. ■
Recall that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} \equiv \begin{pmatrix} a^2x + aby + b^2z & (ad+bc)y \\ 0 & c^2x + dcy + d^2z \end{pmatrix}. \tag{19}$$

Then use $M^{pk} = \frac{1}{\delta^{\frac{1}{2}}} \begin{pmatrix} \left[\frac{pk-1}{2}\right] & \left[\frac{pk}{2}\right] \\ \left[\frac{pk}{2}\right] & \left[\frac{pk+1}{2}\right] \end{pmatrix}$ and $A_\delta \equiv \begin{pmatrix} x_\delta & \delta^{\frac{1}{2}} \\ 0 & z_\delta \end{pmatrix}$ to write out $M^{pk} A_\delta M^{pk}$ and use Cor. 2.4 to obtain the following.

Corollary 2.5

$$A_{\left(\frac{[j+1]}{[j]}\right)} \equiv \sum_{k=0}^{j-1} \frac{[j-k]}{[j]} \begin{pmatrix} \frac{[pk-1]x_\delta + ([pk]+[pk-1]+[1]) + [pk]z_\delta}{\delta} & \delta^{\frac{1}{2}} \\ 0 & \frac{[pk]x_\delta + ([pk]+[pk+1]+[1]) + [pk+1]z_\delta}{\delta} \end{pmatrix}.$$

Remark 2.6 *Corollary 2.5 gives each member of the q-clan as a concrete function of A_δ.*

An easy computation shows that

$$\theta^{-1} = \theta\left(M^{-1} \otimes M^p, \begin{pmatrix} A_\delta & P \\ 0 & 0 \end{pmatrix}\right). \tag{20}$$

Now put $j = -1 \equiv q \pmod{q+1}$ in Eq. 18 and compare with Eq. 20 to obtain

$$A_\delta \equiv \frac{[2]}{\delta} M^p A_{\left(\frac{[1]}{[2]}\right)} M^p, \text{i.e., } \delta A_{\delta^{-1}} \equiv M^{-p} A_\delta M^{-p}. \tag{21}$$

Theorem 2.7 *Replacing the exponent $-p$ with p is equivalent to interchanging x_δ and z_δ in A_δ.*

Proof: $\theta\left(id, I \otimes P, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$ is an involutory automorphism of $G^\otimes$ mapping $(\gamma \otimes \alpha, c)$ to $(\gamma \otimes \alpha P, c)$, i.e., it leaves invariant each $\mathcal{L}_\gamma$. Also, it maps $(\gamma_{t^{\frac{1}{2}}} \otimes \alpha, \alpha A_t \alpha^T)$ to $(\gamma_{t^{\frac{1}{2}}} \otimes \alpha P, \alpha P(PA_tP)(\alpha P)^T)$. So it replaces the q-clan $\mathcal{C} = \{A_t : t \in F\}$ with the q-clan $\mathcal{C}' = \{A'_t = PA_tP : t \in F\}$.

If $\phi = \theta\left(id, I \otimes P, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right)$ and $\theta = \theta\left(id, M \otimes M^{-p}, \begin{pmatrix} 0 & P \\ 0 & M^{-p}A_\delta M^{-p} \end{pmatrix}\right)$, so θ is the putative isomorphism of $G^\otimes$ permuting the members of $\mathcal{J}(\mathcal{C})$ in a cycle of length $q+1$, then $\phi \cdot \theta \cdot \phi$ is an automorphism of $G^{\otimes}$ permuting the members of $\mathcal{J}(\mathcal{C}')$ in a cycle of length $q+1$.

First note $(I \otimes P)\begin{pmatrix} 0 & P \\ 0 & M^{-p}A_\delta M^{-p} \end{pmatrix}(I \otimes P)^T = \begin{pmatrix} 0 & P \\ 0 & PM^{-p}A_\delta M^{-p}P \end{pmatrix} =$ $\begin{pmatrix} 0 & P \\ 0 & M^p(PA_\delta P)M^p \end{pmatrix} = \begin{pmatrix} 0 & P \\ 0 & M^pA'_\delta M^p \end{pmatrix}$.

Then $\phi \cdot \theta \cdot \phi =$

$$\theta\left(id, M \otimes PM^{-p}, (I \otimes P)\begin{pmatrix} 0 & P \\ 0 & M^{-p}A_\delta M^{-p} \end{pmatrix}(I \otimes P)^T\right) \cdot \theta\left(id, I \otimes P, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right) =$$
$$\theta\left(id, M \otimes PM^{-p}P, \begin{pmatrix} 0 & P \\ 0 & M^pA'_\delta M^p \end{pmatrix}\right) = \theta\left(id, M \otimes M^p, \begin{pmatrix} 0 & P \\ 0 & M^pA'_\delta M^p \end{pmatrix}\right). \blacksquare$$

3 Cyclic q-Clans with Involutions

Continue with exactly the same setup as in the previous section, but now in addition consider when the automorphism $\phi = \theta(\sigma, A \otimes B)$ of $G^{\otimes}$ would leave invariant the 4-gonal family $\mathcal{J}(\mathcal{C})$. If $A = \begin{pmatrix} a_4 & a_2 \\ a_3 & a_1 \end{pmatrix}$, such an automorphism must have the form (see Eq. 20 and Theorem 3.1 of [Tensor])

$$\phi = \theta\left(\sigma, A \otimes B, (I \otimes B)\begin{pmatrix} a_4^2 A_{(a_2/a_4)^2} & a_2a_3P \\ 0 & a_3^2 A_{(a_1/a_3)^2} \end{pmatrix}(I \otimes B)^T\right), \tag{22}$$

where $\phi : A(t) \mapsto A(t')$ satisfies

$$t' = \frac{a_1^2 t^\sigma + a_2^2}{a_3^2 t^\sigma + a_4^2}, \quad t \in \tilde{F}. \tag{23}$$

Now suppose ϕ is an involution interchanging $A(\infty)$ and $A(0)$, so that $a_1 = a_4 = 0$, and $A = \begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix}$. If we also assume that ϕ normalizes the group $< \theta >$ of order $q+1$ studied in the previous section, then A conjugates M to a power of M. But $A^{-1}MA = \begin{pmatrix} \delta^{\frac{1}{2}} & a^2 \\ a^{-2} & 0 \end{pmatrix}$ is a power of M if and only if it is equal to $M^{-1} = \begin{pmatrix} \delta^{\frac{1}{2}} & 1 \\ 1 & 0 \end{pmatrix}$ and then $a = 1$, i.e., $A = P$.

But then writing out that $\phi^2 = id$ shows that B must be an involutory matrix for which $BM^{-p}B = M^p$. If p is prime to $q+1$ (as it is for $p = -2/3$, $q = 4^h$), then $BM^{-1}B = M$, and we must have $B = P$ also. So: *the basic additional assumption of this section* is that

$$\phi = \theta\left(id, P\otimes P, \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}\right) \text{ is an automorphism of } G^{\otimes} \text{ leaving } \mathcal{J}(\mathcal{C}) \text{ invariant.} \tag{24}$$

For $t \in \tilde{F}$, $A\left(\frac{[j+1]}{[j]}\right) = \left\{\left(\gamma_{\left(\frac{[j+1]}{[j]}\right)^{\frac{1}{2}}} \otimes \alpha, \alpha A_{\left(\frac{[j+1]}{[j]}\right)}\alpha^T\right) : \alpha \in F^2\right\}$. Since $\left(\gamma_{\left(\frac{[j+1]}{[j]}\right)^{\frac{1}{2}}} \otimes \alpha\right)(P\otimes P) = \left(([j],[j+1])^{\frac{1}{2}} \otimes \frac{1}{\left[\frac{j}{2}\right]}\alpha\right)(P\otimes P) =$ $\left(([j+1],[j])^{\frac{1}{2}} \otimes \frac{1}{\left[\frac{j}{2}\right]}\alpha P\right) = \left(\gamma_{\left(\frac{[j]}{[j+1]}\right)^{\frac{1}{2}}} \otimes \frac{\left[\frac{j+1}{2}\right]}{\left[\frac{j}{2}\right]}\alpha P\right)$, for all $j \not\equiv 0, -1 \pmod{q+1}$, ϕ maps the element $\left(\gamma_{\left(\frac{[j+1]}{[j]}\right)^{\frac{1}{2}}} \otimes \alpha, \alpha A_{\left(\frac{[j+1]}{[j]}\right)}\alpha^T\right)$ of $A(\frac{[j+1]}{[j]})$ to the element $\left(\gamma_{\left(\frac{[j]}{[j+1]}\right)^{\frac{1}{2}}} \otimes \frac{\left[\frac{j+1}{2}\right]}{\left[\frac{j}{2}\right]}\alpha P, \frac{\left[\frac{j+1}{2}\right]}{\left[\frac{j}{2}\right]}\alpha P\left(\frac{[j]}{[j+1]}PA_{\left(\frac{[j+1]}{[j]}\right)}P\right)\left(\frac{\left[\frac{j+1}{2}\right]}{\left[\frac{j}{2}\right]}\alpha P\right)^T\right)$, which is supposed to be in

$A\left(\frac{[j]}{[j+1]}\right)$. And this is the case if and only if $\frac{[j]}{[j+1]}PA_{\left(\frac{[j+1]}{[j]}\right)}P \equiv A_{\left(\frac{[j]}{[j+1]}\right)}$, $j \not\equiv 0, -1 \pmod{q+1}$. Clearly ϕ interchanges $A(0)$ and $A(\infty)$, so we may restate this result as follows:

$$\begin{array}{c} \phi = \theta\left(id, P\otimes P, \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}\right) \text{ is a collineation of } GQ(\mathcal{C}) \text{ iff} \\ [j]PA_{\left(\frac{[j+1]}{[j]}\right)}P \equiv [j+1]A_{\left(\frac{[j]}{[j+1]}\right)} \text{ for all } j \bmod q+1. \end{array} \tag{25}$$

Remark 3.1 *Note that Eq. 25 implies that $PA_tP \equiv tA_{t^{-1}}$ for all nonzero $t \in F$ if ϕ is a collineation.*

Theorem 3.2 *The following are equivalent:*

(i) ϕ *is a collineation of* $GQ(\mathcal{C})$.

(ii) $A_\delta \equiv M^pPA_\delta PM^p$.

(iii) $([p]+[1])x_\delta + [p-1]z_\delta = [p]+[p-1]+[1]$, *where* $A_\delta = \begin{pmatrix} x_\delta & \delta^{\frac{1}{2}} \\ 0 & z_\delta \end{pmatrix}$.

Proof: We know that (i) is equivalent to Eq. 25. In Eq. 25 put $j = 1$, to get the fact that (i) implies

$$PA_\delta P \equiv \delta A_{\delta^{-1}}. \tag{26}$$

By Eq. 21, Eq. 26 implies condition (ii). Hence now we must show that condition (ii) implies Eq. 25. So suppose that $A_\delta \equiv M^p P A_\delta P M^p$, which by Eq. 21 implies $PA_\delta P \equiv \delta A_{\delta^{-1}}$. Then for all $j \geq 1$, $P\left([j]A_{\left(\frac{[j+1]}{[j]}\right)}\right)P \equiv \sum_{k=0}^{j-1}[j-k]PM^{pk}P(PA_\delta P)PM^{pk}P \equiv \sum_{k=0}^{j-1}[j-k]M^{-pk}(\delta M_{\delta^{-1}})M^{-pk} \equiv \sum_{k=0}^{j-1}[j-k]M^{-p(k+1)}A_\delta M^{-p(k+1)} \equiv \sum_{k=1}^{j}[j-k+1]M^{-pk}A_\delta M^{-pk} \equiv \sum_{k=1}^{j}[q-j+k]M^{-pk}A_\delta M^{-pk} \equiv \sum_{i=0}^{j-1}[q-i]M^{-p(j-i)}A_\delta M^{-p(j-i)}$.

So at this point we know that Eq. 26 implies

$$P\left([j]A_{\left(\frac{[j+1]}{[j]}\right)}\right)P \equiv \sum_{i=0}^{j-1}[q-i]M^{-p(j-i)}A_\delta M^{-p(j-i)},\ j \geq 1. \tag{27}$$

Put $j=q$ in Corollary 2.4 to get $0 \equiv \sum_{k=0}^{q-1}[q-k]M^{pk}A_\delta M^{pk}$. Fix j, $0<j\leq q-1$. Then $\sum_{k=0}^{j-1}[q-k]M^{pk}A_\delta M^{pk} \equiv \sum_{k=j}^{q-1}[q-k]M^{pk}A_\delta M^{pk}$. Multiply through on the left and on the right by M^{-pk} to get $\sum_{k=0}^{j-1}[q-k]M^{p(k-j)}A_\delta M^{p(k-j)} \equiv \sum_{k-j}^{q-1}[q-k]M^{p(k-j)}A_\delta M^{p(k-j)}$, i.e.,

$$\begin{aligned}
\textstyle\sum_{i=0}^{j-1}[q-i]M^{-p(j-i)}A_\delta M^{-p(j-i)} &\equiv \textstyle\sum_{i=j}^{q-1}[q-i]M^{p(i-j)}A_\delta M^{p(i-j)} \\
\equiv \textstyle\sum_{k=0}^{q-j-1}[q-j-k]M^{pk}A_\delta M^{pk} &\equiv [q-j]A_{\left(\frac{[q-j+1]}{[q-j]}\right)} \quad (\text{put } i-j=k) \\
\equiv [-j-1]A_{\left(\frac{[-j]}{[-j-1]}\right)} &\equiv [j+1]A_{\left(\frac{[j]}{[j+1]}\right)}. \quad (\text{by Cor. 2.4})
\end{aligned} \tag{28}$$

So by Eqs. 27 and 28 together we see that condition (ii) does imply Eq. 25. This completes a proof that conditions (i) and (ii) are equivalent.

$$M^pP = \frac{1}{\delta^{\frac{1}{2}}}\begin{pmatrix} \left[\frac{p-1}{2}\right] & \left[\frac{p}{2}\right] \\ \left[\frac{p}{2}\right] & \left[\frac{p+1}{2}\right] \end{pmatrix} P = \begin{pmatrix} \frac{\left[\frac{p}{2}\right]}{\left[\frac{1}{2}\right]} & \frac{\left[\frac{p-1}{2}\right]}{\left[\frac{1}{2}\right]} \\ \frac{\left[\frac{p+1}{2}\right]}{\left[\frac{1}{2}\right]} & \frac{\left[\frac{p}{2}\right]}{\left[\frac{1}{2}\right]} \end{pmatrix}. \tag{29}$$

Recall that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_\delta & \delta^{\frac{1}{2}} \\ 0 & z_\delta \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} \equiv \begin{pmatrix} a^2x_\delta + ab\delta^{\frac{1}{2}} + b^2z_\delta & (ad+bc)\delta^{\frac{1}{2}} \\ 0 & c^2x_\delta + cd\delta^{\frac{1}{2}} + d^2z_\delta \end{pmatrix}$.

Hence condition (ii) is equivalent to $\begin{pmatrix} x_\delta & \delta^{\frac{1}{2}} \\ 0 & z_\delta \end{pmatrix} \equiv (M^pP)\,A_\delta\,(M^pP)^T \equiv$ $\begin{pmatrix} \frac{[p]x_\delta + \left[\frac{p}{2}\right]\left[\frac{p-1}{2}\right]\left[\frac{1}{2}\right] + [p-1]z_\delta}{[1]} & \delta^{\frac{1}{2}} \\ 0 & \frac{[p+1]x_\delta + \left[\frac{p+1}{2}\right]\left[\frac{p}{2}\right]\left[\frac{1}{2}\right] + [p]z_\delta}{[1]} \end{pmatrix}$. Comparing first the upper left hand entries, then lower right hand entries, we have

$$
\begin{aligned}
([p]+[1])x_\delta + [p-1]z_\delta &= [p]+[p-1]+[1] \quad = \left[\tfrac{p}{2}\right]\left[\tfrac{p-1}{2}\right]\left[\tfrac{1}{2}\right] && \text{iff} \\
\left[\tfrac{p+1}{2}\right]\left[\tfrac{p-1}{2}\right]x_\delta + \left[\tfrac{p-1}{2}\right]\left[\tfrac{p-1}{2}\right]z_\delta &= \left[\tfrac{p}{2}\right]\left[\tfrac{p-1}{2}\right]\left[\tfrac{1}{2}\right] && \text{iff} \\
\left[\tfrac{p+1}{2}\right]x_\delta + \left[\tfrac{p-1}{2}\right]z_\delta &= \left[\tfrac{p}{2}\right]\left[\tfrac{1}{2}\right] && \text{iff} \\
\left[\tfrac{p+1}{2}\right]\left[\tfrac{p+1}{2}\right]x_\delta + \left[\tfrac{p-1}{2}\right]\left[\tfrac{p+1}{2}\right]z_\delta &= \left[\tfrac{p+1}{2}\right]\left[\tfrac{p}{2}\right]\left[\tfrac{1}{2}\right] && \text{iff} \\
[p+1]x_\delta + ([p]+[1])z_\delta &= [p+1]+[p]+[1].
\end{aligned}
$$

Hence conditions (ii) and (iii) are equivalent. ∎

Warning: In the above proof we divided by $\left[\frac{p-1}{2}\right]$, which is not allowed if $p = 1$.

4 Computer Results

A computer search was performed guided by the results of the previous sections. The basic idea of the search is simply to loop over all the possibilities for the value p and starting matrix A_δ, and then to construct a candidate q-clan $\mathcal{C}$ as outlined in Section 2. The resulting candidate can then be checked directly to verify that it is indeed a proper q-clan.

The first task is simply to choose a single fixed value for δ and compute the function $[a]$. Rather than working in the large field $GF(q^2)$ we first select a suitable $\delta \in GF(q)$ and then precompute the function $[a]$ according to the following recurrence:

$$
\begin{aligned}
[0] &= 0 \\
[1] &= \delta \\
[a+1] &= [a-1] + [a][1]
\end{aligned}
$$

Then we loop through values of p. For each value of p we first compute and store the matrix M^p, which will be repeatedly used. Then an inner loop runs through all the possible pairs of values for x_δ and z_δ. For each such pair we form the matrix

$$
A_\delta = \begin{pmatrix} x_\delta & \delta^{\frac{1}{2}} \\ 0 & z_\delta \end{pmatrix},
$$

and then use the recurrence

$$
A_{\left(\frac{[j+1]}{[j]}\right)} \equiv A_\delta + \frac{[j-1]}{[j]} M^p A_{\left(\frac{[j]}{[j-1]}\right)} M^p
$$

to compute the $q-1$ matrices $A_{\frac{[j+1]}{[j]}}$ as j ranges from 1 to $q-1$.

The list of matrices thus produced is then checked firstly to see if each matrix is itself anisotropic and then to see if the pairwise sums are anisotropic.

This procedure is implemented in the computer algebra system Magma.

For small values of $q \leq 1024$, it is feasible to carry out the process exactly as described above. As expected, for each value of q the procedure finds a q-clan when $p = 5$. However it also finds additional q-clans where $p = 21$, $p = 85$ and $p = 341$ for $q = 64$, $q = 256$ and $q = 1024$ respectively. There are also many choices for x_δ and z_δ that yield a q-clan — in these cases there are $q/2$ choices for x_δ and two choices for z_δ for each x_δ. However, amongst this large number of possibilities there is a unique pair satisyfying the condition

$$[p+1]x_\delta + ([p]+[1])z_\delta = [p+1]+[p]+[1]$$

equivalent to the existence of the special involution.

This evidence lead us to perform a more restricted search for the values $q = 4096$, $q = 16384$ and $q = 65536$. In these cases the search was restricted to $p = (q-1)/3$ and (x_δ, z_δ) pairs that satisfy the involution equation above. In each case a q-clan was successfully constructed. The details of these q-clans are given in the table below.

q	Defining Polynomial	p	δ	x_δ	z_δ
64	$\omega^6+\omega^4+\omega^3+\omega+1$	21	ω	ω^{18}	w^{12}
256	$\omega^8+\omega^4+\omega^3+\omega^2+1$	85	ω^3	ω^{49}	ω^{12}
1024	$\omega^{10}+\omega^6+\omega^5+\omega^3+\omega^2+\omega+1$	341	ω^3	ω^{1007}	ω^{663}
4096	$\omega^{12}+\omega^7+\omega^6+\omega^5+\omega^3+\omega+1$	1365	ω^5	ω^{1356}	ω^{1039}
16384	$\omega^{14}+\omega^7+\omega^5+\omega^3+1$	5461	ω^3	ω^{4705}	ω^{1909}
65536	$\omega^{16}+\omega^5+\omega^3+\omega^2+1$	21845	ω^3	ω^{12933}	ω^{1880}

From the determination of the groups of the Subiaco q-clans in [6], it can be seen that these q-clans are not isomorphic to the Subiaco q-clans, and are therefore new.

This seems to be very compelling evidence for the existence of a new infinite family of q-clans; indeed the process above can be viewed as a recipe for finding a q-clan that has succeeded for every attainable value of q. However finding a "canonical" form for this q-clan amenable to a formal proof seems a very difficult problem.

References

[1] L. Bader, G. Lunardon and S. Payne, On q-clan geometry, $q = 2^e$, *Bull. Belgian Math. Soc., Simon Stevin*, 1(1994), 301 – 328.

[2] F. Buekenhout, editor, Handbook of Incidence Geometry: Buildings and Foundations, North Holland, 1995, xi + 1420 pp.

[3] S. E. Payne, A new infinte family of generalized quadrangles, *Congr. Numer.*, 49(1985), 115 – 128.

[4] S. E. Payne, A tensor product action on q-clan generalized quadrangles with $q = 2^e$, *Lin. Alg. and its Appl.*, 226–228(1995), 115 – 137.

[5] S. E. Payne, The fundamental theorem of q-clan geometry, *Designs, Codes and Cryptography*, to appear.

[6] S. E. Payne, T. Penttila and I. Pinneri, Isomorphisms between Subiaco q-clan geometries, *Bull. Belgian Math. Soc., Simon Stevin*, 2(1995), 197 – 222.

Addresses of Authors:

S. E. Payne
Department of Mathematics, Campus Box 170
University of Colorado at Denver
P.O. box 173364
Denver, Colorado 80217-3364
USA

spayne@carbon.cudenver.colorado.edu

T. Penttila
Department of Mathematics
University of Western Australia
Nedlands 6907
Western Australia
Australia

penttila@maths.uwa.edu.au

G. Royle
Department of Computer Science
University of Western Australia
Nedlands 6907
Western Australia
Australia

gordon@cs.uwa.edu.au