CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364

Fax: (303) 556-8550
Phone: (303) 556-8442
http://www-math.cudenver.edu/

March 1996 UCD/CCM Report No. 80

Minimizing Unconstrained Fuzzy Functions

K. David Jamison and Weldon A. Lodwick

Draft - March 7, 1996

Minimizing Unconstrained Fuzzy Functions

K. David Jamison , Weldon A. Lodwick
Department of Mathematics, Campus Box 170, University of Colorado at Denver, P.O. Box 173364,Denver, CO 80217-3364, U.S.A.
e-mail: kjamison@tiger.cudenver.edu, wlodwick@tiger.cudenver.edu

Abstract: In this paper we introduce a definition for a fuzzy function as a possibility distribution over the set of all functions with certain properties. We then examine some of the implications of this definition. In particular, we show that the image of a fuzzy function is a fuzzy vector (as defined herein). We also show that the set of values at which a real valued fuzzy function achieves its minimum is a fuzzy number and if the fuzzy function is convex, in a particular sense, then the set of vectors for which the function achieves the minimum values is connected. Lastly we discuss minimization of an unconstrained fuzzy function and how one might proceed to defuzzify the solution.

Keywords: Fuzzy number, Fuzzy vector, Fuzzy function

Introduction: This paper considers a fuzzy number as a method of representing uncertainty in a given quantity by defining a possibility distribution for the quantity. For example, *the value is observed to be about 2* might be represented by the triangular fuzzy number (1.5,2,2.5) (see [2] page 15) which defines a possibility distribution for the observation. This concept can be extended to fuzzy vectors in R^n and to fuzzy subsets over the set of all functions between two finite dimensional real vector spaces. Particular fuzzy subsets of functions with some properties to be specified below we call fuzzy functions. We show that they map crisp points to fuzzy vectors. This definition is motivated by considering the case were a given function to be studied is not precisely known. This can occur, for example, if the function is defined by parameters which are fuzzy. For example *the function is linear with slope about 3 and x-intercept near 2.* An advantage of this approach is that it allows meaningful development of useful results such as convexity and optimization under uncertainty as will be shown.

Through out this paper we will identity a fuzzy subset with the symbol $\sim$ over a letter. For example if X is a set, then $\tilde{x}$ may be used to denote a fuzzy subset of X and $\tilde{x}_\alpha$ will denote the α-level of possibility for $\tilde{x}$, i.e. it is the crisp set $\left\{ x \mid \begin{array}{ll} \mu_{\tilde{x}}(x) \geq \alpha & \text{for } \alpha \in (0,1] \\ \mu_{\tilde{x}}(x)>0 & \text{for } \alpha=0 \end{array} \right.$

We denote the set of all fuzzy subsets of X by W(X).

1 Fuzzy Functions with some Illustrative Properties

Definition 1 *Let X be a real finite dimensional vector space with the euclidean norm. A* ***fuzzy vector****, $\tilde{x}$, is a fuzzy subset of X such that 1) $\tilde{x}$ is normal and 2) $\forall\alpha \in [0,1]$ $\tilde{x}_\alpha$ is compact. A* ***convex fuzzy vector****, $\tilde{x}$, is a fuzzy vector such that $\forall\alpha \in [0,1]$ $\tilde{x}_\alpha$ is convex.*

In [4] a convex fuzzy vector is referred to as an approximate quantity. This definition of fuzzy vector is consistent with the definition of fuzzy vector given in [1]. If X=R, the definition of a convex fuzzy vector, the definition of approximate quantity in [4] and fuzzy number in [1], [2] and [9] all coincide.

Example 2 *A vector of fuzzy numbers, $(\tilde{a}_i)$, i=1,n, is a convex fuzzy vector. Each α-level of possibility is $\Pi_{i=1,n}[a_{i,}^{-(\alpha)} a_i^{+(\alpha)}]$ where $\tilde{a}_\alpha=[a_{i,}^{-(\alpha)} a_i^{+(\alpha)}]$. But not all convex fuzzy vectors can be so represented. For example, consider the convex fuzzy vector with*

$$\tilde{x}_\alpha=\{x \mid \|x-x_0\| \leq 1-\alpha \text{ for } \alpha \in [0,1]\}$$

It is desirable to have certain properties hold when defining fuzzy quantities and relationships on them. This may be for intuitive as well as practical reasons. In particular, at a given α-level of possibility, convexity offers computational advantage. For certain distributions it also has intuitive appeal. For example, if it is equally likely that the observation of a single object in R is x or y, then any z between x and y should be of equal or greater possibility. It does not seem to restrictive to expect that if there exists a quantity arbitrarily close to x that is at least α possible, then x is at least α possible. Finally, the possible values that we believe a single finite object can be should be bounded if we assume there exists a single correct measurement in the underling reality . With these properties in mind, we offer the following definition of a fuzzy function.

Definition 3 *Let X, Y be real finite dimensional vector spaces with the usual norm and*
$\mathcal{F}$={f: $\Omega \rightarrow Y \mid \Omega \subset X$ and f is a bounded function over Ω}. On $\mathcal{F}$ we define the norm $\|f\|_{\sup} = \sup_{x\in\Omega} \|f(x)\|$ (see [7]-def.7.14 and note continuity is not used in the norm proof). Let $\tilde{f}$ be a fuzzy subset of $\mathcal{F}$. For $x\in \Omega$ we define $\tilde{f}(x)$ to be the fuzzy subset of Y with membership function
*$\mu_{\tilde{f}(x)}(y) = \sup\left\{\alpha : \mu_{\tilde{f}}(f)=\alpha \text{ and } f(x)=y\right\}$. A **fuzzy function** over Ω is a fuzzy subset $\tilde{f}$ of $\mathcal{F}$ such that 1) $\tilde{f}$ is normal 2) $\forall\alpha \in [0,1]$ $\tilde{f}_\alpha$ is path connected and compact. If $\forall x\in \Omega$ and $\forall\alpha \in [0,1]$, $\tilde{f}_\alpha(x)$ is convex we say $\tilde{f}$ is a **convex fuzzy function**. We write $\tilde{f}$: $\Omega \rightarrow W(Y)$.*

Remark 1 *When we say $\tilde{f}_\alpha$ is path connected we mean $\forall f_1$ and $f_2 \in \tilde{f}_\alpha$ $\exists$ a continuous function (under the supremum norm) F:[0,1]$\rightarrow \mathcal{F}$ such that $F(0)=f_1$ and $F(1)=f_2$.*

Remark 2 *For a fuzzy function to be a convex fuzzy function an equivalent statement is: $\forall x\in \Omega$, $\alpha,\beta \in[0,1]$ and f_1 and $f_2 \in \tilde{f}_\alpha$, $\exists f_3 \in \tilde{f}_\alpha$ such that $f_3(x)=\beta f_1(x)+(1-\beta)f_2(x)$.*

This paper considers only real vector spaces.

Example 4 *One way to write a fuzzy function is to write it as a function of fuzzy parameters where we denote a parameterized fuzzy function of this type by $f(\tilde{a},x)$ where x is a vector in R^n and $\tilde{a}$ is a vector whose entries are fuzzy numbers. For example, let $\tilde{a}$ be the triangular fuzzy number defined by the parameters (1,2,3). We can define a fuzzy function $\tilde{f}(x)=f(\tilde{a},x)=\tilde{a}*x^2$ with x restricted to a compact set in R. More formally, $\mu_{\tilde{f}}(f)=\alpha \Rightarrow f(x)=a*x^2$ where $\mu_{\tilde{a}}(a)=\alpha$. The range of values of $\tilde{f}_{.5}(x)$ are between and including $f(x) = 1.5*x^2$ and $g(x) = 2.5*x^2$ with both f and $g\in \tilde{f}_{.5}$. Hence $\tilde{f}_{.5}(1)=[\frac{3}{2},\frac{5}{2}]$.*

Example 5 *Not all fuzzy functions can be represented as in the prior example. Let f_1, f_2 and f_3 be bounded real-valued functions on Ω with $f_1(x)< f_2(x)<f_3(x)$ $\forall x\in \Omega$. Define $\tilde{f}$ (the fuzzy function equivalent to a triangular fuzzy number) α-level wise by*
$\tilde{f}_\alpha=\{f:\Omega \rightarrow R \mid (1-\alpha)f_1(x)+\alpha f_2(x) \leq f(x) \leq (1-\alpha)f_3(x)+\alpha f_2(x)\ \forall x \in R\}$

This definition of fuzzy function is consistent with the functions which define the linear programing problems examined in [1] and [5]. In these papers, the functions are defined in terms of linear equations in coefficients which are fuzzy numbers. This definition is not equivalent to the definition of fuzzy mapping used in [4] and [9]. A fuzzy mapping is defined as a point to fuzzy set mapping. If the image of a point under a fuzzy mapping is restricted to a fuzzy vector (as defined here) in the range space then the fuzzy mapping is a fuzzy function. The following theorem addresses why we require this restriction.

Remark 3 *Fuzzy functions encompass point to fuzzy vector mappings. For example, assume $\tilde{F}:X\rightarrow W(Y)$ is a fuzzy mapping such that the image at each point is a fuzzy vector (as in [4] and [9]). Then define $\tilde{f}_\alpha=\left\{f \mid \text{ the point } f(x) \in \tilde{F}(x)_\alpha \; \forall x \in X\right\}$. For example consider the point to set mapping taking every real number to the interval [0,1]. The fuzzy function equivalent is the set of all functions $f:R\rightarrow[0,1]$.*

A fuzzy function as defined here allows for more refinement in the definition by restricting the functions in $\tilde{f}_\alpha$. In this sense, a point to fuzzy vector mapping is the least refined form of a fuzzy function in that it contains the least amount of information about the uncertain function being modeled.

The following theorem demonstrates that the definition of a fuzzy function produces the desired result, namely, that it is a point to fuzzy vector mapping.

Theorem 6 *Let $\tilde{f}: \Omega \rightarrow W(Y)$ be a fuzzy function. Then $\tilde{f}(x)$ is a fuzzy vector $\forall x\in \Omega$. If $\tilde{f}$ is a convex fuzzy function then $\tilde{f}(x)$ is a convex fuzzy vector.*

Proof:

Assume $\tilde{f}$ is a fuzzy function.

(Normal) $\tilde{f}$ normal $\Rightarrow \exists f\in\tilde{f}_1 \Rightarrow \forall x\in X$, $y=f(x)\in\tilde{f}_1(x)\Rightarrow\tilde{f}(x)$ normal..

(Compact) Let $(f_n(x))\subset\tilde{f}(x)_\alpha$ where $(f_n)\subset\tilde{f}_\alpha$. $\tilde{f}_\alpha$ compact $\Rightarrow \exists\ (f_{n_i})\subset \tilde{f}_\alpha$ such that $f_{n_i} \rightarrow f\in\tilde{f}_\alpha$. Hence $\|f_{n_i}-f\|_{\sup} \rightarrow 0 \Rightarrow f_{n_i}(x)\rightarrow f(x)\in\tilde{f}_\alpha(x)$.

Hence $\tilde{f}(x)$ is a fuzzy vector. If $\tilde{f}$ is a convex fuzzy function then $\tilde{f}(x)_\alpha$ is convex by definition.□

It was shown in Example 4, that a function with fuzzy parameters may be a fuzzy function. The following two Theorems examine the limits of these representations.

Theorem 7 *Let $f(\tilde{a},x)$:$\Omega \rightarrow W(R)$ represent the fuzzy subset of functions given by $\mu_{f(\tilde{a},x)}(f(a,x))=\mu_{\tilde{a}}(a)$, where $x\in \Omega \subset R^n$ and $\tilde{a}$ is a vector of fuzzy numbers, that is $\tilde{a}=[\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n]^T$ $\tilde{a}_i$ a fuzzy number. Then for each $\alpha \in[0,1]$: $\tilde{f}_\alpha=\left\{f(a,x):\Pi_{i=1,n}[a_{i,}^{-(\alpha)}a_i^{+(\alpha)}]x\Omega \rightarrow R\right\}$. For each $f(a,x)$, define the functions $f_a(x)=f(a,x)$ and $f_x(a)=f(a,x)$ where a and x are held fixed respectively. Assume that $\forall\alpha \in [0,1]$ and $x\in \Omega$, f_x is continuous over $\tilde{a}_\alpha=\Pi_{i=1,n}[a_{i,}^{-(\alpha)} a_i^{+(\alpha)}]$. Then $f(\tilde{a},x)$ is a convex fuzzy function if and only if $\{f_x:\tilde{a}_\alpha \rightarrow R \mid x \in \Omega\}$ is an equicontinuous family of functions and each f_a is bounded over Ω.*

Proof:

$\Rightarrow$

(Equicontinuous) Let $\alpha \in [0,1]$ and assume $\{f_x:\tilde{a}_\alpha \rightarrow R \mid x \in \Omega\}$ is not equicontinuous. Then $\exists\epsilon>0$ such that $\forall n\ \exists\ x_n,a_n$ and b_n with $|a_n-b_n|<\frac{1}{n}$ but $|f_{x_n}(a_n)-f_{x_n}(b_n)| \geq \epsilon$. We know $\tilde{a}_\alpha$ is compact so $\exists$ a subsequence $a_{n_i} \rightarrow a\in\tilde{a}_\alpha$ which implies that $b_{n_i} \rightarrow a$. By assumption $f(a_{n_i},x)\rightarrow f(a,x)$ $\forall x\in \Omega$, similarly for b_{n_i}. We relabel the subscripts. Then, since $\tilde{f}_\alpha$ is compact and $(f(a_n,x))$, $(f(b_{n_i},x))$ are sequences of functions in this compact function set, $\exists$ a convergent subsequence in $\tilde{f}_\alpha$. This implies $f_{a_n} \rightarrow f_a$ and $f_{b_n} \rightarrow f_a$ so that $\|f_{a_n}-f_{b_n}\|_{\sup} \rightarrow 0$. But this is a contradiction since $|f_{x_n}(a_n)-f_{x_n}(b_n)| \geq \epsilon \Rightarrow |f_{a_n}(x_n)-f_{b_n}(x_n)| \geq \epsilon$.

(Bounded) This follows by definition of a fuzzy function.

$\Leftarrow$

(Normal) $\tilde{a}$ normal$\Rightarrow$ $\exists a' \in\tilde{a}_1 \Rightarrow \mu_{f(\tilde{a},x)}(f(a',x))=1$.

(Compact) Let $\alpha \in [0,1]$ and (f_{a_i}) be a sequence of functions in $f(\tilde{a},x)_\alpha$. Then (a_i) is a sequence in $\tilde{a}_\alpha$ which is compact so $\exists(a_{i_n})$ such that $a_{i_n} \rightarrow a\in\tilde{a}_\alpha$ and $f_a \in f(\tilde{a},x)_\alpha$. Let $\epsilon>0$, by definition of equicontinuity $\exists\delta>0$ such that $|a_{i_n}-a| < \delta \Rightarrow |f_x(a_{i_n})-f_x(a)|<\epsilon\ \forall x\in \Omega$. Chose N such that $\forall n \geq N$, $|a_{i_n}-a| < \delta \Rightarrow |f_x(a_{i_n})-f_x(a)|= |f(a_{i_n},x)-f(a,x)|<\epsilon\ \forall x\in \Omega \Rightarrow f_{a_{i_n}} \rightarrow f_a$.

(Path Connected) Let f_a and $f_b \in f(\tilde{a},x)_\alpha$. Then $a,b\in \tilde{a}$ which is a compact and convex so $\exists$ a path from $F:[0,1] \rightarrow R^n$ with $F(0)=a$ and $F(1)=b$. But this also defines a path from f_a to f_b. Let $\epsilon>0$, then from equicontinuity $\exists\delta_1>0$ such that $\forall c,d\in\tilde{a}$, $|c-d| < \delta_1 \Rightarrow |f_x(c)-f_x(d)|<\epsilon\ \forall x\in \Omega$. By continuity of F $\exists\delta_2>0$ such that $\forall x,y\in [0,1]$, $|x-y| < \delta_2 \Rightarrow |F(x)-F(y)|<\delta_1 \Rightarrow |f_x(c)-f_x(d)|<\epsilon$ $\forall x\in \Omega \Rightarrow \|f_c-f_d\|_{\sup}<\epsilon$.

(Pointwise Convex) This follows immediately since each $\tilde{a}_\alpha$ is closed and connected and f_x is continuous$\Rightarrow$ $\{f_x(a) \mid a \in \tilde{a}_\alpha\}$ is closed and connected in R, therefore convex.□

Example 8 *Consider the fuzzy set of bounded functions over R given by* $\tilde{f}(\tilde{a},x)=\tilde{a}^2/(\tilde{a}^2+(1-x\tilde{a})^2)$ *where* $\tilde{a}_\alpha=[0,1]$ $\forall\alpha\in[0,1]$. *Using the notation of the above theorem we note that 1)* $f_a\equiv 0$ *for* $a=0$ *2)* $\forall a\in(0,1]$ $f_a(x)\to 0$ *as* $x\to\infty$ *and 3)* $f_x(a)=1$ *when* $x=\frac{1}{a}$. *Then* $\{f_x:\tilde{a}_\alpha\to R\}$ *cannot be equicontinuous since if we choose* $a\in(0,1]$ *but arbitrarily close to zero and let* $x=\frac{1}{a}$, *then* $\left|f_{\frac{1}{a}}(a)-f_{\frac{1}{a}}(0)\right|=1$. *Thus this is not a representation of a fuzzy function. It also fails the definition since there is no path from* f_a *for* $a\in(0,1]$ *to* f_a *for* $a=0$. *Note, however, that if* x *is restricted to a compact subset of R, the set of functions is equicontinuous and there is a path to* f_a *for* $a=0$ *and in this case the representation is a fuzzy function.*

Theorem 9 *Let* $f(\tilde{a},x):\Omega\to W(R)$ *represent a fuzzy subset of functions where* $\tilde{a}$ *is a vector of fuzzy numbers,* Ω *is compact, and* $f(a,x):\Pi_{i=1,n}[a_{i,}^{-(\alpha)}a_i^{+(\alpha)}]x\Omega\to R$ *is continuous* $\forall x\in\Omega$ *and* $\forall a\in[a_{i,}^{-(\alpha)}a_i^{+(\alpha)}]$, $\forall\alpha\in[0,1]$, *then* $f(\tilde{a},x)$ *is a representation of a convex fuzzy function.*

Proof

We apply the previous theorem. Since f(a,x) is a continuous mapping of a compact set, it is bounded. Also, it is uniformly continuous, so$\forall\epsilon>0$ $\exists\delta>0$ such that $\|(a,x)-(b,y)\|<\delta\Rightarrow\|f(a,x)-f(b,y)\|<\epsilon$. Then $\|a-b\|<\delta\Rightarrow\|(a,x)-(b,x)\|<\delta$ $\forall x\in\Omega\Rightarrow\|f_x(a)-f_x(b)\|<\epsilon$ so $\{f_x:\tilde{a}_\alpha\to R\mid x\in\Omega\}$ is an equicontinuous family of functions.□

2 The Minimum of a Fuzzy Function

Definition 10 *Let* $\tilde{f}:\Omega\to W(R)$ *be a fuzzy function. The* ***fuzzy minimum*** *of* $\tilde{f}$ *over* Ω, *is the fuzzy subset,* $\tilde{m}$, *of R with membership function*

$$\mu_{\tilde{m}}(m)=\begin{cases}\sup\{\alpha\mid\exists\, f\in\tilde{f}_\alpha \text{ with } m=\inf_{x\in\Omega}f(x)\} & \text{if } \exists \text{ at least one such f}\\ 0 & \text{otherwise}\end{cases}$$

Theorem 11 *Let* $\tilde{f}:\Omega\to W(R)$ *be a fuzzy function. Then* $\tilde{m}$, *the fuzzy minimum of* $\tilde{f}$, *is a fuzzy number.*

Proof:

(Normal) $\tilde{f}$ normal $\Rightarrow\exists f'\in\tilde{f}_1$ so that $\inf_{x\in\Omega}f'(x)\in\tilde{m}_1(x)$.

(Compact) Let $(m_n) \subset \tilde{m}_\alpha$ for $\alpha \in [0,1]$. Then for each n, $\exists f_n \in \tilde{f}_\alpha$ with $m_n = \inf_{x\in\Omega} f_n(x)$. Since $\tilde{f}_\alpha$ is compact $\exists (f_{n_i})$ with $f_{n_i} \to f \in \tilde{f}_\alpha$.
Let $m = \inf_{x\in\Omega} f(x)$. Let $\epsilon > 0$. $\exists N$ such that $\forall n_i > N$ $\|f_{n_i} - f\|_{\sup} < \frac{\epsilon}{2}$. Assume that for a particular such n_i, $m_{n_i} \leq m$ (note that the argument that follows still holds if we assume $m \leq m_{n_i}$ if we replace f with f_{n_i} and vice versa were ever they appear). For this n_i $\exists x_{n_i} \in \Omega$ with $f_{n_i}(x_{n_i}) - m_{n_i} < \frac{\epsilon}{2}$. But we also have $|f_{n_i}(x_{n_i}) - f(x_{n_i})| < \frac{\epsilon}{2} \Rightarrow f(x_{n_i}) - m_{n_i} < \epsilon$. So by assumption we have $m_{n_i} \leq m \leq f(x_{n_i}) \Rightarrow |m_{n_i} - m| < \epsilon$. Thus $m_{n_i} \to m$.

(Convex) Let m_1 and $m_2 \in \tilde{m}_\alpha$ for $\alpha \in [0,1]$. Then $\exists f_1, f_2 \in \tilde{f}_\alpha$ with $m_1 = \inf_{x\in\Omega} f_1(x)$ and $m_2 = \inf_{x\in\Omega} f_2(x)$. Since $\tilde{f}_\alpha$ is path connected $\exists$ $G:[0,1] \to \tilde{f}_\alpha$ with G continuous and $G(0)=f_1$ and $G(1)=f_2$. Let $F:[0,1] \to R$ be given by $F(r) = \inf_{x\in\Omega} G(r)(x)$. Let $s \in [0,1]$ and $\epsilon > 0$, G continuous $\Rightarrow \exists$ $\delta > 0$ such that $r \in [0,1]$ and $|r-s| < \delta \Rightarrow \|G(r) - G(s)\| < \frac{\epsilon}{2}$. Then by application of the same argument as in the compactness proof above (replacing f_{n_i} with G(r) and f with G(s)) we have $|F(r) - F(s)| < \epsilon$. Thus F is continuous on [0,1], so $\tilde{m}_\alpha$ is path connected and, thus, convex since $\tilde{m}_\alpha \subset R$.□

We can utilize the common properties that the crisp elements of $\tilde{f}$ might possess to prove statements about $\tilde{f}$.

Definition 12 *Let $\tilde{f}:\Omega \to W(R)$. We say that $\tilde{f}$ is a* ***fuzzy convex function*** *if $\forall \alpha \in [0,1]$ and $\forall f \in \tilde{f}_\alpha$, f is a convex function.*

Definition 13 *Let $\tilde{f}:\Omega \to W(R)$. We say that $\tilde{f}$ is a* ***fuzzy continuous function*** *if $\forall \alpha \in [0,1]$ and $\forall f \in \tilde{f}_\alpha$, f is a continuous function.*

Definition 14 *Let $\tilde{f}:\Omega \to W(R)$, where $\tilde{f}$ is a fuzzy function. Then the* ***fuzzy minimizer*** *of $\tilde{f}$ over Ω is the fuzzy subset $\tilde{S} \subset \Omega$, with membership function;*

$$\mu_{\tilde{S}}(s) = \begin{cases} \sup\{\alpha \mid \exists\, f \in \tilde{f}_\alpha \text{ with } f(s) = \inf_{x\in\Omega} f(x)\} & \text{if } \exists \text{ at least one such f} \\ 0 & \text{otherwise} \end{cases}$$

Theorem 15 *Let $\tilde{f}:\Omega \to W(R)$, where $\tilde{f}$ is a fuzzy convex, fuzzy continuous function on Ω and Ω is a compact, convex subset of R^n. Let $\tilde{S}$ be the fuzzy minimizer of $\tilde{f}$ over Ω. Then $\tilde{S}_\alpha$ is connected $\forall \alpha \in [0,1]$.*

Proof:

Let s_0 and $s_1 \in \tilde{S}_\alpha$. Then $\exists$ f_0 and f_1 $\in \tilde{f}_\alpha$ with $f_0(s_0) = \inf_{x\in\Omega} f_0(x) = m_0$ and $f_1(s_1) = \inf_{x\in\Omega} f_1(x) = m_1$. Since $\tilde{f}$ is a fuzzy function, $\tilde{f}_\alpha$ is path connected so $\exists$ a path $f_0 \to f_1$, let f_β denote an element along this path. In the proof of Theorem 11 we showed that this path provides a path $m_0 \to m_1$ where $m_\beta = \inf_{x\in\Omega} f_\beta(x)$.

Let $\Gamma_\beta = \{s \in \Omega \mid f_\beta(s)=m_\beta\}$ i.e. it is the set of minimizers of f_β over Ω. Since we assumed f_β is a convex function and Ω convex, we know that Γ_β is convex (see [6] Theorem 1 on page 181) and thus connected (convexity$\Rightarrow$path connectedness$\Rightarrow$connectedness, see [7] page 155). We also know that Γ_β is non-empty since each f_β is continuous and bounded on Ω by assumption and Ω is compact. Finally, we know Γ_β is closed. To see this let s be a limit point of Γ_β. Then $s\in \Omega$ since $\Gamma_\beta \subset \Omega$ and Ω is compact (thus closed). Let $(s_i)\subset \Gamma_\beta$ and $s_i \to s$. Then f_β continuous$\Rightarrow f_\beta(s_i)=m_\beta \to f_\beta(s)$, so $f_\beta(s)=m_\beta$ and $s\in \Gamma_\beta$.

We will show that $D(\Gamma_\gamma,\Gamma_\beta)\to 0$ as $\gamma \to \beta$ where $\gamma, \beta \in[0,1]$ and where for two compact sets A,B we define $D(A,B) = \min\{\|a-b\| \mid a \in A,\ b \in B\}$. The minimum can be used here since A and B are compact. Any sequence of points $(a_i)\subset A$ and $(b_i)\subset B$ with $\|a_i-b_i\| \to \inf\{\|a-b\| \mid a \in A,\ b \in B\}$ implies $\exists$ a subsequence converging to a point $(a,b)\in A\text{x}B$ with
$\|a-b\|=\inf\{\|a-b\| \mid a \in A,\ b \in B\}$We will use this to show that our collection of compact connected sets is connected.

Let $\epsilon>0$ and define $T=\{x \in \Omega \mid D(x,\Gamma_\beta)=\epsilon\}$. We show that T is compact. We can choose ϵ small enough so that T is non-empty unless $\tilde{S}_\alpha=\Omega$, in which case we are done since Ω is convex and thus connected. We now show that T is closed. Let t be a limit point of T and $(t_i)\subset T$ with $t_i \to t$. Then for each t_i $\exists m_i \in \Gamma_\beta$ with $D(t_i,m_i)=\epsilon$. Γ_β compact implies $\exists$ a subsequence (m_{i_n}) with $m_{i_n} \to m\in \Gamma_\beta$. Then $\|t_{i_n}-m_{i_n}\| \to \|t-m\|$ so $\|t-m\|=\epsilon$. Thus $D(t,\Gamma_\beta)\leq \epsilon$.. Now assume $\exists m_t \in \Gamma_\beta$ with $\|t-m_t\|<\epsilon$. Since $t_i \to t$ and by continuity of the euclidean norm $\exists t_N$ with $\|t_N-m_t\|<\epsilon$. But this contradicts the fact that $t_N \in T$. Hence so $t\in T$ so T is closed. Now T a closed subset of compact set $\Omega \Rightarrow T$ compact$\Rightarrow \exists t\in T$ with $f_\beta(t)-m_\beta=\min\{f_\beta(x)-m_\beta \mid x \in T\}$ since f_β is continuous. Let $\delta= f_\beta(t)- m_\beta$ then $\delta>0$ since $t\notin \Gamma_\beta$.

We claim that $f_\beta(y)-m_\beta \geq \delta$ $\forall y$ with $D(y,\Gamma_\beta)>\epsilon$.. To see this let $z\in \Gamma_\beta$. Since $D(z,\Gamma_\beta)=0$ and $D(y,\Gamma_\beta)>\epsilon$ and Ω is convex $\exists$ $\lambda \in [0,1]$ with $\lambda z+(1-\lambda)y\in T$. This follows from the convexity of Ω and continuity of the euclidean norm. Then $f_\beta(t)\leq f_\beta(\lambda z+(1-\lambda)y)$ by choice of t and $f_\beta(\lambda z+(1-\lambda)y)\leq \lambda f_\beta(z)+(1-\lambda)f_\beta(y)$ since f_β is convex, so $f_\beta(y)-m_\beta \geq f_\beta(t)- m_\beta=\delta$.

We now show that in a neighborhood of β, $D(\Gamma_\alpha,\Gamma_\beta)\leq \epsilon$. Since $f_\alpha \to f_\beta$ and $m_\alpha \to m_\beta$ as $\alpha \to \beta$ we can chose a neighborhood of β such that $\forall\alpha$ in this neighborhood$\cap[0,1]$ both $\|f_\alpha-f_\beta\|_{\text{sup}} <\frac{\delta}{2}$and $|m_\alpha-m_\beta|<\frac{\delta}{2}$. In this neighborhood, assume $\exists r\in \Gamma_\alpha$ and $D(r,\Gamma_\beta)>\epsilon$. Then $|f_\beta(r)-m_\beta| \geq \delta$ but since $f_\alpha(r)=m_\alpha \Rightarrow$

$|m_\alpha - f_\beta(r)| < \frac{\delta}{2}$ then it must hold that $|m_\alpha - m_\beta| \geq \frac{\delta}{2}$ which is a contradiction. Thus $D(\Gamma_\alpha, \Gamma_\beta) \leq \epsilon$.

We have thus shown that $\cup_{\alpha\in[0,1]}\Gamma_\alpha$ is a connected subset of $\tilde{S}_\alpha$ containing two arbitrary points in $\tilde{S}_\alpha$. So $\tilde{S}_\alpha$ must be connected.□

3 Defuzzification

Suppose we have a fuzzy function which we wish to minimize. Our objective is to decide upon a choice of a crisp x in Ω which provides the best possible outcome. If $\tilde{S}$ is the fuzzy minimizer for the fuzzy function then any choice of $x\in\tilde{S}_1$ will have possibility 1 of being the minimizer but, unless $\tilde{S}$ is a crisp vector in X, there is also a possibility the s will not be a minimizer. A tool for analysis of the possible solutions is the following:

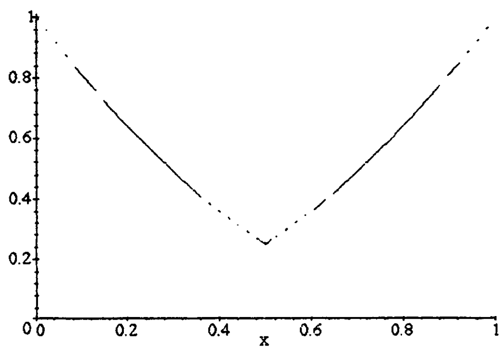
Definition 16 *Let $\tilde{f}:\Omega \rightarrow W(R)$ be a fuzzy function.Define the* ***maximum possible error*** *$emax:\Omega x[0,1] \rightarrow R$ as follows:*
$emax(x,\alpha)=sup\left\{f(x)\text{-}m_f \mid f \in \tilde{f}_\alpha \text{ and } m=inf_{y\in\Omega}f(y)\right\}$

Definition 17 *Let $\tilde{f}:\Omega \rightarrow W(R)$ be a fuzzy function. Define the* ***minimum possible error*** *$emin:\Omega x[0,1] \rightarrow R$ as follows:*
$emin(x,\alpha)=inf\left\{f(x)\text{-}m_f \mid f \in \tilde{f}_\alpha \text{ and } m=inf_{y\in\Omega}f(y)\right\}$

These functions are well defined since each $f\in\tilde{f}_\alpha$ is bounded on Ω and $\tilde{f}_\alpha$ is compact under the supremum norm by definition of a fuzzy function. Thus the supremum is finite since it is taken over a bounded collection of bounded sets.

The maximum possible error gives the maximum possible distance between the function evaluated at x and its minimum value over all functions that are at least α possible. It is natural to want to chose x so as to minimize the maximum possible error. If $\tilde{S}_\alpha$ is non-empty, minimizing at the α-level of possibility can be done by choosing $s\in\tilde{S}_\alpha$ so that $\exists f\in\tilde{f}_\alpha$ with s a minimizer of f and f(s) at the midpoint of $\tilde{m}_\alpha$. The maximum error over all α-level possible functions is then $(m_\alpha^+ - m_\alpha^-)/2$. Minimizing over different α-levels may lead to different solutions. It may be desirable to minimize the maximum possible error for any possible f (α=0). These concepts are illustrated in the following examples.

Example 18 *Consider the fuzzy function defined by* $\tilde{f}(x)=(\tilde{a}-x)^2+\tilde{a}$ *where* $\tilde{a}=[0,1]$, *i.e. it is the interval for all* α*-levels, and* $\Omega=[0,1]$. *Let* $f_a(x)=(x-a)^2-a$ *so* $\tilde{f}_0=\{f_a(x) \mid a \in [0,1]\}$. *Then* $inf_{x\in[0,1]}f_a(x)=a$ *and* $emax(x,0)=sup_{a\in[0,1]}\{(x-a)^2+a-a\}=max(x^2,(x-1)^2)$. *On the otherhand* $emin(x,0)=0\ \forall x\in[0,1]$ *since* $inf_{a\in[0,1]}\{(x-a)^2+a-a\}=0$. *A plot of emax is shown below. Note that the emax function is minimized at x=.5 where the maximum possible error is .25. For this fuzzy function, this is the best choice if the objective is to minimize the maximum possible error.*



$\max(x^2,(x-1)^2)$

Example 19 *Consider the fuzzy function defined by* $\tilde{f}(x)=(\tilde{a}-x)^2+\tilde{b}$ *where a and b are trapezoidal fuzzy numbers represented by (1.8,2,2.2,2.25) and (.8,1,1.2,2) respectively. For this fuzzy function* $\tilde{m}=\tilde{b}$ *and* $\tilde{S}=\tilde{a}$. *For example* $\tilde{m}_1=[1,1.2]$ *and* $\tilde{S}_1=[2,2.2]$ *since* $\tilde{f}_1=\{(a-x)^2+b \mid a \in [2,2.2] \text{ and } b \in [1,1.2]\}$. *Thus* $f(x)=(2-x)^2+1.2$ *is such a function with possibility level 1. It's minimum value is 1.2 and it is minimized at x=2. For x=2 and considering all functions at* α*=1 possibility level, the maximum possible error occurs if a=2.2 so* $f(2)=(2.2-2)^2+b$. *The value of b is immaterial. To see this say that* $f(x)=(2.2-x)^2+1.1$ *which is minimized at x=2.2. Then the error for a choice of x=2 is f(2)-f(2.2)=1.14-1.1=.04. The following table shows the distribution of the values of the maximum possible error function for each choice of x from 2 to 2.2 in steps of .02 and for various possibility levels.*

	α					
x	1	.8	.6	.4	.2	0
2.00	.040	.044	.048	.053	.058	.063
2.02	.032	.036	.040	.044	.048	.053
2.04	.026	.029	.032	.036	.040	.058
2.06	.020	.023	.026	.032	.048	.068
2.08	.014	.017	.026	.040	.058	.078
2.10	.010	.020	.032	.048	.068	.090
2.12	.014	.026	.040	.058	.078	.102
2.14	.020	.032	.048	.068	.090	.116
2.16	.026	.040	.058	.078	.102	.130
2.18	.032	.048	.068	.090	.116	.144
2.20	.040	.058	.078	.102	.130	.160

Selecting from this table, a choice of x=2.1 results in the maximum possible error (.010) being minimized over all level 1 possible functions. But a choice of x=2.02 results in the maximum possible error (.053) being minimized over all possible functions. Which choice is most appropriate would seem to be determined by the context of the problem. In risk management, the later solution might be preferable or a solution for a low α level. In fact, in this context, a problem solution might lie outside the $\tilde{S}_1$ solution set (recall that $\tilde{S}_1$= [2,2.2] in the above example.)

Example 20 *Consider the fuzzy function $\tilde{f}(x,y)=(x-\tilde{a})^2+(y-\tilde{a}^2)^2$ over [0,1]x[0,1] where $\tilde{a}$ is the triangular fuzzy number (.2,.5,.8). For this function $\tilde{m}$=0, a crisp number, since any possible function of the form $f(x,y)=(x-a)^2+(y-a^2)^2$ has minimum value 0 at $(x,y)=(a,a^2)$. So $\tilde{S}_\alpha=\{(x,x^2) \mid x \in \tilde{a}_\alpha\}$. For example, $\tilde{S}_0=\{(x,x^2) \mid x \in [.2,.8]\}$. We see by this example that although $\tilde{S}_\alpha$ is connected it need not be convex. For α=0, the error function is minimized at (.5,.34) where emax((.5,.34),0)=.18. This is because .5=(.2+.8)/2 so x=.5 minimizes $max\{(x-.2)^2,(x-.8)^2\}$ and $.34=(.2^2+.8^2)/2$ so y=.34 minimizes $max\{(y-.2^2)^2,(y-.8^2)^2\}$. Then (.5,.34) minimizes $max\{(x-a)^2+(y-a^2)^2 \mid x \in [.2,.8]\}$. Notice that (.5,.34) is not an element of $\tilde{S}$ so this point minimizes emax but is not a minimizer of any of the possible functions. The point that lies in $\tilde{S}$ and minimizes emax is $(.544,.544^2)$ where $emax((.544,.544^2),0)$=.183916. The possibility that $(.544,.544^2)$ is a minimizer of our fuzzy function is .8533 (since the possibility that a=.544 is .8533).*

Definition 21 *We define the **possible error** for a given x, $\tilde{e}(x)$, as the fuzzy set with α-level defined as* $\tilde{e}_\alpha(x)=[emin(x,\alpha), emax(x,\alpha)]$.

Theorem 22 *Let $\tilde{f}:\Omega \to W(R)$ be a fuzzy function. Then the possible error, $\tilde{e}(x)$, is a fuzzy number.*

Proof

(Normal) Since $\exists f \in \tilde{f}_1$ then $|f(x)-m| \in \tilde{e}(x)_1$ where $m=\inf_{x\in\Omega} f(x)$.

(Compact) Let $(e_i) \subset \tilde{e}(x)_\alpha$. For each e_i $\exists f_i \in \tilde{f}_\alpha$ with $e_i = |f_i(x)-m_i| \in \tilde{e}(x)_\alpha$ where $m_i=\inf_{x\in\Omega} f_i(x)$. Thus $\exists$ a subsequence $f_{i_n} \to f \in \tilde{f}_\alpha$. From theorem 13 $m_{i_n} \to m$ so $|f_i(x)-m_i| \to |f(x)-m| \in \tilde{e}(x)_\alpha$.

(Convex) Let $e_1, e_2 \in \tilde{e}(x)_\alpha$, then $\exists$ $f_1, f_2 \in \tilde{f}_\alpha$ with $e_i = |f_i(x)-m_i|$where $m_i=\inf_{x\in\Omega} f_i(x)$. $\exists$ a path $G:[0,1] \to \tilde{f}_\alpha$ mapping f_1 to f_2. Then $G_x:[0,1] \to R$ defined by $G_x(a)=G(a)(x)$ is continuous as is the function $F:[0,1] \to R$ from Theorem 13 (mapping to the infimum of G(a)). Then $H:[0,1] \to R$ defined by $H(a)=|G_x(a)-F(a)|$ is continuous with $H(0)=e_1$ and $H(1)=e_2$ so $\forall \lambda \in (0,1)$ $\exists r \in [0,1]$ with $H(r)=\lambda e_1+(1-\lambda)e_2$.□

We see that the possible error is a possibility distribution for the error that may occur for a given choice of x as the function input value. An objective in choosing x may be to achieve the most favorable possible error distribution. To make such a choice one needs a method of rating these distributions. We suggest one possible method is to assign a positive real number to a given distribution. The value assigned should consider the relative value to the decision maker of one possible outcome over another and the possibility of that outcome occurring. For example, one possible assignment is as follows:

$$m(\tilde{e}(x))= \left(\frac{\int |x|^p \mu_{\tilde{e}}(x)dx}{\int \mu_{\tilde{e}}(x)dx}\right)^{\frac{1}{p}} \text{ if } \int \mu_{\tilde{e}}(x)dx \neq 0,\ |x| \text{ otherwise}$$

This assignment weights the possible values by using x^p as a weighting function. For example if p=2 and if the possible error distribution for x is the interval [a,b] and for y the interval $[a-\epsilon, b+\epsilon]$ for small $\epsilon>0$ all values with possibility one, then the assignment for x will be less than the assignment for y. This makes the error distribution for x more favorable than that for y which might be desirable for a particular decision maker. The bias can be eliminated or changed to a bias in favor of y by modifying the definition. For example if p=1 instead of 2 then the assignment for x and y are equal and the choice between them is neutral. Using this assignment one can then redefine the minimization problem as:

For fuzzy function $\tilde{f}$, find x which minimizes $m(\tilde{e}(x))$.

Example 23 *As a final example we look again at the fuzzy function in example 19 defined by* $\tilde{f}(x)=(\tilde{a}-x)^2+\tilde{b}$ *where a and b are trapezoidal fuzzy numbers represented by (1.8,2,2.2,2.25) and (.8,1,1.2,2) respectively. Now we wish is find the minimum of* $m(\tilde{e}(x))$ *with p=1 for this function. This can be handled analytically in a piecewise mannor. We will only show the calculations for the interval in R where we know the answer lies. On the interval* $2.025 \leq x \leq 2.1$,
for $(.15\alpha+4.05)/2>x$, $\tilde{e}_\alpha(x)=[0,(-.05\alpha+2.25-x)^2]$
for $(.15\alpha+4.05)/2\leq x$, $\tilde{e}_\alpha(x)=[0,(.2\alpha+1.8-x)^2]$
Then $m(\tilde{e}(x))=\int_{(2x-4.05)/.15}^{1}\alpha(-.05\alpha+2.25-x)^2 d\alpha +$
$\int_0^{(2x-4.05)/.15}\alpha(.2\alpha+1.8-x)^2 d\alpha$
$=\frac{-20}{3}(2.2-x)^3+5(\frac{-5}{3}x+3.6)^3+\frac{-5}{3}(1.8-x)^3$.
In the interval of evaluation, this function is minimized at $x=2.056777$ *where* $m(\tilde{e}(x))=.03409$. *We consider this x to be the solution.*

4 Conclusion

The definition of fuzzy function provides a convenient method of representing model uncertainty while maintaining the underling properties of the model. It provides a method for determining solutions which might provide better results for the specified problem than simply looking at a single possible model parameter value. Future research will examine constrained minimization problems with more complex functions, alternative methods of defuzzifing the results and classical analysis of fuzzy functions. We will also look further at measures of possibility distributions as discussed above.

5 References

[1] J.J. Buckley, Joint solution to fuzzy programming problems, *Fuzzy Sets and Systems 72* (1995)
[2] A. Kaufmann and M.M.Gupta, *Introduction to Fuzzy Arithmetic theory and Applications* (Van Nostrand Reinhold, New York, 1991)
[3] E. Kreyszig, *Introductory Functional Analysis with Applications* (John Wiley & Sons, 1978)
[4] B.S. Lee and Sung Jin Cho, A fixed point theorem for contractive-type fuzzy mappings, *Fuzzy Sets and Systems* 61 (1994)

[5] W.A. Lodwick, Analysis of structure in fuzzy linear programming, *Fuzzy Sets and Systems* 38 (1990)

[6] D.G. Luenberger, *Introduction to Linear and Nonlinear Programming* (Addison-Wesley Publishing Company, Reading, MA 1972)

[7] J. R. Munkres, *Topology, a first course* (Prentice-Hall, Inc. Englewood Cliffs, New Jersey 1975)

[8] W. Rudin, *Principles of Mathematical Analysis* (McGraw Hill Book Company, 1976)

[9] D. Zhang and C. Guo, Fuzzy integrals of set-valued mappings and fuzzy mappings, *Fuzzy Sets and Systems* 75 (1995)

CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364

Fax: (303) 556-8550
Phone: (303) 556-8442
http://www-math.cudenver.edu/

64. C. Liu, C. Liao, Z. Liu, X. Zheng and S. McCormick, "Multilevel Adaptive Methods for Turbulent Combustion with Detailed Chemistry: Part II."
65. L.P. Franca and A. Russo, "Deriving Upwinding, Mass Lumping and Selective Reduced Integration by Residual-Free Bubbles."
66. J.H. Bramble, A.V. Knyazev and Joseph E. Pasciak, "A Subspace Preconditioning Algorithm For Eigenvector/Eigenvalue Computation."
67. S.C. Billups, S.P. Dirkse and M.C. Ferris, "A Comparison of Algorithms for Large Scale Mixed Complementarity Problems."
68. P. Vaněk, R. Tezaur, M. Brezina and J. Křížková, "Two-level Method with Coarse Space Size Independent Convergence."
69. P. Vaněk and S. Ghosal, "A New Technique for Construction of Image Pyramids."
70. L.P. Franca and A. Russo, "Mass Lumping Emanating from Residual-Free Bubbles."
71. L.P. Franca and A. Russo, "Unlocking with Residual-Free Bubbles."
72. C. Liu, Z. Liu and G. Xiong, "Direct Numerical Simulation for the Receptivity and the Whole Process of Transition Around 2-D Airfoils."
73. R. Tezaur, P. Vaněk and M. Brezina, "Two-Level Method for Solids on Unstructured Meshes."
74. T.F. Russell, D.W. Dean, T.H. Illangasekare, R. Mapa and J. Garcia, "Upscaling of Dispersivity in Modeling of Solute Transport: Mathematical Theory and Laboratory Experiments."
75. T.F. Russell, R.W. Healy, R.G. Striegl, G.L. Hutchinson and G.P. Livingston, "Analytical Solution for the Problem of 1-Dimensional Diffusion into a Static Chamber."
76. T.F. Russell and R.V. Trujillo, "The Finite Volume Element Method For Elliptic and Parabolic Equations."
77. R.W. Healy and T.F. Russell, "Solution of the Advection-Dispersion Equation in Two Dimensions by a Finite-Volume Eulerian-Lagrangian Localized Adjoint Method."
78. Z. Cai, R.R. Parashkevov, T.F. Russell and X. Ye, "Domain Decomposition for a Mixed Finite Element Method in Three Dimensions."
79. S.E. Payne, T. Pentilla and G.F. Royle, "Building a Cyclic q-Clan."