STABILIZATION TECHNIQUES AND SUBGRID SCALES CAPTURING

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Abstract

We present an overview of stabilized finite element methods and of the standard Galerkin method enriched with *residual-free* bubble functions. The inadequacy of the standard Galerkin method using piecewise polynomials is discussed for different applications; the treatment using stabilized methods in their different versions is reviewed; and the connection to the standard Galerkin method with richer subspaces follows using the subgrid method or the residual-free-bubbles viewpoint. We close with a discussion on how to approximate the exact problem suggested by residual-free bubbles.

1 The standard Galerkin method and some of its failures

The standard Galerkin method can be roughly described as being an approximation of the variational formulation of a PDE (or system of PDE's) in a space of functions that is spanned by piecewise polynomials. This simple idea presents several advantages: first, the discrete system of equations that arise from such an approximation is going to be "banded" since the piecewise polynomials can be constructed to have a "small" support, and therefore the matrices involved are sparse. Second, taking derivatives and integrating polynomials is a very attractive task for any first year calculus student, and the simplicity of the implementation of the method for the most cumbersome PDE or system of PDE's seems straightforward. Third, the mathematical analysis seems to be possible without a lot of sophistication (at least if we have an elliptic problem, and we disregard technicalities referring to domain shape, etc.).

No wonder there was a boost of this methodology in the mid 60's and early 70's, and a general feeling that this was the way to approximate PDE's in general, and a confirmation of the expectations were available for a variety of structural problems which are elliptic.

However in the midst of this success the experts were aware that there were problems in applying this recipe to all problems under the sun. We will describe a few of those examples as the failures of the standard Galerkin method.

First let us describe the Galerkin method for an abstract boundary value problem given by

$$Au = f \qquad u \in V\,, \tag{1.1}$$

where for concreteness we take $V = H^1_0(\Omega)$. We then have

$$< Au, v >= a(u,v) =< u, A^* v > \qquad \forall u, v \in V\,, \tag{1.2}$$

and the variational formulation corresponding to (1.1) is given by:

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = (f,v) \text{ for all } v \in V. \end{cases} \tag{1.3}$$

The classical Galerkin method then consists of taking a finite-dimensional subspace of V, say $V_h \subset V$, which is spanned by continuous piecewise polynomials (often the choice is piecewise linears) and using the same variational formulation, given by eq. (1.3), in V_h, namely:

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a(u_h,v_h) = (f,v_h) \text{ for all } v_h \in V_h. \end{cases} \tag{1.4}$$

Error estimates are readily available if we can find two constants $\alpha > 0$ and $0 < M < \infty$ such that

$$\alpha ||v||^2_V \le a(v,v) \qquad \forall v \in V \tag{1.5}$$

$$a(v,w) \le M ||v||_V ||w||_V \qquad \forall v, w \in V. \tag{1.6}$$

Indeed from (1.5), (1.6) we have the optimal error bound:

$$||u - u_h||_V \le \frac{M}{\alpha} \inf_{v_h \in V_h} ||u - v_h||_V\,, \tag{1.7}$$

and the recipe is successful if $M/\alpha \approx 1$. An example of a problem where this theory is immediately applicable is the Poisson equation that governs several problems of physical interest. However the list does not go on very far without complications, and we now briefly illustrate some of the failures of the standard Galerkin method.

A) Failure # 1: Advection-dominated problems.

Here the problem consists in finding a scalar valued function $u(x)$ (temperature, for example) in a domain Ω, such that

$$- \varepsilon \Delta u + \boldsymbol{a} \cdot \nabla u = f \qquad \text{in}\, \Omega \tag{1.8}$$

subject (for the sake of simplicity) to the boundary condition

$$u = 0 \qquad \text{on}\, \partial\Omega, \tag{1.9}$$

where the flow velocity field $\boldsymbol{a}$ and the source function f are given. The variational formulation of this problem is:

$$\begin{cases} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ \varepsilon \displaystyle\int_\Omega \nabla u \cdot \nabla v \, d\Omega + \int_\Omega (\boldsymbol{a} \cdot \nabla u) \, v \, d\Omega = \int_\Omega f v \, d\Omega \qquad \forall v \in H_0^1(\Omega). \end{cases} \tag{1.10}$$

When we take the subspace of piecewise linears we discover that the method produces spurious oscillations throughout the domain if $h|\boldsymbol{a}| >> \varepsilon$. One indication why we should expect trouble comes from the realization that the optimal error bound (1.7) is practically useless in this case since by definition our bilinear form (in the simplest case when $\boldsymbol{a}$ is constant) yields:

$$a(v, v) = \varepsilon ||\nabla u||^2_{L^2}, \tag{1.11}$$

and therefore, from (1.5), $\alpha = \varepsilon$ times the Poincaré's constant (which is of order one), while M in (1.6) is of the same size as $|\boldsymbol{a}|$. Thus, for small values of ε the right-hand-side of (1.7) is very large, implying that the error can be very large! (And it is, with few special exceptions!!!)

B) Failure # 2: The Stokes' problem.

Here we wish to compute the velocity $\mathbf{u}$ and the pressure p such that

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in}\, \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in}\, \Omega \\ \mathbf{u} = \mathbf{0} & \text{on}\, \partial\Omega \end{cases} \tag{1.12}$$

where the source function $\mathbf{f}$ is given and the viscosity is taken to be one, for simplicity. The variational formulation of this problem is:

$$\begin{cases} \text{find } u \in (H_0^1(\Omega))^n \text{ and } p \in L^2(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\Omega - \int_\Omega p \nabla \cdot \mathbf{v} \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\Omega \qquad \forall \mathbf{v} \in (H_0^1(\Omega))^n \\ \displaystyle\int_\Omega q \nabla \cdot \mathbf{u} \, d\Omega = 0 \qquad \forall q \in L^2(\Omega), \end{cases} \tag{1.13}$$

where n is the number of space dimensions. If, say, $\mathbf{u}$ and p are approximated by continuous piecewise linears then the Galerkin method *fails for every mesh-size h!* As in the previous example we may be tempted to look at the optimal error bound (1.7) and in this case the bilinear form given by the sum of the left-hand-sides of (1.13) gives

$$a(\,(\mathbf{u},p)\,,\,(\mathbf{u},p)\,) = ||\nabla\mathbf{u}||^2_{L^2(\Omega)} \tag{1.14}$$

which does not give any information about the pressure variable p. We know from numerical experiments that this choice of approximations for velocity-pressure yields a poor method. The mathematical theory to analyze constrained problems is known as mixed method theory, and it provides a better framework for proving stability and convergence of suitable pairs of velocity-pressure (this theory is not covered by the simple-minded estimate (1.14)). The mixed method theory will also confirm that the present choice of equal-order linears leads to an unstable method (e.g., see [6, 10, 29, 31, 41, 48] and references therein).

C) Failure # 3: Reissner-Mindlin Plates.

Here we are interested in computing the rotations $\boldsymbol{\theta}$, vertical displacement w and shear strain $\boldsymbol{\gamma}$ of a plate of thickness t governed by:

$$\begin{cases} \nabla\cdot\mathbf{C}\boldsymbol{\varepsilon}(\boldsymbol{\theta}) - \boldsymbol{\gamma} = \mathbf{0} & \text{in}\,\Omega \\ \nabla\cdot\boldsymbol{\gamma} = f & \text{in}\,\Omega \\ \nabla w - \boldsymbol{\theta} - t^2\boldsymbol{\gamma} = \mathbf{0} & \text{in}\,\Omega \end{cases} \tag{1.15}$$

where f is the distributed load on the surface of the plate, $\mathbf{C}$ is the fourth-order tensor of elastic moduli and $\boldsymbol{\varepsilon}(\boldsymbol{\theta})$ is the symmetric part of the gradient of the rotation vector, given by

$$\boldsymbol{\varepsilon}(\boldsymbol{\theta}) = \frac{\nabla\boldsymbol{\theta} + (\nabla\boldsymbol{\theta})^T}{2}\,. \tag{1.16}$$

We append to (1.15) the boundary conditions:

$$\boldsymbol{\theta} = \mathbf{0} \qquad \text{on}\,\partial\Omega \tag{1.17}$$

$$w = 0 \qquad \text{on}\,\partial\Omega \tag{1.18}$$

and we set, for the sake of simplicity, a certain number of physical constants to one. The variational formulation of this problem is:

$$\begin{cases} \text{find } \boldsymbol{\theta} \in (H^1_0(\Omega))^2,\ w \in H^1_0(\Omega) \text{ and } \boldsymbol{\gamma} \in (L^2(\Omega))^2 \text{ such that} \\ \displaystyle\int_\Omega \mathbf{C}\boldsymbol{\varepsilon}(\boldsymbol{\theta})\cdot\boldsymbol{\varepsilon}(\boldsymbol{\eta})\,d\Omega + \int_\Omega \boldsymbol{\gamma}\cdot(\nabla v - \boldsymbol{\eta})\,d\Omega = \int_\Omega f v\,d\Omega \\ \qquad\qquad\qquad\qquad \text{for all } \boldsymbol{\eta} \in (H^1_0(\Omega))^2,\ v \in H^1_0(\Omega) \\ \displaystyle -\int_\Omega \boldsymbol{\delta}\cdot(\nabla w - \boldsymbol{\theta})\,d\Omega + t^2\int_\Omega \boldsymbol{\delta}\cdot\boldsymbol{\gamma}\,d\Omega = 0 \qquad \text{for all } \boldsymbol{\delta} \in (L^2(\Omega))^2. \end{cases} \tag{1.19}$$

If $\boldsymbol{\theta}$ and w are approximated by continuous piecewise linears and $\boldsymbol{\gamma}$ by either continuous piecewise linears or piecewise constants then the Galerkin method *fails* for $h >> t$, a situation common to thin plates. As in the previous example the bilinear form given by the sum of the left-hand-sides of (1.19) yields

$$a(\,(\boldsymbol{\theta}, w, \boldsymbol{\gamma})\,,\,(\boldsymbol{\theta}, w, \boldsymbol{\gamma})\,) = (\mathbf{C}\varepsilon(\boldsymbol{\theta}), \varepsilon(\boldsymbol{\theta})) + t^2 ||\boldsymbol{\gamma}||^2_{L^2(\Omega)} \tag{1.20}$$

which does not give any information on the displacement variable w and has a coefficient t^2 (that may be small for thin plates) multiplying the L^2-norm of the strains $\boldsymbol{\gamma}$. We may be led to believe that this method does not have stability for displacements and has poor stability for strains. Again referring to the more sophisticated mixed method analysis this particular choice of approximation is confirmed to be inadequate (see [6, 10, 31, 48] and references therein). For specially designed elements that will work in this case see e.g. [1, 7, 11].

D) Failure # 4: Dirichlet problem with Lagrange multipliers.

Here we wish to find the scalar variable u solution of the Dirichlet problem:

$$\begin{cases} -\Delta u = f & \text{in}\, \Omega \\ \quad\; u = g & \text{on}\, \Gamma = \partial\Omega \end{cases} \tag{1.21}$$

and the Lagrange multiplier

$$\lambda = \frac{\partial u}{\partial n} \qquad \text{on}\, \Gamma, \tag{1.22}$$

where f and g are given functions. The variational formulation for this problem is (see [2]):

$$\begin{cases} \text{find } u \in H^1(\Omega) \text{ and } \lambda \in H^{-1/2}(\Gamma) \text{ such that} \\ \displaystyle\int_\Omega \nabla u \cdot \nabla v \, d\Omega - \int_\Gamma \lambda v \, ds = \int_\Omega f v \, d\Omega \qquad \forall v \in H^1(\Omega), \\ \displaystyle\int_\Gamma u\mu \, ds = \int_\Gamma g\mu \, ds \qquad \forall \mu \in H^{-1/2}(\Gamma). \end{cases} \tag{1.23}$$

The Galerkin method will fail if the "grid for λ" is too fine compared with the "grid for u" (see always [2]). Also as in the previous example the bilinear form given by the sum of the left-hand-sides of (1.23) yields

$$a(\,(u, \lambda)\,,\,(u, \lambda)\,) = ||\nabla u||^2_{L^2(\Omega)}, \tag{1.24}$$

which does not give any information on the Lagrange multiplier variable λ.

In the next two sections we will shortly discuss two strategies to deal with the shortcomings pointed out above, that have been developed in the past decade (although the first appearance of SUPG goes back to 1979). The first one consists of modifying the bilinear form a associated with the problem so that enhanced numerical stability is achieved without compromising consistency. This approach is known under several names, starting with SUPG for example 1 above, then Galerkin-least-squares and more recently as stabilized methods (see [4, 5, 9, 14, 16, 17, 18, 20, 21, 22, 23, 24, 28, 30, 33, 34, 35, 36, 37, 38, 45, 46, 47] and references therein). The second approach consists of using the standard Galerkin method enriched with special functions (and in particular we shall focus our attention on the so-called residual-free bubbles). The idea is to enlarge the space of functions to deal with the particular problem at hand so that our mesh, say coarse mesh, is able to deal with the effects of the unresolvable scales. This is also known as a subgrid model approach (see [3, 8, 12, 13, 15, 19, 25, 26, 27, 32, 39, 40, 42, 43, 44] and references therein).

2 Stabilized methods

To roughly describe the general idea of the method and its variations let us reconsider our abstract variational problem

$$Au = f \qquad u \in V, \tag{2.1}$$

and its variational formulation

$$a(u,v) = (f,v) \qquad \forall v \in V. \tag{2.2}$$

The idea is to modify the bilinear form $a(u,v)$ so that consistency is preserved and stability is increased. We consider two basic variants:

$$a(u_h,v_h) + \sum_K \tau_K (Au_h - f, Av_h)_K = (f,v_h) \qquad \forall v_h \in V_h\,, \tag{2.3}$$

or,

$$a(u_h,v_h) - \sum_K \tau_K (Au_h - f, A^* v_h)_K = (f,v_h) \qquad \forall v_h \in V_h\,, \tag{2.4}$$

where τ_K in either case is a stability parameter that depends on the element size and on the application.

If we apply these strategies to each problem discussed in the previous section we obtain:

A) Advection-dominated problems:

$$\begin{cases} -\varepsilon \Delta u + \boldsymbol{a} \cdot \nabla u = f & \text{in}\,\Omega \\ u = 0 & \text{on}\,\partial\Omega \end{cases} \tag{2.5}$$

The stabilized variational formulation is:

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ \varepsilon \displaystyle\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega + \int_\Omega \boldsymbol{a} \cdot \nabla u_h \, v_h \, d\Omega + S = \int_\Omega f v_h \, d\Omega \qquad \forall v_h \in V_h, \end{cases} \tag{2.6}$$

where S is the additional stabilizing term that, for this example, should add stability in the streamline direction, i.e. a term of the form

$$\tau \int_{\Omega} (\boldsymbol{a} \cdot \nabla u_h)\, (\boldsymbol{a} \cdot \nabla v_h)\, d\Omega\,. \tag{2.7}$$

A term of the form (2.7) is needed for stability, but as it stands it would destroy consistency. Thus we consider either

$$S = \tau \sum_K \int_K (-\varepsilon \Delta u_h + \boldsymbol{a} \cdot \nabla u_h - f)\, (\boldsymbol{a} \cdot \nabla v_h)\, d\Omega\,, \tag{2.8}$$

or

$$S = \tau \sum_K \int_K (-\varepsilon \Delta u_h + \boldsymbol{a} \cdot \nabla u_h - f)(-\varepsilon \Delta v_h + \boldsymbol{a} \cdot \nabla v_h)\, d\Omega\,, \tag{2.9}$$

or

$$S = -\tau \sum_K \int_K (-\varepsilon \Delta u_h + \boldsymbol{a} \cdot \nabla u_h - f)(-\varepsilon \Delta v_h - \boldsymbol{a} \cdot \nabla v_h)\, d\Omega\,. \tag{2.10}$$

Either of these options will work, as surveyed in [21]. Notice that, here and in the following examples, for the sake of simplicity, we took the simpler form $\tau \sum_K (\ldots)$ rather than the (better) $\sum_K \tau_K (\ldots)$ as in (2.3)-(2.4).

B) The Stokes' problem:

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in}\, \Omega \\ \nabla \cdot \mathbf{u} = 0 & \text{in}\, \Omega \\ \mathbf{u} = \mathbf{0} & \text{on}\, \partial\Omega \end{cases} \tag{2.11}$$

The stabilized variational formulation for this problem is:

$$\int_{\Omega} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h\, d\Omega - \int_{\Omega} p_h\, \nabla \cdot \mathbf{v}_h\, d\Omega + S_1 = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h\, d\Omega \qquad \forall \mathbf{v}_h\,, \tag{2.12}$$

$$\int_{\Omega} q_h\, \nabla \cdot \mathbf{u}_h\, d\Omega + S_2 = 0 \qquad \forall q_h\,, \tag{2.13}$$

where

$$S_1 = \tau_1 \sum_K \int_K (-\Delta \mathbf{u}_h + \nabla p_h - \mathbf{f})(-\Delta \mathbf{v}_h)\, d\Omega\,, \tag{2.14}$$

$$S_2 = \tau_2 \sum_K \int_K (-\Delta \mathbf{u}_h + \nabla p_h - \mathbf{f}) \cdot \nabla q_h\, d\Omega\,, \tag{2.15}$$

or some variants (see [23] for a survey of stabilized methods for the Stokes problem). The term that is crucial for stability is contained in S_2 and it is essentially

$$\tau_2 \sum_K \int_K \nabla p_h \cdot \nabla q_h\, d\Omega\,. \tag{2.16}$$

C) Reissner-Mindlin Plates:

$$\begin{cases} \nabla \cdot \mathbf{C}\varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = \mathbf{0} & \text{in}\,\Omega \\ \nabla \cdot \boldsymbol{\gamma} = f & \text{in}\,\Omega \\ \nabla w - \boldsymbol{\theta} - t^2\boldsymbol{\gamma} = \mathbf{0} & \text{in}\,\Omega, \end{cases} \tag{2.17}$$

where

$$\varepsilon(\boldsymbol{\theta}) = \frac{\nabla\boldsymbol{\theta} + (\nabla\boldsymbol{\theta})^T}{2}, \tag{2.18}$$

and with the boundary conditions

$$\boldsymbol{\theta} = \mathbf{0} \qquad \text{on}\,\partial\Omega, \tag{2.19}$$

$$w = 0 \qquad \text{on}\,\partial\Omega. \tag{2.20}$$

The stabilized variational formulation for this problem is:

$$\int_\Omega \mathbf{C}\varepsilon(\boldsymbol{\theta}_h) \cdot \varepsilon(\boldsymbol{\eta}_h)\, d\Omega + \int_\Omega \boldsymbol{\gamma}_h \cdot (\nabla v_h - \boldsymbol{\eta}_h)\, d\Omega + S_1 = \int_\Omega f v_h \, d\Omega \qquad \forall \boldsymbol{\eta}_h, \forall v_h\,, \tag{2.21}$$

$$-\int_\Omega \boldsymbol{\delta}_h \cdot (\nabla w_h - \boldsymbol{\theta}_h)\, d\Omega + t^2 \int_\Omega \boldsymbol{\delta}_h \cdot \boldsymbol{\gamma}_h \, d\Omega + S_2 = 0 \qquad \forall \boldsymbol{\delta}_h\,, \tag{2.22}$$

where

$$S_1 = \tau_1 \sum_K \int_K (\nabla w_h - \boldsymbol{\theta}_h - t^2\boldsymbol{\gamma}_h) \cdot (\nabla v_h \ldots)\, d\Omega\,, \tag{2.23}$$

$$S_2 = \tau_2 \sum_K \int_K (\Delta\boldsymbol{\theta}_h - \boldsymbol{\gamma}_h) \cdot \boldsymbol{\delta}_h \, d\Omega\,, \tag{2.24}$$

or variants (see [34] for a family of methods and [11, 45] for some convergent low-order stabilized elements).

D) Dirichlet problem with Lagrange multipliers:

$$\begin{cases} -\Delta u = f & \text{in}\,\Omega \\ u = g & \text{on}\,\Gamma = \partial\Omega \\ \lambda = \dfrac{\partial u}{\partial n} & \text{on}\,\Gamma. \end{cases} \tag{2.25}$$

The stabilized variational formulation for this problem is:

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, d\Omega - \int_\Gamma \lambda_h v_h \, ds + S_1 = \int_\Omega f v_h \, d\Omega \qquad \forall v_h\,, \tag{2.26}$$

$$\int_\Gamma u_h \mu_h \, ds + S_2 = \int_\Gamma g \mu_h \, ds \qquad \forall \mu_h\,. \tag{2.27}$$

The key term for stability is in S_2 and it is given by $\tau_1 \int_\Gamma \lambda_h \mu_h \, ds$ (which is not consistent). Then we may consider

$$(\Delta u_h - f, \Delta v_h)_K, \tag{2.28}$$

$$(\frac{\partial u_h}{\partial n} - \lambda_h, \frac{\partial v_h}{\partial n} - \mu_h)_{\Gamma \cap \partial K}, \tag{2.29}$$

$$(u - g, v)_{\Gamma \cap \partial K}. \tag{2.30}$$

All of these additional terms are suggested in [5] and a method using this approach in the framework of domain decomposition methods is presented in [4] with an abstract theory for stabilized methods.

In our discussion above we avoided the crucial question: How do we select τ?

The general strategy to find it may be described as follows: first you guess the form of it, then you perform an error analysis to confirm that the order of the parameter yields optimal or quasi-optimal estimates. If this is not the case, guess again, until you get the best possible estimate. Then perform numerical experiments for your choices and try to cover a wide range of problems of interest. If all works then you have a method!

While this may seem unsatisfactory, this is how most of these methods have been designed. The interesting result is that the methods obtained by this procedure sometimes may be robust enough so that changing the choice of the parameters does not affect the accuracy of the results. For some versions of the method this will not be true, but again for sufficiently fine meshes (or meshes that are adaptively refined) all versions will lead to accurate methods with very minor differences in the numerical results.

A wide variety of problems have been treated by stabilized methods and various authors advocate one parameter or another. We refer to [21, 22, 23, 24, 33, 34, 35, 36, 37, 38] for some choices of τ available.

3 Bubble Stabilization

The other main strategy developed in the past decade is based in the same variational formulation as the continuous problem, but now approximated by a richer subspace than the original one.

The idea can be roughly described as follows: given a grid $\mathcal{C}_h$, and a finite element space V_h, try to increase V_h (suitably) in order to increase stability.

Typically this is achieved by adding *bubbles* to each element, which are eliminated afterwards by *static condensation.* To make ideas precise let us first define the term *bubble*:

Definition: A *bubble* (in V, on $\mathcal{C}_h$) is a function $\phi \in V$ such that $\text{supp}(\phi)$ is contained in a single element.

The definition stresses the dependence on the underlying space V that describes the variable to be approximated. In Figure 1 we illustrate some possibilities depending on the space V. Now we wish to consider the static condensation

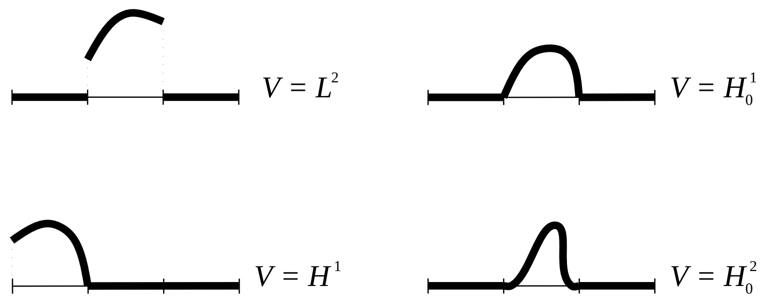

Figure 1. Some bubble functions

procedure to shed some light on the effect of eliminating the bubbles and simply writing the final method in terms of the reduced space of polynomials that we started with. If we denote the new augmented space of functions by V_h, then

$$V_h = V_L \oplus V_B\,, \tag{3.1}$$

where V_L is spanned by continuous piecewise linears and V_B, by bubble functions. Then members of V_h can be decomposed as follows:

$$u_h = u_L + u_B, \qquad\qquad v_h = v_L + v_B\,. \tag{3.2}$$

If we substitute into the variational formulation

$$a(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h\,, \tag{3.3}$$

then we have

$$a(u_L + u_B, v_L + v_B) = (f, v_L + v_B) \qquad \forall v_L \in V_L\,, \forall v_B \in V_B\,. \tag{3.4}$$

If we take first $v_L = 0$ then the variational formulation reduces to

$$\begin{aligned} a(u_B, v_B) &= (f, v_B) - a(u_L, v_B) \\ &= < f - Au_L, v_B > \qquad \forall v_B \in V_B, \end{aligned} \tag{3.5}$$

which can be "solved", for any $u_L \in V_L$, with solution given by

$$u_B = M_B(f - Au_L)\,, \tag{3.6}$$

where M_B is a bounded linear operator from V' to V_B. If we now take $v_B = 0$ in (3.4) and use (3.6) we get a method in terms of the reduced space V_L only:

$$a(u_L, v_L) + < M_B(f - Au_L), A^* v_L >= (f, v_L)\,. \tag{3.7}$$

If for a minute we disregard the second term on the left-hand side, the method reduces to the standard Galerkin method using piecewise linears. The second term represents the effect of *adding V_B and then eliminating it by static condensation.*

The question now is: what does M_B look like?

For the advective-diffusive equation, $-\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u = f$, if f and $\boldsymbol{a}$ are piecewise constants and V_B is spanned by a single function b_K in each element K, then

$$< M_B(f - Au_L), A^* v_L >= \sum_K \frac{(\int_K b_K \, d\Omega)^2}{\varepsilon(\int_K |\nabla b_K|^2 \, d\Omega)} \frac{1}{|K|} \int_K (f - Au_L)(A^* v_L) \, d\Omega \,. \tag{3.8}$$

The term multiplying the last integral may be identified with the stability parameter τ_K of SUPG as shown in [8]. More generally we have that (Baiocchi-Brezzi-Franca [3]) with a suitable choice of the bubble space V_B, it is possible to make $< M_B u, v >$ equal to any bilinear form ρ, provided

$$0 \leq \rho \leq c_0 I \,, \tag{3.9}$$

with c_0 depending on the problem and on V_L. Also, if we consider $M_B = \tau I$ where I is the identity operator then we see that bubble stabilization suggests perturbation terms of the form

$$-\sum_K \int_K \tau_K (Au - f) A^* v \, d\Omega \,, \tag{3.10}$$

where τ_K depends on V_B.

As the presentation of the stabilized method left open the definition of the stability parameter τ, here we now have the basic question: *How to select the bubbles?*

A recent (and promising) point of view can be justified by the following argument:

1. The major reason of failure is the inadequate treatment of fine scales.
2. If rich enough, the bubbles (or, more generally, V_B) should be able to deal adequately with fine scales.
3. The stabilizing term represents the effect of *fine* (unresolvable) scales onto *coarse* (resolvable) ones.
4. Conceptually, we should take V_B as large as possible.

This last item suggests, ideally, that $V_L \oplus V_B = V$. In this case u_L would be the interpolant of the exact solution. This is, in particular, possible for all linear one-dimensional problems, but not, in general, in two or three dimensions. On the other hand, the basic idea of having V_B made of bubbles is very convenient, since it allows an *element-by-element* computation of the additional stabilizing term (the second term in the left-hand-side of (3.7)), although some experiments with spaces V_B made with functions having support in two or more elements indicate that this might also be an interesting line of future developments.

If, however, we stick to the idea of one-element bubbles (as in our definition), the largest choice we can take for V_B is $V_B = \oplus_K H_0^1(K)$. We can easily see that this is equivalent to selecting V_B as the space spanned by *residual-free-bubbles*, which are defined to satisfy the governing equations in strong form, i.e.,

$$Au_B = -(Au_L - f) \qquad \text{in } K, \tag{3.11}$$

subject to zero Dirichlet boundary condition on the element boundary

$$u_{B|K} = 0 \qquad \text{on } \partial K\,. \tag{3.12}$$

The problem given by equations (3.11)-(3.12) is addressed by solving instead [25, 26, 27]:

$$\begin{cases} A\varphi_{i,K} \;=\; -A\psi_{i,K} & \text{in } K, \\ \quad \varphi_{i,K} \;=\; 0 & \text{on } \partial K, \end{cases} \tag{3.13}$$

where the $\psi_{i,K}$'s are the n_{en} local basis functions for u_L and

$$\begin{cases} A\varphi_{f,K} \;=\; f & \text{in } K, \\ \quad \varphi_{f,K} \;=\; 0 & \text{on } \partial K. \end{cases} \tag{3.14}$$

Thus, if $u_{L|K} = \sum_{i=1}^{n_{en}} c_{i,K}\psi_{i,K}$ then

$$u_{B|K} = \sum_{i=1}^{n_{en}} c_{i,K}\varphi_{i,K} + \varphi_{f,K}\,, \tag{3.15}$$

with the same coefficients $c_{i,K}$'s.

Thus given a problem we should solve (3.13)-(3.14) to find the bubble basis functions φ's to determine the space of residual-free-bubbles V_B which, in turn, will produce the "optimal form" for the stabilizing term. This presents a systematic procedure to derive discretizations without the aforementioned open questions.

Notice that, in several cases, the functions $\{\varphi_{1,K}, \ldots, \varphi_{i,n_{en}}, \varphi_{f,K}\}$ might not be linearly independent, so that in practice we have to deal with less than $n_{en}+1$ bubbles. In particular, for the model problem (2.5), with piecewise constant $\boldsymbol{a}$ and f, we only need one function (that we call b_K^r) for every K in (3.13)-(3.14). The function b_K^r satisfies

$$a(b_K^r, v) = (1, v) \qquad \forall v \in H_0^1(K) \tag{3.16}$$

so that (3.8) now simply becomes

$$< M_B(f - Au_L), A^* v_L >= \sum_K \widehat{\tau}_K \int_K (f - Au_L)(A^* v_L)\, d\,\Omega, \tag{3.17}$$

where

$$\widehat{\tau}_K = \frac{1}{|K|}\int_K b^r_K \, d\,\Omega. \tag{3.18}$$

The parameter $\widehat{\tau}_K$ multiplying the last integral in (3.17) may be identified with the stability parameter τ_K of SUPG in the advective limit as shown in [15]. For the limit case $\varepsilon \to 0$, a reasonable value for $\widehat{\tau}_K$ can be guessed (as in [15]) by taking, instead of b^r_K in (3.18) the (limit) solution of $\boldsymbol{a}\cdot\nabla b_K = -1$ in K with $b_K = 0$ on the inflow part of ∂K. This design of $\widehat{\tau}_K$ is effective.

However, for more general ε, even this simple case presents relevant difficulties for the actual computation of (3.18).

A possible alternative can be represented by the use of the pseudo residual-free-bubbles introduced in [13]. Very briefly, the basic ideas are the following:

1. For every P internal to K, we partition K into three triangles K_i having a common vertex in P, and we set b_P to be the bubble which is linear in each K_i.

2. We solve (approximately) (3.16) in the one-dimensional space spanned by b_P. The solution $B_P = \alpha(P)b_P$ is given by

$$\alpha(P) = \frac{(1, b_P)}{a(b_P, b_P)}. \tag{3.19}$$

3. For choosing P, we now minimize

$$J(P) = \int_K |-\varepsilon\Delta B_P + \boldsymbol{a}\cdot\nabla B_P - 1| \, d\,\Omega\,, \tag{3.20}$$

   where the integral is in the sense of measures.

4. The actual minimization is done on a single oriented segment, depending on $\boldsymbol{a}$ and K. Among the minimizers, we take "the first one."

5. If P^* is the chosen minimizer of (3.20), we then define the "pseudo residual-free-bubble" as B_{P^*} and we have

$$\widehat{\tau}^{\psi}_K = \frac{1}{|K|}\int B^*_P \, d\,\Omega\,, \tag{3.21}$$

   similarly to (3.18).

For further details, we refer to [13]. We just explicitly remark that the practical computation of P^* is very cheap, and that, in the limit for $\varepsilon \to 0$ (for fixed K and $\boldsymbol{a}$) (3.18) and (3.21) give the same result.

Bibliography

1. D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner-Mindlin plate*, SIAM J. Numer. Anal., 26 (1989), pp. 1276–1290.

2. I. Babuška, *The finite element method with Lagrangian multipliers*, Numer. Math., 20 (1973), pp. 179–192.

3. C. Baiocchi, F. Brezzi, and L. Franca, *Virtual bubbles and the Galerkin-least-squares method*, Comput. Methods Appl. Mech. Engrg., 105 (1993), pp. 125–141.

4. C. Baiocchi, F. Brezzi, and D. Marini, *Stabilization of Galerkin methods and applications to domain decomposition*, in Future Tendencies in Computer Science, Control and Applied Mathematics, A. Bensoussan and J.-P. Verjus, eds., vol. 653 of Lecture Notes in Computer Science, Springer-Verlag, 1992, pp. 345–355. Proceedings of the International Conference on the Occasion of the 25th Anniversary of INRIA, Paris, France, December 1992.

5. H. Barbosa and T. J. R. Hughes, *Boundary Lagrange multipliers in finite element methods: error analysis in natural norms*, Numer. Math., 62 (1992), pp. 1–16.

6. K.-J. Bathe, *Finite Element Procedures*, Prentice-Hall, Englewood Cliffs, New Jersey, 1996.

7. F. Brezzi, K.-J. Bathe, and M. Fortin, *Mixed interpolated elements for Reissner-Mindlin plates*, Int. J. Numer. Methods Eng., 28 (1989), pp. 1787–1801.

8. F. Brezzi, M. Bristeau, L. Franca, M. Mallet, and G. Rogé, *A relationship between stabilized finite element methods and the Galerkin method with bubble functions*, Comput. Methods Appl. Mech. Engrg., 96 (1992), pp. 117–129.

9. F. Brezzi and J. Douglas, *Stabilized mixed methods for the Stokes problem*, Numer. Math., 53 (1988), pp. 225–236.

10. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, vol. 15 of Springer Series in Computational Mathematics, Springer-Verlag, Berlin, New-York, 1991.

11. F. Brezzi, M. Fortin, and R. Stenberg, *Quasi-optimal error bounds for approximation of shear-stresses in Mindlin-Reissner plate models*, Math. Models Meth. Appl. Sci., 1 (1991).

12. F. Brezzi, L. P. Franca, T. J. R. Hughes, and A. Russo, $b = \int g$, Comput. Methods Appl. Mech. Engrg., (1996). To appear.

13. F. Brezzi, D. Marini, and A. Russo, *Pseudo residual-free bubbles and stabilized methods*. To appear in the Proceedings of the Third ECCOMAS Computational Fluid Dynamics Conference, September 9–13, 1996, Paris, France.

14. F. Brezzi and J. Pitkäranta, *On the stabilization of finite element approximations of the Stokes problem*, in Efficient Solutions of Elliptic Systems, Notes on Numerical Fluid Mechanics, W. Hackbusch, ed., vol. 10, Viewig, 1984, pp. 11–19.

15. F. Brezzi and A. Russo, *Choosing bubbles for advection-diffusion problems*, Math. Models Meth. Appl. Sci., 4 (1994), pp. 571–587.

16. A. N. Brooks and T. J. R. Hughes, *Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 32 (1982), pp. 199–259.

17. J. Douglas and J. Wang, *An absolutely stabilized finite element method for the Stokes problem*, Math. Comp., 52 (1989), pp. 495–508.

18. L. P. Franca and E. G. D. do Carmo, *The Galerkin gradient least-squares method*, Comput. Methods Appl. Mech. Engrg., 74 (1989), pp. 41–54.

19. L. P. Franca and C. Farhat, *Bubble functions prompt unusual stabilized finite element methods*, Comput. Methods Appl. Mech. Engrg., 123 (1995), pp. 299–308.

20. L. P. Franca and S. L. Frey, *Stabilized finite element methods: II. The incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 99 (1992), pp. 209–233.

21. L. P. Franca, S. L. Frey, and T. J. R. Hughes, *Stabilized finite element methods: I. Application to the advective-diffusive model*, Comput. Methods Appl. Mech. Engrg., 95 (1992), pp. 253–276.

22. L. P. Franca and T. J. R. Hughes, *Two classes of mixed finite element methods*, Comput. Methods Appl. Mech. Engrg., 69 (1988), pp. 89–129.

23. L. P. Franca, T. J. R. Hughes, and R. Stenberg, *Stabilized finite element methods for the Stokes problem*, in Incompressible Computational Fluid Dynamics-Trends and Advances, M. D. Gunzburger and R. Nicolaides, eds., Cambridge University Press, 1993, pp. 87–107.

24. L. P. Franca and A. L. Madureira, *Element diameter free stability parameters for stabilized methods applied to fluids*, Comput. Methods Appl. Mech. Engrg., 105 (1993), pp. 395–403.

25. L. P. Franca and A. Russo, *Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles*, Math. Models Meth. Appl. Sci., (1996). To appear.

26. ———, *Mass lumping emanating from residual-free bubbles*, Comput. Methods Appl. Mech. Engrg., (1996). To appear.

27. ———, *Unlocking with residual-free bubbles*, Comput. Methods Appl. Mech. Engrg., (1996). To appear.

28. L. P. Franca and R. Stenberg, *Error analysis of some Galerkin least squares methods for the elasticity equations*, SIAM J. Numer. Anal., 28 (1991), pp. 1680–1697.

29. V. Girault and P. A. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, vol. 5 of Springer Series in Computational Mathematics, Berlin, New-York, 1986.

30. P. Hansbo and A. Szepessy, *A velocity-pressure streamline diffusion finite element method for the incompressible Navier-Stokes equation*, Comput. Methods Appl. Mech. Engrg., 84 (1990), pp. 175–192.

31. T. J. R. Hughes, *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*, Prentice-Hall, Englewood Cliffs, New Jersey, 1987.

32. ———, *Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origin of stabilized methods*, Comput. Methods Appl. Mech. Engrg., 127 (1995), pp. 387–401.

33. T. J. R. Hughes and L. P. Franca, *A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: symmetric formulations that converge for all velocity/pressure spaces*, Comput. Methods Appl. Mech. Engrg., 65 (1987), pp. 85–96.

34. ———, *A mixed finite element formulation for Reissner-Mindlin plate theory: uniform convergence of all higher-order spaces*, Comput. Methods Appl. Mech. Engrg., 67 (1988), pp. 223–240.

35. T. J. R. Hughes, L. P. Franca, and M. Balestra, *A new finite element formulation for computational fluid dynamics: V. Circumventing the babuška-brezzi condition: A stable Petrov-Galerkin formulation of the Stokes problem accommodating equal-order interpolations*, Comput. Methods Appl. Mech. Engrg., 59 (1986), pp. 85–99.

36. T. J. R. Hughes, L. P. Franca, and G. M. Hulbert, *A new finite element formulation for computational fluid dynamics: VIII. The Galerkin-least-squares method for advective-diffusive equations*, Comput. Methods Appl. Mech. Engrg., 73 (1989), pp. 173–189.

37. C. Johnson, *Numerical solution of partial differential equations by the finite element method*, Cambridge University Press, Cambridge, 1987.

38. C. Johnson, U. Nävert, and J. Pitkäranta, *Finite element methods for linear hyperbolic problem*, Comput. Methods Appl. Mech. Engrg., 45 (1984), pp. 285–312.

39. M. Lesoinne, C. Farhat, and L. P. Franca, *Unusual stabilized finite element methods for second order linear differential equations*, in Proceedings of the Ninth International Conference on Finite Elements in Fluids - New Trends and Applications, M. M. Cecchi, K. Morgan, J. Periaux, B. A. Schrefler, and O. C. Zienkiewicz, eds., Venice, Italy, October 1995, pp. 377–386.

40. R. Pierre, *Simple C^0 approximations for the computation of incompressible flows*, Comput. Methods Appl. Mech. Engrg., 68 (1988), pp. 205–227.

41. O. Pironneau, *Finite Element Methods for Fluids*, John Wiley,New York, 1989.

42. A. Russo, *Residual free bubbles and stabilized methods*, in Proceedings of the Ninth International Conference on Finite Elements in Fluids - New Trends and Applications, M. M. Cecchi, K. Morgan, J. Periaux, B. A. Schrefler, and O. C. Zienkiewicz, eds., Venice, Italy, October 1995, pp. 1607–1615.

43. ——, *Bubble stabilization of finite element methods for the linearized incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., (1996). To appear.

44. ——, *A posteriori error indicators via bubble functions*, Math. Models Meth. Appl. Sci., 6 (1996), pp. 33–41.

45. R. Stenberg, *A new finite element formulation for the plate bending problem*, in Asymptotic Methods for Elastic Structures, P. Ciarlet, L. Trabucho, and J. M. Viano, eds., Walter de Gruyter & Co., 1995, pp. 209–221.

46. T. E. Tezduyar, J. Liou, and M. Behr, *A new strategy for finite element computations involving moving boundaries and interfaces – The DSD/ST procedure: I. The concept and the preliminary numerical tests*, Comput. Methods Appl. Mech. Engrg., 94 (1992), pp. 339–352.

47. T. E. Tezduyar, J. Liou, M. Behr, and S. Mittal, *A new strategy for finite element computations involving moving boundaries and interfaces – The DSD/ST procedure: II. Computation of free-surface flows, two-liquid flows, and flows with drifting cylinders*, Comput. Methods Appl. Mech. Engrg., 94 (1992), pp. 353–372.

48. O. C. Zienkiewicz and R. L. Taylor, *The Finite Element Method*, McGraw-Hill, London, 4th ed., 1989.