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"The language of
truth is simple"
Euripides

Abstract

In this paper we show the equivalence between the variational multiscale and the residual-free bubbles concepts.

1 Introduction

The subject of stabilized finite element methods has existed now for over sixteen years. The basic reason for introducing stabilized methods is that straightforward application of the Galerkin version of the finite element method to certain problems of mathematical physics and engineering yields numerical approximations that are deficient in that they do not inherit the stability properties of the continuous problem. Stabilized methods constitute a systematic methodology for improving stability behavior without compromising accuracy. As such, stabilized methods have provided fundamental solutions to the problem of discrete approximations in several practically important areas. Perhaps the most notable area is fluid dynamics. Convection-diffusion operators have perplexed numerical analysts for decades. Historically they have been treated by methods which have either compromised stability (e.g., central differences) or accuracy (e.g., upwinding). In fact, one often hears in some circles of computational fluid dynamicists that stability and accuracy are in competition, and that one must be sacrificed to attain the other. Stabilized methods represent a refutation of this ancient religion. This has been proven mathematically and verified numerically by numerous investigators.

In recent years several efforts have been undertaken to better understand the theoretical foundations and origins of stabilized methods. Two approaches have achieved particular success.

The first is based on the identification of stabilized methods with Galerkin finite element methods employing finite element spaces enriched with so-called "bubble" functions. This observation may be found in several works, namely, [13, 3, 1]. Subsequently it was realized that strict equivalence of stabilized methods and bubble function methods could only be accomplished with specially constructed bubbles [2]. The most recent rendition of this philosophy derives the bubbles from certain element-level boundary-value problems. These are referred to as *residual-free bubbles* [4, 14, 8, 9, 10]. They have provided a satisfactory basis for the derivation of stabilized methods with excellent properties and employ well established and accepted concepts throughout their construction. The conceptual viewpoint is to attack the original problem first with Galerkin's method involving standard and simple polynomial finite element spaces (e.g., linear triangles) and then improve the approximation, and correct any deficiencies with regard to stability, by systematically enriching the space with residual-free bubbles. Alternatively, the method can be viewed as simply the Galerkin finite element method applied to the enlarged finite element space consisting of the standard polynomials *and* residual-free bubbles. This observation obviates any possible criticism of the resulting stabilized method because its construction emanates from such a universally accepted derivational procedure, namely, Galerkin's method for a particular finite-dimensional subspace enjoying good properties for the problem under consideration.

Another approach which has also provided a firm theoretical foundation for the construction of stabilized methods is the *variational multiscale method* presented recently [11]. The philosophy of this method is different from the previous one. As already pointed out, the standard Galerkin finite element method with simple polynomial spaces is an *inadequate* numerical paradigm for many practically important problems, in particular, those involving fine scale features that are numerically unresolvable due to the length scale of elements composing the mesh. It is acknowledged in this view that even if one is uninterested in resolving, or "seeing", the fine scale features, their effect on the coarse, or resolvable, scales *must* be accounted for in order to accurately calculate the coarse scales. Thus the implementation of the variational multiscale procedure consists of two steps: the first is *purely* non-numerical – the original problem is decomposed into two subproblems. One involves solving for the fine scales in terms of the coarse scales. The result is substituted into the second subproblem which results in a modified problem involving only the coarse scales. This is sometimes referred to as a *subgrid-scale model.* This problem turns out to be a suitable one for presentation to the standard Galerkin finite element method employing simple polynomial-based elements. Because unresolved scales have been removed, a successful approximation naturally follows. Application of the Galerkin method to the modified problem is the second and final step of the variational multiscale approach.

Various practical approximations can be made within the variational multiscale procedure. For example, one can assume that the fine scale phenomena exists only in the interior of element subdomains, before introduction of, and regardless of the particular approximating finite element spaces to be introduced in the Galerkin step. This assumption leads to a framework which permits an identification with stabilized methods and, of particular interest herein, an *equivalence* with residual-free bubbles.

The situation can be summarized in a commutative diagram. The residual-free bubble approach takes the left and lower path whereas the variational multiscale approach takes the upper and right path. Interestingly, stabilized methods may be viewed as a short-cut, diagonal path through the diagram.

In the remainder of this paper we make these ideas precise. In Section 2 we show the aforementioned equivalence between these two methodologies; in Section 3 we present the classical application to the convection-diffusion operator and finally we draw some conclusions in Section 4.

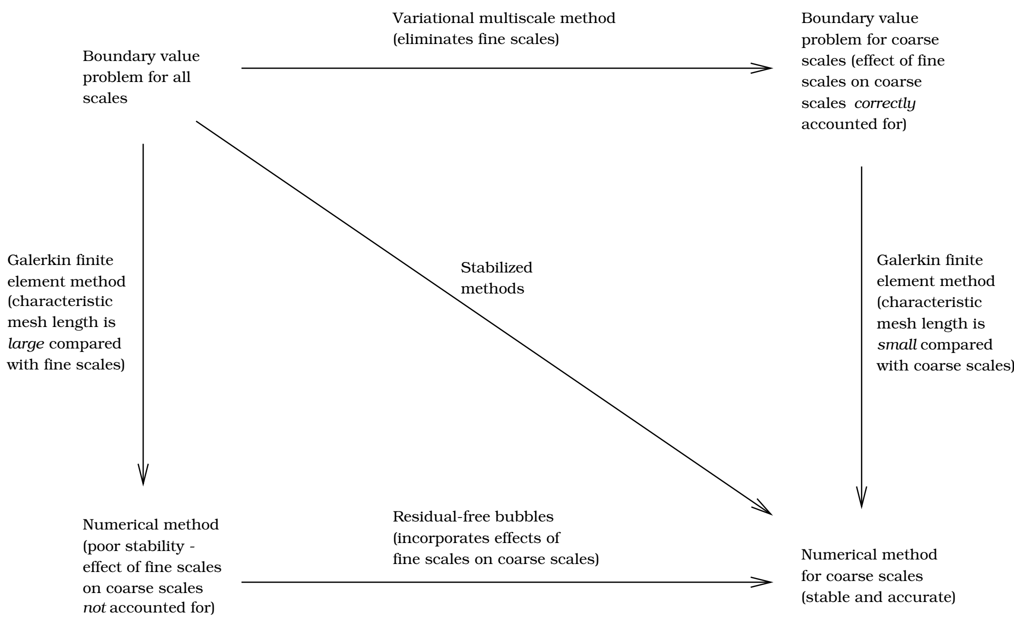

Figure 1: Commutative diagram

2 Equivalence

We consider an abstract variational problem

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = F(v) \text{ for all } v \in V \end{cases} \tag{1}$$

where V is a Hilbert space, $a(\cdot,\cdot)$ is a continuous and coercive bilinear form on V, and $F(\cdot)$ is a continuous linear form on V. We assume that (1) is the variational formulation of a second order elliptic boundary value problem

$$\begin{cases} \mathcal{L}u = f \quad \text{in } \Omega \\ \quad u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2}$$

with $a(u,v) = (\mathcal{L}u, v)$, $F(v) = (f, v)$ where $(u,v) = \int_\Omega uv$ is the usual $L^2(\Omega)$-inner product. In this case, we have $V = H_0^1(\Omega)$. The classical Galerkin approximation of problem (1) consists of taking a finite-dimensional subspace V_h of V and then solving problem (1) in V_h:

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a\,(u_h, v_h) = F(v_h) \text{ for all } v_h \in V_h. \end{cases} \tag{3}$$

To define a finite element approximation of problem (1), let $\mathcal{T}_h = \{K\}$ be a discretization of Ω into element subdomains K, and let $h_{\mathrm{K}} = \mathrm{diam}(K)$, $h = \max_{\mathrm{K}} h_{\mathrm{K}}$. Then the finite element space V_h consists of continuous functions which are polynomial of prescribed degree on each $K \in \mathcal{T}_h$.

The choice of V_h described above works very well, as, for instance, in the case $\mathcal{L}u = -\Delta u$, but it may also lead to a disaster, as in the case when $\mathcal{L}$ is a convection-dominated convection-diffusion operator: as pointed out in the Introduction, it is well-known that in this case, if h is not small enough, spurious oscillations are generated in the numerical solution, and the convection mechanism propagates these oscillations all over the domain.

Both the variational multiscale and the residual-free bubbles viewpoints make use of a so-called *augmented* formulation: the piecewise-polynomial space V_h will be suitably enlarged to cope with the problems mentioned in the Introduction. For this reason, from now on we will denote the "initial space" V_h by V_{R}^h, where the subscript $(\cdot)_{\mathrm{R}}$ and the superscript $(\cdot)^h$ are related to the variational multiscale and the residual-free bubbles viewpoint respectively. To avoid some minor complications, we restrict ourselves to the case of *continuous, piecewise linear* elements, i.e. we set

$$V_{\mathrm{R}}^h = \left\{ v_{\mathrm{R}} \in H_0^1(\Omega), \quad v_{\mathrm{R}|K} \text{ is linear for each } K \in \mathcal{T}_h \right\}. \tag{4}$$

For the sake of clarity, we rewrite problem (3) in terms of V_{R}^h:

$$\begin{cases} \text{find } u_{\mathrm{R}} \in V_{\mathrm{R}}^h \text{ such that} \\ a\,(u_{\mathrm{R}}, v_{\mathrm{R}}) = F(v_{\mathrm{R}}) \text{ for all } v_{\mathrm{R}} \in V_{\mathrm{R}}^h. \end{cases} \tag{5}$$

Let V_{U}^b be a closed subspace of $H_0^1(\Omega)$ (not necessarily of finite dimension) such that

$$V_{\mathrm{R}}^h \cap V_{\mathrm{U}}^b = \{0\} \tag{6}$$

and define the augmented space V_h as

$$V_h = V_{\mathrm{R}}^h \oplus V_{\mathrm{U}}^b. \tag{7}$$

Our aim is to compute the effect of $V_{\rm U}^b$ onto $V_{\rm R}^h$, and then write a problem in terms of $V_{\rm R}^h$ only. Using the decomposition (7), we have that any $v_h \in V_h$ can be split into a resolvable part $v_{\rm R} \in V_{\rm R}^h$ and an unresolvable part $v_{\rm U} \in V_{\rm U}^b$ in a unique way:

$$v_h = v_{\rm R} + v_{\rm U} \in V_{\rm R}^h \oplus V_{\rm U}^b. \tag{8}$$

In turn, the variational problem (3) can also be split:

$$\begin{cases} \text{find } u_h = u_{\rm R} + u_{\rm U} \in V_{\rm R}^h \oplus V_{\rm U}^b \text{ such that} \\ a\left(u_{\rm R} + u_{\rm U}, v_{\rm R}\right) = F(v_{\rm R}) \quad \text{for all } v_{\rm R} \in V_{\rm R}^h \\ a\left(u_{\rm R} + u_{\rm U}, v_{\rm U}\right) = F(v_{\rm U}) \quad \text{for all } v_{\rm U} \in V_{\rm U}^b. \end{cases} \tag{9}$$

Using the bilinearity of $a\left(\cdot,\cdot\right)$, the second equation in (9) can be written as

$$a\left(u_{\rm U}, v_{\rm U}\right) = -\left[a\left(u_{\rm R}, \cdot\right) - F(\cdot)\right](v_{\rm U}) \quad \text{for all } v_{\rm U} \in V_{\rm U}^b. \tag{10}$$

Problem (10) can be "solved" for any $u_{\rm R} \in V_{\rm R}^h$ and the solution can be formally written as

$$u_{\rm U} = M\left(\mathcal{L}u_{\rm R} - f\right) \tag{11}$$

where M is a bounded linear operator from $H^{-1}(\Omega)$ to $H_0^1(\Omega)$.

Remark 1 (for the mathematically inclined reader). The operator M appearing in (11) is obtained in the following way. We start from $(\mathcal{L}u_{\rm R} - f)$ which is an element of $H^{-1}(\Omega)$, i.e. a bounded linear functional on $H_0^1(\Omega)$; by restricting $(\mathcal{L}u_{\rm R} - f)$ to $V_{\rm U}^b \subset H_0^1(\Omega)$ we obtain a bounded linear functional on $V_{\rm U}^b$, i.e. an element of the topological dual of $V_{\rm U}^b$. We solve the variational problem (10) getting a solution $u_{\rm U} \in V_{\rm U}^b$; finally, $V_{\rm U}^b \hookrightarrow H_0^1(\Omega)$ with continuous injection. ■

Using (11) in the first equation of (9), we can write a variational problem in terms of $(u_{\rm R}, v_{\rm R})$ only:

$$a\left(u_{\rm R}, v_{\rm R}\right) + \underbrace{a\left(M\left(\mathcal{L}u_{\rm R} - f\right), v_{\rm R}\right)}_{\text{effect of the space } V_{\rm U}^b} = (f, v_{\rm R}) \quad \text{for all } v_{\rm R} \in V_{\rm R}^h. \tag{12}$$

Comparing (12) with (5), we see that the process of enlarging $V_{\rm R}^h$ and then projecting the solution back can be seen as a *modification* of the original bilinear form.

Remark 2 Clearly, if to augment V_{R}^h we could use a space V_{U}^b so large that

$$V_h = V_{\mathrm{R}}^h \oplus V_{\mathrm{U}}^b = H_0^1(\Omega), \tag{13}$$

then $u_{\mathrm{R}} + u_{\mathrm{U}}$ would be the *exact* solution, and u_{R} its projection on V_{R}^h. For continuous functions, the projection on V_{R}^h coincides with the linear interpolation on $\mathcal{T}_h$, so that, if the exact solution is regular enough, u_{R} would be the linear interpolant of the exact solution. ∎

The next two subsections describe the variational multiscale and the residual-free bubbles approach respectively.

2.1 The variational multiscale approach

Since we are using low order polynomials in V_{R}^h to approximate problem (1), the details inside each element K will be invisible to us or *unresolvable*. We do not wish to describe these unresolvable scales in detail; instead, we wish to compute their *effect* on the resolvable scales. In this subsection, the space of continuous piecewise linear elements V_{R}^h will be called the space of *resolvable* scales, and $V_{\mathrm{U}}^b = V_{\mathrm{U}}$ will be the space containing the *unresolvable* scales. We make here a crucial *ansatz*:

$$\boxed{V_{\mathrm{U}} = \oplus_{\mathrm{K}} H_0^1(K).} \tag{14}$$

In other words, we take into account only those unresolvable scales vanishing on the boundaries of the elements, in the hope that their effect (on V_{R}^h) can be representative enough of the effect of *all* unresolvable scales. This assumption is quite strong, and up to now its validity can only be justified *a posteriori* by seeing that in many cases it is general enough to give good approximation schemes. Needless to say, for one-dimensional problems, the choice (14) implies (13) (for V_{R}^h defined as in (4)). Hence, in view of Remark 2, (14) will give $u_{\mathrm{R}} = u$ at the nodes. Unfortunately, this is not the case for two- and three-dimensional problems, that deserve a more carefully analysis. We hope to pursue this subject in future work.

Now we solve problem (1) on $V_h = V_{\mathrm{R}}^h \oplus V_{\mathrm{U}}$. We have, as in (8), that any $v_h \in V_h$ can be split into a resolvable part $v_{\mathrm{R}} \in V_{\mathrm{R}}^h$ and into an unresolvable part $v_{\mathrm{U}} \in V_{\mathrm{U}}$ in a unique way:

$$v_h = v_{\mathrm{R}} + v_{\mathrm{U}} \in V_{\mathrm{R}}^h \oplus V_{\mathrm{U}}, \tag{15}$$

but now in addition v_{U} can be uniquely decomposed among the K's:

$$v_{\mathrm{U}} = \sum_{\mathrm{K}} v_{\mathrm{U,K}}, \qquad v_{\mathrm{U,K}} \in H_0^1(K). \tag{16}$$

Then, the variational problem (1) in V_h can be written as follows:

$$\begin{cases} \text{find } u_h = u_{\mathrm{R}} + u_{\mathrm{U}} = u_{\mathrm{R}} + \sum_{\mathrm{K}} u_{\mathrm{U,K}} \in V_{\mathrm{R}}^h \oplus V_{\mathrm{U}} \text{ such that} \\ a\left(u_{\mathrm{R}} + u_{\mathrm{U}}, v_{\mathrm{R}}\right) = F(v_{\mathrm{R}}) \quad \text{for all } v_{\mathrm{R}} \in V_{\mathrm{R}}^h \\ a\left(u_{\mathrm{R}} + u_{\mathrm{U,K}}, v_{\mathrm{U,K}}\right)_K = F(v_{\mathrm{U,K}})_K \quad \text{for all } K \in \mathcal{T}_h,\ v_{\mathrm{U,K}} \in H_0^1(K), \end{cases} \tag{17}$$

where the subscript $(\cdot)_K$ indicates that the integrals involved are restricted to the element K. Consider the first equation of (17); using the decomposition (16) on u_{U} and the bilinearity of $a\left(\cdot,\cdot\right)$, it can be written as

$$a\left(u_{\mathrm{R}}, v_{\mathrm{R}}\right) + \sum_{\mathrm{K}} a\left(u_{\mathrm{U,K}}, v_{\mathrm{R}}\right)_K = \left(f, v_{\mathrm{R}}\right) \quad \text{for all } v_{\mathrm{R}} \in V_{\mathrm{R}}^h. \tag{18}$$

The term $\sum_{\mathrm{K}} a\left(u_{\mathrm{U,K}}, v_{\mathrm{R}}\right)_K$ represents the effect of the unresolvable scales $u_{\mathrm{U,K}}$ onto the resolvable ones u_{R}. We can give this term a different expression, observing that

$$a\left(u, v\right)_K = \left(\mathcal{L}u, v\right)_K = \left(u, \mathcal{L}_{\mathrm{K}}^* v\right)_K \tag{19}$$

where $\mathcal{L}_{\mathrm{K}}^*$ is the formal adjoint of $\mathcal{L}$ on K (with zero boundary conditions on ∂K). We then have

$$\sum_{\mathrm{K}} a\left(u_{\mathrm{U,K}}, v_{\mathrm{R}}\right)_K = \sum_{\mathrm{K}} \left(u_{\mathrm{U,K}}, \mathcal{L}_{\mathrm{K}}^* v_{\mathrm{R}}\right)_K. \tag{20}$$

We now use the second equation in (17) to determine $u_{\mathrm{U,K}}$ in terms of u_{R}. By linearity, we can rewrite it as

$$a\left(u_{\mathrm{U,K}}, v_{\mathrm{U,K}}\right)_K = -\left[a\left(u_{\mathrm{R}}, \cdot\right) - F(\cdot)\right]_K \left(v_{\mathrm{U,K}}\right) \quad \text{for all } v_{\mathrm{U,K}} \in H_0^1(K) \tag{21}$$

or, using the differential operator,

$$\begin{cases} \mathcal{L}u_{\mathrm{U,K}} = -[\mathcal{L}u_{\mathrm{R}} - f] & \text{in } K \\ u = 0 & \text{on } \partial K. \end{cases} \tag{22}$$

For each u_{R}, problem (21) (or (22)) has always a unique solution $u_{\mathrm{U,K}} \in H_0^1(K)$ which can be written as

$$u_{\mathrm{U,K}} = M_{\mathrm{K}}\left(\mathcal{L}u_{\mathrm{R}} - f\right), \tag{23}$$

where M_{K} is a bounded linear operator from $H^{-1}(K)$ to $H_0^1(K)$. Substituting (23) into (18) the equation for u_{R} becomes:

$$a\left(u_{\mathrm{R}}, v_{\mathrm{R}}\right) + \sum_{\mathrm{K}} a\left(M_{\mathrm{K}}\left(\mathcal{L}u_{\mathrm{R}} - f\right), v_{\mathrm{R}}\right)_K = \left(f, v_{\mathrm{R}}\right), \tag{24}$$

or, using (19):

$$a\left(u_{\mathrm{R}}, v_{\mathrm{R}}\right)+\underbrace{\sum_{\mathrm{K}}\left(M_{\mathrm{K}}\left(\mathcal{L} u_{\mathrm{R}}-f\right), \mathcal{L}_{\mathrm{K}}^{*} v_{\mathrm{R}}\right)_{K}}_{\text{effect of U-scales onto R-scales}}=\left(f, v_{\mathrm{R}}\right) \quad \text{for all } v_{\mathrm{R}} \in V_{\mathrm{R}}^{h}. \tag{25}$$

Remark 3 If the operator M_{K} is taken to be equal to $-\tau\,\mathrm{I}$ (minus the stability constant τ times the identity operator) then the method given by (25) coincides with the *unusual stabilized method* suggested in [2] and discussed in [6] . ■

Problem (22) can also be "solved" by the Green's function technique. For $y \in K$, let g_y^{K} be the solution of

$$\begin{cases} \mathcal{L} g_y^{\mathrm{K}}=\delta_y & \text{in } K \\ \quad g_y^{\mathrm{K}}=0 & \text{on } \partial K, \end{cases} \tag{26}$$

so that, for any reasonable ϕ, the function $w(y)=\int_K g_y^{\mathrm{K}}(x) \phi(x)\, dx$ solves the boundary value problem

$$\begin{cases} \mathcal{L} w=\phi & \text{in } K \\ \quad w=0 & \text{on } \partial K. \end{cases} \tag{27}$$

Then we have

$$M_{\mathrm{K}}\left(\mathcal{L} u_{\mathrm{R}}-f\right)(x)=-\int_K g_y^{\mathrm{K}}(x)\left[\mathcal{L} u_{\mathrm{R}}-f\right](y)\, dx\, dy \tag{28}$$

so that the perturbation term appearing in (25) can be written as

$$\sum_{\mathrm{K}}\left(M_{\mathrm{K}}\left(\mathcal{L} u_{\mathrm{R}}-f\right), \mathcal{L}_{\mathrm{K}}^{*} v_{\mathrm{R}}\right)_{K}=-\sum_{\mathrm{K}} \int_{K \times K} g_y^{\mathrm{K}}(x)\left[\mathcal{L} u_{\mathrm{R}}-f\right](y)(\mathcal{L}_{\mathrm{K}}^{*} v_{\mathrm{R}})(x)\, dx\, dy. \tag{29}$$

Now we show how the same results can be obtained using the residual-free bubbles approach.

2.2 The residual-free bubbles approach

The presentation follows [8]. We denote by B_{K} a finite-dimensional subspace of $H_0^1(K)$ (to be determined later) and by $V_{\mathrm{U}}^b=V^b=\oplus_{\mathrm{K}} B_{\mathrm{K}}$ the space used to enlarge V_{R}^h. We solve problem (3) on $V_h=V_{\mathrm{R}}^h \oplus V^b$. We have, as in (8), that any $v_h \in V_h$

can be split into a linear part $v_{\rm R} \in V_{\rm R}^h$ and into a bubble part $v_{\rm B} \in V^b$ in a unique way:

$$v_h = v_{\rm R} + v_{\rm B} \in V_{\rm R}^h \oplus V^b, \tag{30}$$

but in addition (as in the previous Section) the bubble part $v_{\rm B}$ can be uniquely decomposed among the K's:

$$v_{\rm B} = \sum_{\rm K} v_{\rm B,K}, \qquad v_{\rm B,K} \in B_{\rm K}. \tag{31}$$

Thus, the variational problem (1) in V_h can be written as follows:

$$\begin{cases} \text{find } u_h = u_{\rm R} + u_{\rm B} = u_{\rm R} + \sum_{\rm K} u_{\rm B,K} \in V_{\rm R}^h \oplus V^b \text{ such that} \\ a\left(u_{\rm R} + u_{\rm B}, v_{\rm R}\right) = F(v_{\rm R}) \quad \text{for all } v_{\rm R} \in V_{\rm R}^h \\ a\left(u_{\rm R} + u_{\rm B,K}, v_{\rm B,K}\right)_K = F(v_{\rm B,K})_K \quad \text{for all } K \in \mathcal{T}_h, v_{\rm B,K} \in B_{\rm K}, \end{cases} \tag{32}$$

where, as before, the subscript $(\cdot)_K$ indicates that the integrals involved are restricted to the element K. We rewrite the last equation in (32) as

$$a\left(u_{\rm B,K}, v_{\rm B,K}\right)_K = -\left[a\left(u_{\rm R}, \cdot\right) - F(\cdot)\right]_K\left(v_{\rm B,K}\right) \quad \text{for all } v_{\rm B,K} \in B_{\rm K}. \tag{33}$$

We are now ready for selecting $B_{\rm K}$. For each K, the residual-free bubbles space $B_{\rm K}$ is defined by the property that equation (33) actually holds for any test function in the whole $H_0^1(K)$. This requirement identifies for each $u_{\rm R} \in V_{\rm R}^h$ a function $u_{\rm B,K} \in H_0^1(K)$ as the solution of the variational problem

$$\begin{cases} \text{find } u_{\rm B,K} \in H_0^1(K) \text{ such that} \\ a\left(u_{\rm B,K}, v\right)_K = -\left[a\left(u_{\rm R}, \cdot\right) - F(\cdot)\right]_K(v) \text{ for all } v \in H_0^1(K). \end{cases} \tag{34}$$

In other words, we also have the operator $M_{\rm K} : H^{-1}(K) \to H_0^1(K)$ presented in (23):

$$u_{\rm B,K} = M_{\rm K}\left[a\left(u_{\rm R}, \cdot\right) - F(\cdot)\right]_K = M_{\rm K}\left(\mathcal{L}u_{\rm R} - f\right). \tag{35}$$

We have then to ensure that $B_{\rm K}$ contains $u_{\rm B,K}$, solution of (34), for each $u_{\rm R} \in V_{\rm R}^h$, i.e. $B_{\rm K}$ must contain $M_{\rm K}\left(\mathcal{L}u_{\rm R} - f\right)$ for $u_{\rm R}$ ranging in $V_{\rm R}^h$. This, together with the requirement for $B_{\rm K}$ to be a linear space, leads to the following definition:

$$B_{\rm K} = \left\{M_{\rm K}\left(\mathcal{L}v_{\rm R} - \lambda f\right),\ v_{\rm R} \in V_{\rm R}^h,\ \lambda \in \mathbb{R}\right\}. \tag{36}$$

The elements of $B_{\rm K}$ are the *residual-free bubbles* of [8]. We now look more closely at $B_{\rm K}$. For each $K \in \mathcal{T}_h$, let $N_{\rm K}$ be the number of local degrees of freedom of the space V^h_{R}, i.e. $N_{\mathrm{K}} = \dim\{v_{\mathrm{R}|K},\ v_{\mathrm{R}} \in V^h_{\mathrm{R}}\}$; for instance, $N_{\mathrm{K}} = 3$ for continuous, piecewise linear elements in dimension two. Let $\{\varphi_i,\ i = 1, \dots, N_{\mathrm{K}}\}$ be the local shape functions and let $b_{\mathrm{K},i} \in H^1_0(K)$ be the solution of the following boundary value problem:

$$\begin{cases} \mathcal{L}b_{\mathrm{K},i} = -\mathcal{L}\varphi_i & \text{in } K \\ b_{\mathrm{K},i} = 0 & \text{on } \partial K. \end{cases} \tag{37}$$

Finally, let $b^f_{\mathrm{K}} \in H^1_0(K)$ solve

$$\begin{cases} \mathcal{L}b^f_{\mathrm{K}} = f & \text{in } K \\ b^f_{\mathrm{K}} = 0 & \text{on } \partial K. \end{cases} \tag{38}$$

Then by linearity we have

$$B_{\mathrm{K}} = \mathrm{span}\{b_{\mathrm{K},1}, \dots, b_{\mathrm{K},N_K}, b^f_{\mathrm{K}}\} \tag{39}$$

so that

$$\dim(B_{\mathrm{K}}) \leq N_{\mathrm{K}} + 1. \tag{40}$$

In this way we have seen that the space of residual-free bubbles is finite-dimensional, and its dimension depends on the range space of a certain local operator. Notice that the same situation had already been found in [2], looking for possible sets of bubbles (not residual-free, in general) that might produce a prescribed stabilizing term.

2.3 Equivalence of the two approaches

Comparing the expression for the "unresolvable scale part" $u_{\mathrm{U,K}}$ in (23) and the one for the "bubble part" $u_{\mathrm{B,K}}$ in (35), we see that the variational multiscale and the residual-free bubbles approaches actually coincide, in that they yield the same solution in the enlarged space and hence the same equation on the original space. On the other hand, since both (23) and its residual-free bubbles counterpart (35) are unsolvable in practice, with only few exceptions, one approach or the other might suggest different ways for computing an "approximate solution".

In the next Section, we will see on an example how this machinery can be applied in practice.

3 A classical example

In this Section we will show how we can reproduce the SUPG stabilization scheme by the methodology described above, by considering, for the sake of simplicity, $\mathcal{L}$ to be a linear scalar convection-diffusion operator, i.e.

$$\mathcal{L}u = -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u, \tag{41}$$

where the diffusion term ε is a positive constant, and both the convection field $\boldsymbol{a}$ and the right-hand-side f are piecewise constant with respect to the triangulation $\mathcal{T}_h$. We assume that the operator $\mathcal{L}$ is *convection-dominated*, i.e. that $\varepsilon \ll |\boldsymbol{a}|$. For this particular problem the present analysis is going to recover exactly the same results (and philosophy!) of [4], but now in a much wider context.

As already pointed out, the exact solution of (41) can exhibit boundary and internal *layers*, i.e. very narrow regions where the solution and its derivatives change abruptly. If the discretization scale h is too big to resolve the layers, then a classical finite element method will yield a solution with large numerical oscillations spreading all over the domain, completely unrelated to the true solution. To properly resolve the layers, the discretization parameter h must be of the same size of the ratio between diffusion and convection: $h \approx \varepsilon/|\boldsymbol{a}|$. In many problems, this choice would lead to a huge number of degrees of freedom, making the discretization intractable. As explained in the introduction, stabilization methods have been invented to cope with this kind of problem. One of the most popular stabilization method nowadays is the so-called SUPG method first described in [5], where SUPG stands for Streamline-Upwind Petrov/Galerkin. This method consists in adding to the original bilinear form $a(\cdot,\cdot)$ a term which introduces a suitable amount of artificial viscosity in the direction of streamlines, but without upsetting consistency. In the case of the operator (41) and homogeneous Dirichlet boundary conditions, we have

$$a(u,v) = \varepsilon\int_\Omega \nabla u\cdot\nabla v + \int_\Omega (\boldsymbol{a}\cdot\nabla u)\, v \tag{42}$$

and if V_h is the space of continuous, piecewise linear elements the SUPG method reads as

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) + \sum_K \tau_K \int_K (\boldsymbol{a}\cdot\nabla u_h - f)(\boldsymbol{a}\cdot\nabla v_h) = F(v_h) \text{ for all } v_h \in V_h, \end{cases} \tag{43}$$

where τ_K is a stabilization parameter depending on the local character of the discretization: in elements whose diameter is not small enough to resolve all scales, $\tau_K \approx h_K/|\boldsymbol{a}_{|K}|$ and elsewhere $\tau_K \approx 0$. More precisely, we can introduce a mesh Péclet number in the following way:

$$\text{for each } K \in \mathcal{T}_h, \quad \mathrm{Pe}_K = \frac{|\boldsymbol{a}_{|K}| h_K}{6\varepsilon} \tag{44}$$

and then define τ_K accordingly to the size of Pe_K:

$$\tau_K = \begin{cases} \dfrac{h_K}{2|\boldsymbol{a}_{|K}|} & \text{if } \mathrm{Pe}_K \geq 1 \\ \dfrac{h_K^2}{12\varepsilon} & \text{if } \mathrm{Pe}_K < 1. \end{cases} \tag{45}$$

This scheme leads to a reasonable numerical solution, where of course layers are not resolved, but they are very well localized and out of the layers the accuracy is very good. We refer to [7, 12] for further details.

We now apply the methodology presented in the previous Section. We will use equation (29) to determine the modification of the bilinear form, and then we will describe the residual-free bubbles space B_K.

Since the coefficients of the operator are piecewise constant, for each $K \in \mathcal{T}_h$ we simply have

$$\mathcal{L}_K^* u = -\varepsilon \Delta u - \boldsymbol{a}_{|K} \cdot \nabla u. \tag{46}$$

Recall that V_R^h is the space of continuous, piecewise linear elements on $\mathcal{T}_h$. Then

$$(\mathcal{L} u_R - f)_{|K} = (\boldsymbol{a} \cdot \nabla u_R - f)_{|K} = \text{constant} \tag{47}$$

and

$$(\mathcal{L}_K^* v_R)_{|K} = -(\boldsymbol{a} \cdot \nabla v_R)_{|K} = \text{constant}. \tag{48}$$

We have then

$$\begin{aligned} &-\int_{K\times K} g_y^K(x)\,[\mathcal{L} u_R - f](y)\,(\mathcal{L}_K^* v_R)(x)\,dx\,dy = \\ &(\boldsymbol{a} \cdot \nabla u_R - f)_{|K}\,(\boldsymbol{a} \cdot \nabla v_R)_{|K} \int_{K\times K} g_y^K(x)\,dx\,dy = \\ &\left[\int_K (\boldsymbol{a} \cdot \nabla u_R - f)\,(\boldsymbol{a} \cdot \nabla v_R)\right] \frac{1}{|K|} \int_{K\times K} g_y^K(x)\,dx\,dy. \end{aligned} \tag{49}$$

The resulting scheme on V_{R}^h is then

$$\begin{cases} \text{find } u_{\mathrm{R}} \in V_{\mathrm{R}}^h \text{ such that} \\ a\,(u_{\mathrm{R}}, v_{\mathrm{R}}) + \sum_K \widehat{\tau}_{\mathrm{K}} \int_K (\boldsymbol{a} \cdot \nabla u_{\mathrm{R}} - f)(\boldsymbol{a} \cdot \nabla v_{\mathrm{R}}) = F(v_{\mathrm{R}}) \text{ for all } v_{\mathrm{R}} \in V_{\mathrm{R}}^h, \end{cases} \tag{50}$$

where

$$\widehat{\tau}_{\mathrm{K}} = \frac{1}{|K|} \int_{K\times K} g_y^{\mathrm{K}}(x)\,dx\,dy. \tag{51}$$

We see that the SUPG scheme (43) and (50) have an identical structure; we need only to compare the two constants τ_{K} and $\widehat{\tau}_{\mathrm{K}}$. We have

$$\int_{K\times K} g_y^{\mathrm{K}}(x)\,dx\,dy = \int_K \left[\int_K g_y^{\mathrm{K}}(x) 1_y\,dy \right] dx \tag{52}$$

and by (27) the function b_{K} defined as

$$b_{\mathrm{K}}(x) = \int_K g_y^{\mathrm{K}}(x) 1_y\,dy, \tag{53}$$

solves the following boundary value problem on K:

$$\begin{cases} -\varepsilon \Delta b_{\mathrm{K}} + \boldsymbol{a}_{|K} \cdot \nabla b_{\mathrm{K}} = 1 & \text{in } K \\ b_{\mathrm{K}} = 0 & \text{on } \partial K. \end{cases} \tag{54}$$

We have then

$$\widehat{\tau}_{\mathrm{K}} = \frac{1}{|K|} \int_K b_{\mathrm{K}}. \tag{55}$$

We will show now that in this example the residual-free bubbles space B_{K} defined in (36) is generated by the single function b_{K}, solution of problem (54) (this is the reason why we chose equation (53) as the title of the present paper). For, by equations (23), (35), and (47) we have

$$u_{\mathrm{U,K}} = u_{\mathrm{B,K}} = M_{\mathrm{K}}\left(\mathcal{L} u_{\mathrm{R}} - f\right) = -\left(\boldsymbol{a} \cdot \nabla u_{\mathrm{R}} - f\right)_{|K}\, M_{\mathrm{K}}(-1), \tag{56}$$

and by (54)

$$M_{\mathrm{K}}(-1) = b_{\mathrm{K}}. \tag{57}$$

We have then

$$u_{\mathrm{B,K}} = -\left(\boldsymbol{a} \cdot \nabla u_{\mathrm{R}} - f\right)_{|K} b_{\mathrm{K}}, \tag{58}$$

so that

$$B_K = \text{span}\{b_K\}, \qquad (59)$$

and $\dim B_K = 1$. In a more general situation (variable coefficients and right-hand-side), the dimension of B_K could grow up to 4, in agreement with (40). Finally, if we compare (25) with (50) we (obviously) get again (55).

Both approaches leave us with the problem of evaluating, possibly in some approximate way, the integral of b_K appearing in (55). For strongly convection-dominated cases (the most interesting ones) we can argue as in [4]. If $\varepsilon \ll |\boldsymbol{a}_{|K}|\, h_K$, the solution of (54) will be very close to a pyramid with one (or two) almost vertical faces on the outflow boundary of K (the element boundary layer). The remaining faces of this pyramid have slope $1/|\boldsymbol{a}_{|K}|$ in the direction of $\boldsymbol{a}_{|K}$. Hence, if we define $\widehat{h}_K$ as the length of longest segment parallel to $\boldsymbol{a}_{|K}$ and contained in K, we have

$$\int_K b_K \approx \text{Volume of the pyramid} = \frac{|K|}{3}\frac{\widehat{h}_K}{|\boldsymbol{a}_{|K}|}, \qquad (60)$$

so that

$$\widehat{\tau}_K = \frac{1}{|K|}\int_K b_K \approx \frac{\widehat{h}_K}{3\,|\boldsymbol{a}_{|K}|}. \qquad (61)$$

Using a scaling argument (see [14]), we can also show that when ε is large with respect to $|\boldsymbol{a}_{|K}| h_K$, we have

$$\frac{1}{|K|}\int_K b_K \approx C\frac{h_K^2}{\varepsilon} \qquad (62)$$

where C still depends on K and h but can be uniformly bounded from above and from below if we have a regular family of triangulations.

If we compare (45) and (61) we see that the values of τ_K and $\widehat{\tau}_K$ are very close in all regimes; indeed, the theoretical results for the SUPG method of [7, 12] also hold with $\widehat{\tau}_K$ in place of τ_K, and the numerical experiments give results of similar quality.

4 Conclusions

In this note we have shown that two apparently distinct approaches recently introduced in [11], [4, 8] respectively are indeed equivalent, or simply an integral apart as the title of the present paper suggests. Therefore both of them share the same attributes and shortcomings.

On the list of attributes we should first point out that these strategies offer a systematic derivation of discretization methods. Some of these coincide with stabilized methods, and in other applications numerical tricks such as mass lumping and selective reduced integration emanate from these derivations [9, 10]. Another interesting attribute is that the unresolved part serves as an error indicator [15]. The largest this part in a certain region (or in an element), the more we need to refine in that region, providing key information to adaptive strategies.

The main criticism refers to the need of the analytical solution of the unresolved part of the solution. While this can be dealt with in simple element geometries for certain linear problems, generalizations are not straightforward. We do consider different possibilities:

- Instead of solving the actual boundary-value-problem at the element level, we may only consider certain limits known to cause numerical pathologies (e.g., viscosity going to zero in convection-dominated flow problems, as described in Section 3);
- approximation of variable coefficients may also simplify the solution at the element level;
- in the Green's function viewpoint, approximate Green's functions may replace the actual exact Green's solution;
- other possibilities are currently under consideration.

We should note that the effect of the unresolved part needs only to be considered in an integral sense and therefore actual analytical computations may be avoided in pursuing the computation of this effect. This may be the ultimate way to derive discretizations to several classes of problems that have not yet been addressed satisfactorily.

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