A SCALABLE SUBSTRUCTURING METHOD BY LAGRANGE MULTIPLIERS FOR PLATE BENDING PROBLEMS

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Abstract. We present a new Lagrange multiplier based domain decomposition method for solving iteratively systems of equations arising from the finite element discretization of plate bending problems. The proposed method is essentially an extension of the FETI substructuring algorithm to the biharmonic equation. The main idea is to enforce continuity of the transversal displacement field at the subdomain crosspoints throughout the preconditioned conjugate gradient iterations. The resulting method is proved to have a condition number that does not grow with the number of subdomains, and grows at most polylogarithmically with the number of elements per subdomain. These optimal properties hold for numerous plate bending elements that are used in practice including the HCT, DKT, and a class of non-locking elements for the Reissner-Mindlin plate models. Computational experiments are reported and shown to confirm the theoretical optimal convergence properties of the new domain decomposition method. Computational efficiency is also demonstrated with the numerical solution in 45 iterations and 105 seconds on a 64-processor IBM SP2 of a plate bending problem with almost one million degrees of freedom.

Key words. Domain decomposition, biharmonic equation, plates and shells, parallel computing, substructuring, duality, non-overlapping, convergence theory

1. Introduction. The FETI (Finite Element Tearing and Interconnecting) method is a domain decomposition algorithm derived from a hybrid variational principle and designed for the iterative solution of systems of equations arising from the finite element discretization of self-adjoint elliptic partial differential equations. In this method, a given spatial domain is "torn" into *non-overlapping* subdomains where an incomplete solution of the primary field is first evaluated using a direct solver. Next, intersubdomain field continuity is enforced via Lagrange multipliers applied at the subdomain interfaces. This "gluing" phase generates a smaller size symmetric *dual* problem where the unknowns are the Lagrange multipliers, and which is best solved by a preconditioned conjugate gradient (PCG) algorithm. The FETI method was developed in [8, 13, 14], and discussed in detail in the monograph [15]. In contrast with other related domain decomposition methods using Lagrange multipliers as unknowns [17, 27], the FETI method distinguishes itself with the treatment of the null spaces of the subdomain stiffness matrices (rigid body modes) associated with the so-called floating subdomains, i.e., subdomains without a sufficient number of essential boundary conditions to prevent singularities of the local stiffness matrix. Resolving the rigid body modes leads to a small "coarse" problem that is solved in each PCG iteration. It was recognized in [12] and proved mathematically in [23] that solving this coarse problem accomplishes a global exchange of information between the subdomains and results in a method which, for elasticity problems, has a condition number that grows only polylogarithmically with the number of elements per subdomain, and is bounded independently of the number of subdomains. Extension to time-dependent problems was done in [9]. However for plate bending problems,

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the condition number was observed to grow fast with the number of elements per subdomain [12]. This is caused by the fact that plate bending is a fourth order problem, while the FETI domain decomposition method "tears" the approximate solution at subdomain crosspoints, which is suitable only for second order problems.

In this paper, we cure this limitation and extend the FETI methodology to obtain a non-overlapping domain decomposition method for plate bending problems. This new method has the properties one usually looks for in iterative substructuring methods: the condition number can be bounded independently of the number of subdomains, and it grows only polylogarithmically with the number of elements per subdomain. The computational cost per iteration is proportional to the solution of a boundary value problem in each subdomain, plus the solution of a sparse *coarse problem* with only few variables per subdomain. Such methods are commonly referred to as *scalable* and *quasi-optimal*, though, of course, for very large number of subdomains, the solution of the coarse problem would dominate.

The key idea of our method is to enforce the continuity of the approximate solution at the subdomain crosspoints throughout the iterations by adding the corresponding Lagrange multipliers to the coarse problem. A similar idea was employed in the Balancing Domain Decomposition (BDD) method for plates [20], where approximate continuity at crosspoints is enforced by adding new basis functions to the original coarse space [21, 22] in order to keep the energy of the approximate solution minimal with respect to displacements that are solutions for point loads at the subdomain crosspoints. The distinguishing features of both the present method and the method from [20] is that they are *non-overlapping* and work for standard finite elements used in everyday engineering practice. It should be noted that while the underlying ideas of FETI and BDD are in a way dual, FETI is not the BDD method applied to the dual problem.

For other domain decomposition methods for the biharmonic equation and plate bending see, for example, [4, 28].

The remainder of this paper is organized as follows. Derivation of the algorithm and theory in an abstract form are presented in Section 2. Selection of algorithm components for bending elements is specified in Section 2.4. In Section 3.2, we prove that the condition number of the new FETI method for plate bending problems is bounded polylogarithmically in H/h, where H and h denote respectively the characteristic subdomain and element sizes. The paper is concluded in Section 4 with the discussion of computational results that confirm the optimal convergence properties of the proposed Lagrange multiplier based domain decomposition method and demonstrate its efficiency and parallel performance.

Extensions to shells, implementation issues, and further computational results will be published elsewhere [10, 11]. For an alternative derivation and formulation of the present method, based on the concept of coarse optimality of a dual objective function, see an earlier version of this paper, which is available as a report [24].

2. Formulation of the Algorithm. All spaces are subspaces of $\mathbb{R}^n$. The inner product $\langle u, v\rangle = u^T v$ serves also as duality pairing. The ℓ^2 norm is denoted by $\|u\| = \langle u, u\rangle^{1/2}$. For a symmetric positive semidefinite matrix A, the induced seminorm is denoted by $\|u\|_A = \langle Au, u\rangle^{1/2}$. This is a norm if A is positive definite. The superscript $^+$ denotes pseudoinverse, defined as follows.

DEFINITION 2.1. *Let A be a linear operator. The pseudoinverse A^+ is any linear operator such that if $a \in \operatorname{Im} A$ then $AA^+a = a$.*

The pseudoinverse is not in general unique. However, our algorithms will be invariant to a specific choice of the pseudoinverse. If A is symmetric operator on a finite dimensional space, A^+ can be chosen to be also symmetric from the spectral decomposition

$$A = \sum_t \frac{1}{t} v_t v_t^T, \qquad Av_t = tv_t, \qquad v_t^T v_t = 1, \tag{2.1}$$

as

$$A^+ = \sum_{t \neq 0} \frac{1}{t} v_t v_t^T.$$

If A is positive semidefinite, denote

$$A^\alpha = \sum_{t>0} t^\alpha v_t v_t^T.$$

In particular, with this notation, $A = A^{1/2}A^{1/2}$, $A^+ = A^{-1/2}A^{-1/2}$, and Ker $A^\alpha =$ Ker A for any real α.

2.1. Problem Setting. Let Ω be a domain in $\mathbb{R}^2$ decomposed into N_s non-overlapping subdomains $\Omega_1,\ \Omega_2,\ \ldots,\ \Omega_{N_s}$. Let u_s be the vector of degrees of freedom for subdomain Ω_s corresponding to a conforming finite element discretization of a plate bending problem defined on Ω, such that each subdomain is a union of some of the elements. Let u_s, K_s, and f_s, be the vector of degrees of freedom, the local stiffness matrix, and the load vectors, respectively, associated with the subdomain Ω_s. We will use the block notation

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N_s} \end{bmatrix}, \quad f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N_s} \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & 0 & \ldots & 0 \\ 0 & K_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & K_{N_s} \end{bmatrix}.$$

The local stiffness matrix K_s is always positive semidefinite and it is singular for a *floating subdomain*, that is, one without sufficient essential boundary conditions to prevent the subdomain stiffness matrix K_s from being singular. Let Z_s be the matrix with linearly independent columns that generate the kernels of K_s, Im $Z_s =$ Ker K_s. If K_s is regular, Z_s is a void matrix. Denote

$$Z = \begin{bmatrix} Z_1 & 0 & \ldots & 0 \\ 0 & Z_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & Z_{N_s} \end{bmatrix};$$

so,

$$\text{Im } Z = \text{Ker } K, \qquad \text{Ker } Z = \{0\}.$$

Let $B = [B_1, B_2, \ldots, B_{N_s}]$ be a given matrix such that $Bu = 0$ expresses the condition that the values of the degrees of freedom associated with two or more subdomains coincide. Also, assume that B has nonzero entries only for rows that correspond to degrees of freedom on subdomain interfaces.

Denote by W the space of all vectors of degrees of freedom, and by Λ the space of the vectors of values of the continuity constraint; thus,

$$u = [u_1, u_2, \dots u_{N_s}]^T \in W$$
$$K : W \to W, \qquad B : W \to \Lambda.$$

The problem to be solved is the minimization of the strain energy of the system subject to intersubdomain continuity conditions

$$\mathcal{E}(u) = \frac{1}{2} u^T K u - f^T u \to \min \qquad \text{subject to } Bu = 0. \tag{2.2}$$

We assume that the global structure is not floating, that is, the solution of (2.2) is unique. From (2.2), this assumption is equivalent to the assumption that

$$\text{Ker } K \cap \text{Ker } B = \{0\}. \tag{2.3}$$

2.2. Algebraic formulation. Introducing Lagrange multipliers λ for the constraint $Bu = 0$, the problem (2.2) becomes

$$\begin{array}{rcrcl} Ku & + & B^T\lambda & = & f \\ Bu & & & = & 0 \end{array} \tag{2.4}$$

A solution u of the first equation in (2.4) exists if and only if $f - B^T\lambda \in \text{Im } K$. Hence,

$$u = K^+(f - B^T\lambda) + Z\alpha \quad \text{if} \quad f - B^T\lambda \perp \text{Ker } K, \tag{2.5}$$

where α is to be determined. Substituting u from (2.5) into the second equation of (2.4) yields

$$BK^+(f - B^T\lambda) + BZ\alpha = 0. \tag{2.6}$$

Introducing P, the ℓ^2 projection onto $(\text{Im } BZ)^\perp$, equation (2.6) multiplied by P together with (2.5) show that λ satisfies the system of equations

$$\begin{array}{r} P(F\lambda - d) = 0 \\ G^T\lambda = e, \end{array} \tag{2.7}$$

where

$$G = BZ, \quad F = BK^+B^T, \quad d = BK^+f, \quad P = I - G(G^TG)^{-1}G^T, \quad e = Z^Tf. \tag{2.8}$$

See Lemma 3.1 for justification that G^TG is nonsingular.

The original FETI method [13] is the method of preconditioned conjugate gradients for the equation, equivalent to (2.7),

$$PF\lambda = Pd, \tag{2.9}$$

where the initial approximation λ_0 is chosen so that $G^T\lambda_0 = e$, and all search directions are in Ker G^T. We will show later (Section 3.1) that the solution of (2.7) is unique up to an addition of vector from Ker B^T, so the conjugate gradients method for (2.9) actually runs in the factorspace modulo Ker B^T.

The physical interpretation of the Lagrange multipliers λ is *interface forces and moments*. From (2.5) and the definition of F in (2.8), the residual $P(F\lambda - d) =$

$-Bu$ has the interpretation of *jumps of the values of degrees of freedom* between subdomains. The condition $f - B^T\lambda \perp \text{Ker } K$ means that the action of the loads and intersubdomain forces and moments does not excite rigid body motions.

The generalized FETI method is obtained by forcing the iterates to satisfy also a weighted residual condition. That is, we require throughout the iterations that

$$C^T P(F\lambda - d) = 0, \qquad G^T\lambda = e, \tag{2.10}$$

where C is another given matrix. Search directions that preserve (2.10) form the space

$$V' = \{\lambda \in \Lambda | G^T\lambda = 0, C^T PF\lambda = 0\}. \tag{2.11}$$

The corresponding space of residuals is

$$V = \{\lambda \in \text{Im } B | G^T\lambda = 0, C^T\lambda = 0\}. \tag{2.12}$$

To complete the description of elements of the generalized FETI method, we introduce the following operators. Given $\bar\lambda \in \Lambda$, we define

$$\mathcal{P}(\bar\lambda, \bar d, \bar e) = \bar\lambda + G\alpha + C\beta, \tag{2.13}$$

where α, β solve the system

$$\begin{array}{lcrcr} G^T F(\bar\lambda + G\alpha + C\beta) & + & G^T G\mu & = & G^T \bar d \\ C^T F(\bar\lambda + G\alpha + C\beta) & + & C^T G\mu & = & C^T \bar d \\ G^T(\bar\lambda + G\alpha + C\beta) & & & = & \bar e. \end{array} \tag{2.14}$$

We define

$$Q\bar\lambda = \mathcal{P}(\bar\lambda, 0, 0). \tag{2.15}$$

The generalized FETI method is now the method conjugate gradients for operator $PF : V' \to V$, preconditioned by $QDQ^T : V \to V'$, where the $D : \Lambda \to \Lambda$ is a given operator symmetric on V. In Section 3.1 we will show that Q^T is a projection and $\text{Im } Q^T = (\text{Ker } Q)^\perp = V$, the application of Q^T on V can be omitted, and one obtains the following algorithm.

ALGORITHM 2.2 (Generalized FETI). *Given an initial* $\bar\lambda_0$, *compute the initial approximation*

$$\lambda_0 = \mathcal{P}(\bar\lambda_0, d, e) \tag{2.16}$$

from (2.14), and compute the initial residual

$$r_0 = P(F\lambda_0 - d).$$

Repeat for $k = 1, 2, \ldots$ *until convergence:*

$$\begin{aligned} z_{k-1} &= Dr_{k-1} \\ y_{k-1} &= Qz_{k-1} \\ \xi_k &= r_{k-1}^T y_{k-1} \\ p_k &= y_{k-1} + \frac{\xi_k}{\xi_{k-1}} p_{k-1} \qquad (p_1 = y_0) \\ \nu_k &= \frac{\xi_k}{p_k^T PF p_k} \\ \lambda_k &= \lambda_{k-1} + \nu_k p_k \\ r_k &= r_{k-1} - \nu_k PF p_k \end{aligned}$$

Of course, the original FETI algorithm is obtained when Q is replaced by the identity, and the initial approximation λ_0 is required to satisfy only $G^T\lambda_0 = e$.

2.3. The Dirichlet Preconditioner. Decompose the space of all degrees of freedom W into the space of degrees of freedom on subdomain interface and degrees of freedom internal to subdomains:

$$W = \bar{W} \times \dot{W}.$$

In the corresponding block notation,

$$B = [\bar{B}, 0], \quad \bar{B} : \bar{W} \to \Lambda,$$

since B has nonzero entries for interface degrees of freedom only. Also,

$$Z = \begin{bmatrix} \bar{Z} \\ \dot{Z} \end{bmatrix},$$

and we have

$$G = BZ = \bar{B}\bar{Z}, \qquad \text{Ker } B^T = \text{Ker } \bar{B}^T.$$

Let S be the Schur complement of K obtained by elimination of degrees of freedom internal to subdomains. Then

$$F = BK^+B^T = \bar{B}S^+\bar{B}^T \tag{2.17}$$

and Ker S = Im $\bar{Z}$. It is well known that evaluation of the matrix-vector product S^+u reduces to the solution of independent *Neumann problems* on all subdomains. Inspired by (2.17), we choose $D = \bar{B}S\bar{B}^T$, giving the preconditioner

$$QD = Q\bar{B}S\bar{B}^T. \tag{2.18}$$

This preconditioner is called the *Dirichlet preconditioner*, since evaluating the matrix-vector product Sr is equivalent to solving independent *Dirichlet problems* on all subdomains.

2.4. Method Selection for Plate Problems. The columns of C are chosen as vectors with a one at the position of the Lagrange multiplier that enforces the continuity of the transversal displacement at a crosspoint, and zeroes elsewhere. A crosspoint is defined as an interface node adjacent to at least three subdomains or to two subdomains and the complement of Ω.

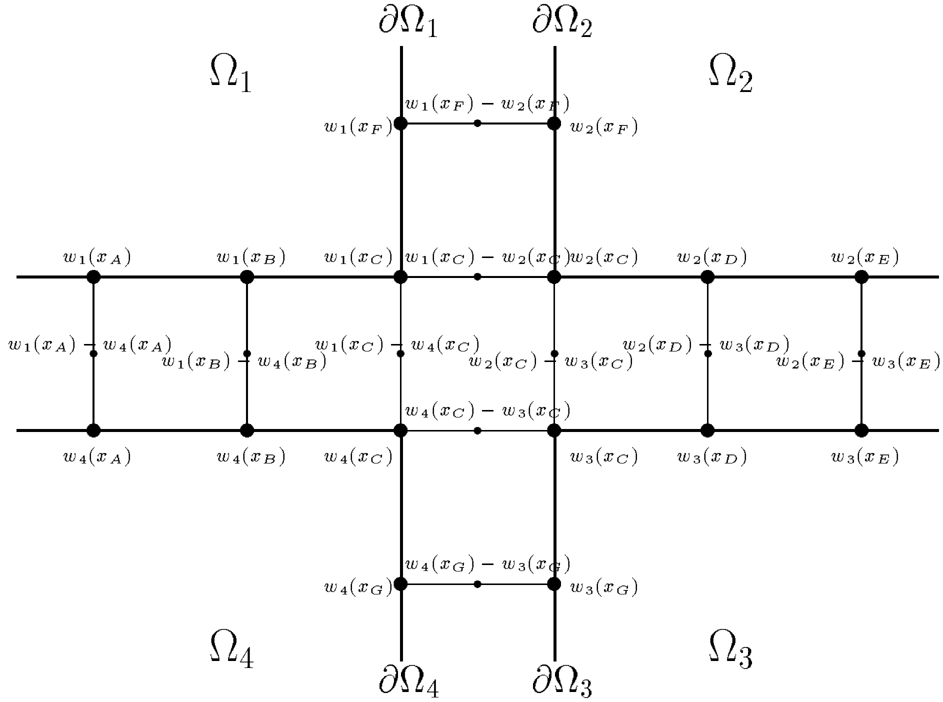
We define $\bar{B}$ as follows, cf., Fig. 2.1. For a node x_i on an edge $\partial\Omega_r \cap \partial\Omega_s$, let

$$\begin{aligned} (\bar{B}w)_{rs}(x_i) &= \sigma_{rs}(w_r(x_i) - w_s(x_i)), \\ (\nabla\bar{B}w)_{rs}(x_i) &= \sigma_{rs}(\nabla w_r(x_i) - \nabla w_s(x_i)), \end{aligned} \tag{2.19}$$

where $\sigma_{rs} = 1$ or $\sigma_{rs} = -1$. Here, $\nabla w_r(x_i)$ means the values of ∂_1 and ∂_2 degrees of freedom at node x_i. In particular, the entries of $\bar{B}$ are $-1, 0, +1$, and they are constant along an edge between two subdomains.

Remark 2.3. *In an earlier paper [9], we have studied the case of time-dependent problems where the subdomain stiffness matrices K_s are perturbed by the addition of a multiple of the subdomain mass matrix, thus making the new local matrix positive*

FIG. 2.1. *Definition of* $\bar{B}$



definite. Consequently, all matrices Z_s *are void and the natural coarse problem is lost in time-dependent applications. The methodology developed in [9] for reintroducing a coarsening operator in the FETI algorithm for dynamics problems is a special case of the present generalization where* C *is taken to be the matrix* G before the perturbation, *that is,* $C = [B_s \tilde{Z}_s]$ *where the columns of* $\tilde{Z}_s$ *are the basis of the kernel of the local stiffness matrix of the subdomain* Ω_s*. The selection of* C *in [8] and the reason why the preconditioner works are quite different from here.*

3. Analysis.

3.1. Algebraic results. In this section, we establish some important algebraic properties of the generalized FETI algorithm.

LEMMA 3.1. $(G^T G)^{-1}$ *exists.*

Proof. Let $G\alpha = BZ\alpha = 0$. Then $Z\alpha \in \text{Ker } B$, and $Z\alpha \in \text{Ker } K$ by the definition of Z. It follows from (2.3) that $Z\alpha = 0$, hence $\alpha = 0$, since Z was assumed to be of full rank. See also [15, Theorem 5.4]. ☐

THEOREM 3.2. *The solution* λ *of (2.7) is unique up to addition of a vector from* Ker B^T. *Any solution* λ *of (2.7) yields the same solution* u *of (2.2) by (2.5) with* $\alpha = -(G^T G)^{-1} G^T (d - F\lambda)$.

Proof. The relation between λ and α follows by a direct computation. To prove uniqueness of λ, it is sufficient to show that

$$\text{Ker } PF \cap \text{Ker } G^T = \text{Ker } F \cap \text{Ker } G^T = \text{Ker } B^T. \tag{3.1}$$

First, $B^T\lambda = 0$ implies $F\lambda = 0$ and $G^T\lambda = 0$, so

$$\text{Ker } B^T \subset \text{Ker } F \cap \text{Ker } G^T \subset \text{Ker } PF \cap \text{Ker } G^T.$$

Conversely, assume $G^T\lambda = 0$ and $PF\lambda = 0$. Then, since $G^T = Z^T B^T$, the first equation implies $B^T\lambda \perp \text{Im } Z = \text{Ker } K$. Thus $B^T\lambda \in \text{Im } K$. From the definition of P, $G^T\lambda = 0$ is equivalent to $P\lambda = \lambda$. From $PF\lambda = 0$, we obtain

$$0 = \lambda^T PF\lambda = \lambda^T F\lambda = (B^T\lambda)^T K^+ B^T\lambda.$$

Since K^+ is positive semidefinite, this implies that $B^T\lambda \in \text{Ker } K^+ = \text{Ker } K$. Together, $B^T\lambda \in \text{Ker } K \cap \text{Im } K = \{0\}$. □

Non-uniqueness of the multipliers λ corresponds to redundant intersubdomain continuity constraints, which will occur naturally at crosspoints of more than two subdomains.

Denote the space of the Lagrange multipliers as the factorspace

$$\tilde{\Lambda} = \Lambda / \text{Ker } B^T.$$

The operators P, F, and G^T induce operators on $\tilde{\Lambda}$, which will be denoted by the same symbols. To avoid confusion, all nullspaces will refer to the space Λ, not the factorspace $\tilde{\Lambda}$. For example, the nullspace of the induced operator G^T on $\tilde{\Lambda}$ will be denoted by $\text{Ker } G^T / \text{Ker } B^T$. It is easy to see that F is symmetric and positive semidefinite. The next lemma shows that the associated quadratic form is in fact positive definite on $\text{Ker } G^T / \text{Ker } B^T$.

LEMMA 3.3. *The operator PF is symmetric and positive definite on $\text{Ker } G^T / \text{Ker } B^T$.*

Proof. For $u, v \in \text{Ker } G^T$, we have from the definition of P,

$$\langle PFPu, v\rangle = \langle PFu, v\rangle = \langle Fu, v\rangle,$$

which proves that PF is symmetric positive semidefinite on $\text{Ker } G^T$. From (3.1), it follows that PF is nonsingular on $\text{Ker } G^T / \text{Ker } B^T$. □

Denote the associated subspaces of the factorspace $\tilde{\Lambda}$ as

$$\tilde{V}' = \{u + \text{Ker } B^T | u \in V'\} = V' / \text{Ker } B^T, \qquad \tilde{V} = \{u + \text{Ker } B^T | u \in V\}.$$

Note that V and $\tilde{V}$ are isomorphic: since $V \subset \text{Im } B$ and $\text{Im } B \cap \text{Ker } B^T = \{0\}$, each class of $\tilde{V}$ contains exactly one element of V.

We will need several properties of the spaces $\tilde{V}$ and $\tilde{V}'$.

LEMMA 3.4. $\tilde{V} = PF\tilde{V}'$

Proof. From the definition, clearly $PFV' \subset V$, hence $PF\tilde{V}' \subset \tilde{V}$. To show that $\tilde{V} \subset PF\tilde{V}'$, let $u + \text{Ker } B^T \in \tilde{V}$. Since PF is a bijection on $\text{Ker } G^T / \text{Ker } B^T$ by Lemma 3.3, there is $\tilde{\lambda} \in \text{Ker } G^T / \text{Ker } B^T$ such that $PF\tilde{\lambda} = u + \text{Ker } B^T$. It follows that $\tilde{\lambda} \in V'$. □

LEMMA 3.5. *The space $\tilde{V}'$ is the dual of $\tilde{V}$ with the duality pairing $\langle\cdot,\cdot\rangle$.*

Proof. Any $\lambda \in \tilde{V}'$ defines a linear functional λ' on $\tilde{V}$ by $\lambda'(v) = \langle\lambda, v\rangle$. Let λ' be an arbitrary linear functional on $\tilde{V}$. From Lemma 3.4, it follows that $\lambda' PF$ is a linear functional on $\tilde{V}'$, and, from Riesz representation theorem, there is a unique $\lambda \in \tilde{V}'$ such that $\lambda'(PFv) = \langle\lambda, v\rangle$ for all $v \in \tilde{V}'$. From Lemma 3.3, PF is a bijection on $\tilde{V}$, and it follows that the mapping between a linear functional λ' and its representation $\lambda \in \tilde{V}'$ is an isomorphism. □

LEMMA 3.6. $\tilde{\Lambda} = \tilde{V}' \oplus \tilde{V}^{\perp}$.

Proof. From Lemma 3.3, PF is symmetric positive definite on Ker G^T/ Ker B^T. Hence, $(PF)^{-1}$ is also symmetric, positive definite on Ker G^T/ Ker B^T, and, using Lemma 3.4, it follows that

$$\tilde{\Lambda} = \tilde{V}' \oplus (\tilde{V}')^{\perp_{(PF)^{-1}}} = \tilde{V}' \oplus ((PF)\tilde{V})^{\perp_{(PF)^{-1}}} = \tilde{V}' \oplus \tilde{V}^{\perp}, \tag{3.2}$$

which was to be proved. □

We show that the operator Q defined by (2.15) is a projection with some special properties.

LEMMA 3.7. *$Q : \bar{\lambda} \to \mathcal{P}(\bar{\lambda}, 0, 0)$ is a projection that satisfies*

$$Q : \tilde{\Lambda} \to \tilde{\Lambda}, \quad Q^2 = Q, \quad \text{Im } Q = \tilde{V}', \quad \text{Ker } Q = \tilde{V}^{\perp}. \tag{3.3}$$

Furthermore, Q can be written as

$$Q = I - \begin{bmatrix} G & C & 0 \end{bmatrix} M^{+} \begin{bmatrix} G^T F \\ C^T F \\ G^T \end{bmatrix},$$

where

$$M = \begin{bmatrix} G^T FG & G^T FC & G^T G \\ C^T FG & C^T FC & C^T G \\ G^T G & G^T C & 0 \end{bmatrix} = \begin{bmatrix} G^T & 0 \\ C^T & 0 \\ 0 & G^T \end{bmatrix} \begin{bmatrix} F & I \\ I & 0 \end{bmatrix} \begin{bmatrix} G & C & 0 \\ 0 & 0 & G \end{bmatrix}. \tag{3.4}$$

Proof. It is easy to see from the definition of Q that $\lambda = Q\bar{\lambda}$ satisfies the system of equations

$$\begin{array}{lcrcl} G^T F\lambda & + & G^T G\mu & = & 0, \\ C^T F\lambda & + & C^T G\mu & = & 0, \\ G^T \lambda & & & = & 0. \end{array} \tag{3.5}$$

The first equation implies $PF\lambda = F\lambda + G\mu$. Hence, from the other two equations, $G^T\lambda = 0$ and $C^T PF\lambda = 0$. Therefore, $\lambda \in V'$. Also $\lambda - \bar{\lambda} = G\alpha + C\beta$, so it holds that $\lambda - \bar{\lambda} \perp V$.

On the other hand, let $\lambda \in V'$, from the definition of P, there exists μ such that $PF\lambda = F\lambda + G\mu$. From the definition of V', $G^T\lambda = 0$ and $C^T PF\lambda = 0$, hence we obtain (3.5). Let $\lambda - \bar{\lambda} \perp V$, so for any $u \in \text{Im } B$, $G^T u = 0$ and $C^T u = 0$ implies $\langle \lambda - \bar{\lambda}, u\rangle = 0$. Consequently, there exist α and β such that for all $u \in \text{Im } B$,

$$\langle \lambda - \bar{\lambda}, u\rangle = \langle G\alpha, u\rangle + \langle C\beta, u\rangle,$$

which implies that

$$\lambda = \bar{\lambda} + G\alpha + C\beta + \gamma, \qquad \gamma \in (\text{Im } B)^{\perp} = \text{Ker } B^T \tag{3.6}$$

Substituting (3.6) into (3.5), the definition of Q (2.15) shows that $\lambda \in Q(\bar{\lambda} + \text{Ker } B^T)$.

Finally, the equation for Q is verified by a direct computation. □

Similarly, we verify that the initial approximation satisfies (2.10):

LEMMA 3.8. *$\lambda_0 = \mathcal{P}(\bar{\lambda}_0, d, e)$ is defined for any $\bar{\lambda}_0 \in \Lambda$ and satisfies*

$$C^T P(F\lambda_0 - d) = 0, \qquad G^T\lambda_0 = e. \tag{3.7}$$

Proof. As in the preceding proof, λ_0 satisfies (3.7) if and only if there is a μ that satisfies

$$\begin{array}{rcccl} G^T F\lambda_0 & + & G^T G\mu & = & G^T d \\ C^T F\lambda_0 & + & C^T G\mu & = & C^T d \\ G^T \lambda_0 & & & = & e. \end{array} \tag{3.8}$$

Thus, from the definition of $\mathcal{P}$, $\lambda_0 = \bar{\lambda}_0 + G\alpha + C\beta$ satisfies (3.7). The system above can also be written as

$$M \begin{bmatrix} \alpha \\ \beta \\ \mu \end{bmatrix} = X^T \begin{bmatrix} d - F\bar{\lambda}_0 \\ G(G^T G)^{-1} e - \bar{\lambda}_0 \end{bmatrix}, \qquad X = \begin{bmatrix} G & C & 0 \\ 0 & 0 & G \end{bmatrix}.$$

From the factorization (3.4), it follows that Ker M = Ker X. Using symmetry of M, we have Im $M = (\text{Ker } M)^\perp = (\text{Ker } X)^\perp = \text{Im } X^T$; hence, (3.8) has a solution. ☐

For the purpose of analysis, we equip the space $\tilde{V}$ with the norm

$$\|v\|_{\tilde{V}} = \|\bar{B}^T v\|_S = \langle S\bar{B}^T v, \bar{B}^T v\rangle^{1/2}. \tag{3.9}$$

Since $\bar{B}^T v \perp$ Ker S for $v \in V$, (3.9) indeed defines a norm rather than only seminorm. The dual space $\tilde{V}'$ is equipped with the dual norm

$$\|\lambda\|_{\tilde{V}'} = \sup_{v \in \tilde{V}} \frac{\langle \lambda, v\rangle}{\|v\|_{\tilde{V}}}. \tag{3.10}$$

That (3.10) defines a norm rather than seminorm follows from Lemma 3.5. From the definition of $\tilde{V}'$ follows immediately the property

$$\|\lambda\|_{\tilde{V}'} = \sup_{v \in \tilde{V}} \frac{\langle \lambda, v\rangle}{\|\bar{B}^T v\|_S} = \sup_{w \in \bar{W},\ \bar{B}w \in \tilde{V}} \frac{\langle \lambda, \bar{B}w\rangle}{\|\bar{B}^T \bar{B}w\|_S}. \tag{3.11}$$

The norm on $\tilde{V}$ was chosen so that the preconditioner QD is trivially coercive and bounded:

LEMMA 3.9. *For all* $v \in \tilde{V}$, $\langle v, QDv\rangle = \|v\|^2_{\tilde{V}}$.

Proof. Let $v \in \tilde{V} = \text{Im } Q^T$. Since Q^T is a projection, we have by definition of the preconditioner D,

$$\langle v, QDv\rangle = \langle Q^T v, \bar{B}S\bar{B}^T v\rangle = \langle v, \bar{B}S\bar{B}^T v\rangle = \langle \bar{B}^T v, S\bar{B}^T v\rangle,$$

which was to be shown. ☐

Coercivity and boundedness of the system operator PF on $\tilde{V}'$ will be estimated using the following lemma.

LEMMA 3.10. *For all* $\lambda \in \tilde{V}'$, $\langle \lambda, F\lambda\rangle = \sup_{w \in \bar{W}, w \perp \text{Ker } S} \frac{\langle \lambda, \bar{B}w\rangle^2}{\|w\|_S^2} = \sup_{w \in \bar{W}} \frac{\langle \lambda, \bar{B}w\rangle^2}{\|w\|_S^2}$.

Proof. Let $\lambda \in \tilde{V}'$. Then

$$\begin{aligned} \langle \lambda, F\lambda\rangle &= \langle S^+ \bar{B}^T\lambda, \bar{B}^T\lambda\rangle = \langle S^{-1/2}\bar{B}^T\lambda, S^{-1/2}\bar{B}^T\lambda\rangle \\ &= \|S^{-1/2}\bar{B}^T\lambda\|^2 = \sup_{x \in \bar{W}} \frac{\langle S^{-1/2}\bar{B}^T\lambda, x\rangle^2}{\|x\|^2} = \sup_{\substack{x \in \bar{W},\ x = x_1 + x_2 \\ x_1 \in \text{Ker } S\ x_2 \perp \text{Ker } S}} \frac{\langle \bar{B}^T\lambda, S^{-1/2}x\rangle^2}{\|x_1 + x_2\|^2} \\ &= \sup_{x_2 \in \bar{W},\ x_2 \perp \text{Ker } S} \frac{\langle \bar{B}^T\lambda, S^{-1/2}x_2\rangle^2}{\|x_2\|^2} \end{aligned}$$

since $S^{-1/2}x_1 = 0$ and $||x||^2 = ||x_1||^2 + ||x_2||^2$. Now write any $w \in \bar{W}$ as

$$w = w_1 + w_2, \qquad w_1 \in \text{Ker } S, \qquad w_2 = S^{-1/2}x_2 \perp \text{Ker } S.$$

From the definition of $\tilde{V}'$, $\lambda \in \tilde{V}'$ implies that $\langle \bar{B}^T\lambda, w_1\rangle = 0$, hence $\langle \bar{B}^T\lambda, w_2\rangle = \langle \bar{B}^T\lambda, w\rangle = \langle \lambda, \bar{B}w\rangle$. It follows that

$$\langle \lambda, F\lambda\rangle = \sup_{w_2 \in \bar{W},\ w_2 \perp \text{Ker } S} \frac{\langle \bar{B}^T\lambda, w_2\rangle^2}{\langle w_2, Sw_2\rangle} = \sup_{w\in\bar{W}} \frac{\langle \lambda, \bar{B}w\rangle^2}{||w||_S^2},$$

which was to be proved. □

The following simple lemma will be the basis of our estimates. It will allow to reduce estimates of norms to estimates of boundedness and coercivity. The proof follows a standard argument and it is presented for completeness only.

LEMMA 3.11. *Let X be a Banach space, X' the dual of X, and $A : X \to X'$ a linear operator such that*

$$\langle y, Ax\rangle = \langle x, Ay\rangle, \qquad \forall x, y \in X \tag{3.12}$$

$$c_A||x||_X^2 \le \langle x, Ax\rangle \le C_A||x||_X^2, \qquad \forall x \in X \tag{3.13}$$

with constants $C_A, c_A > 0$. Then

$$||A||_{X\to X'} \le C_A, \qquad ||A^{-1}||_{X'\to X} \le \frac{1}{c_A}.$$

Proof. From (3.12),

$$||A||_{X\to X'} = \sup_{x\in X} \frac{||Ax||_{X'}}{||x||_X} = \sup_{x,\tilde{x}\in X} \frac{\langle Ax, \tilde{x}\rangle}{||x||_X||\tilde{x}||_X} = \sup_{x\in X} \frac{\langle Ax, x\rangle}{||x||_X^2} \le C_A.$$

From (3.13),

$$\frac{1}{||A^{-1}||_{X'\to X}} = \inf_{x\in X} \frac{||Ax||_{X'}}{||x||_X} = \inf_{x\in X} \sup_{\tilde{x}\in X} \frac{\langle Ax, \tilde{x}\rangle}{||x||_X||\tilde{x}||_X} \ge \inf_{x\in X} \frac{\langle Ax, x\rangle}{||x||_X^2} \ge c_A,$$

concluding the proof. □

It is well known [18] that after k iterations of the preconditioned conjugate gradient method, the energy norm or the error $|||e||| = \langle PFe, e\rangle^{1/2}$ is reduced by a factor of at least $2((\sqrt{\kappa}-1)/(\sqrt{\kappa}+1))^k$, where κ is the condition number. The condition number is given in our case by

$$\kappa = \kappa(QDPF) = \frac{\lambda_{\max}(QDPF)}{\lambda_{\min}(QDPF)}, \tag{3.14}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues. We are now ready to prove an abstract bound on κ.

THEOREM 3.12. *Assume there exist constants C_1, C_2 such that*
(i) for any $\lambda \in V'$ and $w \in \bar{W}$ such that $\bar{B}w \in V$, there is $\tilde{w} \in \bar{W}$ such that

$$\langle \lambda, \bar{B}\tilde{w}\rangle = \langle \lambda, \bar{B}w\rangle, \quad \text{and} \quad ||\tilde{w}||_S^2 \le C_1||\bar{B}^T\bar{B}w||_S^2; \tag{3.15}$$

(ii) for any $\lambda \in V'$ and $w \in \bar{W}$, $w \perp$ Ker S, there is $\tilde{w} \in \bar{W}$ such that $\bar{B}\tilde{w} \in V$,

$$\langle \lambda, \bar{B}\tilde{w}\rangle = \langle \lambda, \bar{B}w\rangle, \quad \text{and} \quad ||\bar{B}^T\bar{B}\tilde{w}||_S^2 \le C_2||w||_S^2. \tag{3.16}$$

Then $\kappa(QDPF) \le C_1 C_2$.

Proof. Lemma 3.11 applied to the operator QD together with Lemma 3.9 give

$$\|QD\|^2_{\tilde V \to \tilde V'} \le 1, \qquad \|(QD)^{-1}\|^2_{\tilde V' \to \tilde V} \le 1. \tag{3.17}$$

(3.15) and (3.16) imply the same properties with V and V' replaced by the factorspaces $\tilde V$ and $\tilde V'$. From assumption (i), we have, for any $\lambda \in V'$,

$$\sup_{w \in \bar W} \frac{\langle \lambda, \bar B w\rangle^2}{\|w\|_S^2} \ge \frac{1}{C_1} \sup_{w \in \bar W, \bar B w \in V} \frac{\langle \lambda, \bar B w\rangle^2}{\|\bar B^T \bar B w\|_S^2} \tag{3.18}$$

while assumption (ii) gives the converse bound

$$\sup_{w \in \bar W, w \perp \mathrm{Ker}\ S} \frac{\langle \lambda, \bar B w\rangle^2}{\|w\|_S^2} \le C_2 \sup_{w \in \bar W, \bar B w \in V} \frac{\langle \lambda, \bar B w\rangle^2}{\|\bar B^T \bar B w\|_S^2}. \tag{3.19}$$

Using (3.11) and Lemma 3.10, we see that the inequalities (3.18) and (3.19) imply the inequality

$$\frac{1}{C_1}\|\lambda\|^2_{\tilde V'} \le \langle \lambda, PF\lambda\rangle \le C_2 \|\lambda\|^2_{\tilde V'}, \quad \forall \lambda \in \tilde V'.$$

By application of Lemma 3.11 to the operator PF, we obtain

$$\|PF\|^2_{\tilde V' \to \tilde V} \le C_2, \qquad \|(PF)^{-1}\|^2_{\tilde V \to \tilde V'} \le C_1. \tag{3.20}$$

From (3.17) and (3.20), we have

$$\|QDPF\|_{\tilde V' \to \tilde V'} \le \|QD\|_{\tilde V \to \tilde V'} \|PF\|_{\tilde V' \to \tilde V} \le C_2$$

and

$$\|(QDPF)^{-1}\|_{\tilde V' \to \tilde V'} \le \|(QD)^{-1}\|_{\tilde V' \to \tilde V} \|(PF)^{-1}\|_{\tilde V \to \tilde V'} \le C_1.$$

The result follows. □

3.2. Convergence Estimates for Plate Bending. The theory presented in this section requires that the plate bending finite element satisfy the following *approximate parametric variational principle* recently formulated in [20]:

ASSUMPTION 3.13 ([19, 20]). *We consider finite elements with displacements and rotations at the vertices only, and assume that there exist constants $c_1 > 0$, c_2 such that if the plate thickness t satisfies $0 < t \le h$, then for each element T, the local stiffness matrix K_T satisfies*

$$c_1 K_T^{HCT} \le K_T \le c_2 K_T^{HCT} \tag{3.21}$$

where K_T^{HCT} is the HCT element level stiffness matrix of the biharmonic equation [6], with the rotations interpreted as derivatives of the transversal displacement in the HCT element.

That is, as the thickness of the plate goes to zero, the stiffness matrix of the element should be spectrally equivalent to that of the HCT element for the biharmonic equation. The HCT element is a C^1 element that uses cubic splines for values on element sides, linear interpolation for normal derivatives on the sides, and piecewise polynomial extension into the element interior [5]. Assumption 3.13 is proved in [19] for the particular case of the DKT element [2]. Assumption 3.13 also holds for the following general class of non-locking $P1$ Reissner-Mindlin elements.

THEOREM 3.14 ([19]). *Assume that the energy functional for an element T is spectrally equivalent to*

$$\int_T |\nabla\theta|^2\,dx + \frac{1}{t^2+h^2}\int_T |\theta - \nabla u|^2\,dx \tag{3.22}$$

with $u \in P_1(T), \theta \in (P_1(T))^2$, $h = \mathrm{diam}(T)$, u the transversal displacement, and θ the rotation. Then (3.21) holds.

Elements with the energy functional of the form (3.22) include the DKT element as restated in [26]. It should be noted that for the related Timoshenko beam element, the thin limit is exactly the discretization by cubic splines of the biharmonic equation [16].

Let us consider the biharmonic boundary value problem in a variational form,

$$u \in \mathcal{V}: \quad a(u,v) = f(v), \quad \forall v \in \mathcal{V},$$

where

$$a(u,v) = \int_\Omega \partial_{11}u\partial_{11}v + \partial_{12}u\partial_{12}v + \partial_{22}u\partial_{22}v, \qquad \mathcal{V} = H_0^2(\Omega).$$

The subdomains Ω_s, $s = 1, ..., N_s$, are assumed to form a regular triangulation, i.e. they are generated from a reference domain (square) $\hat{\Omega}$ of unit diameter by mappings F_s, so that $\Omega_s = F_s(\hat{\Omega}_s)$. These mappings are assumed to satisfy

$$\|\partial F_s\| \le CH, \qquad \|\partial F_s^{-1}\| \le CH^{-1} \tag{3.23}$$

where ∂F_s is the mapping Jacobian and $\|.\|$ is the Euclidean $\mathbb{R}^2$ matrix norm. In other words, the subdomains are assumed to have a regular shape and diameter $O(H)$. Without loss of generality, we assume $H < 1$.

Furthermore, we assume that the problem is discretized using reduced HCT elements. The general case of plate bending then follows from spectral equivalence of the local element stiffness matrices following Assumption 3.13. Let $V_h(\Omega) \subset \mathcal{V}$ denote the corresponding finite element space, and h denote the characteristic element size. Each subdomain Ω_s is assumed to be a union of some of the elements. The degrees of freedom are values of the transversal displacement and its derivatives (rotations) at the nodal points of the discretization. If w denotes a vector of degrees of freedom, we denote $I_{HCT}w$ the corresponding finite element function (cf. [20]). It is well known [20] that

$$\|w\|_S^2 \le c_2 \sum_{s=1}^{N_s} |\nabla I_{HCT} w|^2_{H^{1/2}(\partial\Omega_s)} \quad \forall w \in W, \tag{3.24}$$

$$c_1 \sum_{s=1}^{N_s} |\nabla I_{HCT} w|^2_{H^{1/2}(\partial\Omega_s)} \le \|w\|_S^2 \le c_2 \sum_{s=1}^{N_s} |\nabla I_{HCT} w|^2_{H^{1/2}(\partial\Omega_s)} \quad \forall w \perp \mathrm{Ker}\ S, \tag{3.25}$$

where the positive constants c_1, c_2 are independent of the characteristic mesh size h and the subdomain diameter H. The constants c_1, c_2 may depend on the regularity of the shape of subdomains. For definitions of Sobolev spaces, see, for example [1].

Throughout this section, $c, C, c_1, c_2, c_3, c_4, c_5$ and c_6 denote positive constants independent of H and h. We now summarize some well-known results and inequalities in a form suitable for our purposes.

LEMMA 3.15 ([20]). *Let $x \in \partial\Omega_s$ be a vertex of a subdomain Ω_s. For any $u \in V_h(\Omega_s)$ such that $u(x) = 0$, define $z \in V_h(\Omega_s)$ by $z(x) = u(x) = 0$, $\nabla z(x) = \nabla u(x)$, and $z(y) = 0$, $\nabla z(y) = 0$ at all other nodes y of $\partial\Omega_s$. Then*

$$\|\nabla z\|^2_{H^{1/2}(\partial\Omega_s)} \leq C\left(1 + \log\frac{H}{h}\right)\left(\|\nabla u\|^2_{H^{1/2}(\partial\Omega_s)} + \frac{1}{H}\|\nabla u\|^2_{L^2(\partial\Omega_s)}\right).$$

The following estimate of the trace norm of the extension by zero is proved as in [3, Lemma 3.5].

LEMMA 3.16. *There exists a constant C such that if the support of $u \in V_h(\partial\Omega_s)$ is contained in a segment σ of $\partial\Omega_s$ of length τ, then*

$$|u|^2_{H^{1/2}(\partial\Omega_s)} \leq |u|^2_{H^{1/2}(\sigma)} + C\left(1 + \log\frac{\tau}{h}\right)\|u\|^2_{L^\infty(\sigma)}.$$

The following estimate is a modification of the previous lemma.

LEMMA 3.17. *Let the support of $u \in V_h(\partial\Omega_s)$ be contained in a segment of $\partial\Omega_s$ and x_1 and y_1 be its endpoints. Let x_2 and y_2 be the nodal points next to x_1 and y_1 respectively in the segment. Let (x_2, y_2) denote the segment between x_2 and y_2. Let the length of this segment be $\tau \geq h$. Assume that the function u satisfies the condition $\|u\|_{L^\infty((x_1,y_1))} \leq c\|u\|_{L^\infty((x_2,y_2))}$. Then*

$$|u|^2_{H^{1/2}(\partial\Omega_s)} \leq |u|^2_{H^{1/2}((x_2,y_2))} + C\left(1 + \log\frac{\tau}{h}\right)\|u\|^2_{L^\infty((x_2,y_2))}.$$

Proof. By the previous lemma, the inequality holds with (x_2, y_2) replaced by the segment (x_1, y_1). By the definition of the $H^{1/2}$ seminorm, we have

$$|u|^2_{H^{1/2}((x_1,y_1))} = |u|^2_{H^{1/2}((x_2,y_2))} + \int_{x1}^{x2}\int_{x1}^{y2}\frac{|u(x) - u(y)|^2}{\|x - y\|^2}dxdy + \int_{y1}^{y2}\int_{x1}^{y2}\frac{|u(x) - u(y)|^2}{\|x - y\|^2}dxdy.$$

Using the fact that $|u(x) - u(y)| \leq \min\{\frac{c_1}{h}c\|u\|_{L^\infty((x_2,y_2))}\|x - y\|, 2c\|u\|_{L^\infty((x_2,y_2))}\}$, the last two terms are bounded by $c_2\|u\|_{L^\infty((x_2,y_2))}$. ☐

We will also need a straightforward extension of the discrete Sobolev inequality of Dryja [7] to piecewise polynomial functions of order $p > 1$ [25].

LEMMA 3.18. *Let $p \geq 1$. Then there exists a constant $C = C(p)$ such that for every u continuous on $\sigma \subset \partial\Omega_s$ such that $u \in P_p$ on the side of every element T,*

$$\|u\|^2_{L^\infty(\sigma)} \leq C\left(1 + \log\frac{H}{h}\right)\left(|u|^2_{H^{1/2}(\sigma)} + \frac{1}{H}\|u\|^2_{L^2(\sigma)}\right).$$

The following lemma establishes a useful inequality between discrete and Sobolev norms.

LEMMA 3.19. *Let $u \in V_h(\partial\Omega_i)$, U be the corresponding vector of degrees of freedom, and $u(x_0) = 0$ for some $x_0 \in \partial\Omega_i$. Then,*

$$h\|U\|^2 \leq c(1 + H^2)\|\nabla u\|^2_{L^2(\partial\Omega_i)}$$

Proof. Let E be an element edge on $\partial\Omega_i$ and x_1 and x_2 be its endpoints. Each component of ∇u is a polynomial of degree at most two on E. Since all norms in a finite dimensional space are equivalent, it holds that

$$\begin{aligned} h(\|\nabla u(x_1)\|^2 + \|\nabla u(x_2)\|^2) &\le c_1 \min_{f \in P_2 \times P_2, f(x_1)=\nabla u(x_1), f(x_2)=\nabla u(x_2)} \|f\|^2_{L^2(E)} \\ &\le c_1 \|\nabla u\|^2_{L^2(E)}. \end{aligned}$$

Summing over all edges of the boundary, we obtain

$$h\|U\|^2 \le \frac{c_1}{2}\|\nabla u\|^2_{L^2(\partial\Omega_i)} + h\|U_0\|^2,$$

where U_0 denotes the displacement degrees of freedom. We show that $h\|U_0\|^2 < c_2 H^2 \|\nabla u\|^2_{L^2(\partial\Omega_i)}$. Since $u(x_0) = 0$, we can write for any meshpoint x on the boundary

$$u(x) = \int_{(x_0,x)} \nabla u(y) \cdot \tau(y) dy,$$

where $(x_0, x) \subset \partial\Omega_i$ is the part of the boundary between x_0 and x, and τ is the tangential vector to the boundary. Squaring, using the Cauchy inequality, and considering that the length of (x_0, x) is bounded by $c_2 H$, we obtain

$$u(x)^2 \le c_2 H \|\nabla u\|^2_{L^2(\partial\Omega_i)}.$$

The result follows by summing over all meshpoints on the boundary. □

The next two lemmas contain the principal technical estimates.

LEMMA 3.20. *For all $\lambda \in V'$ and all $w \in \bar{W}$ such that $\bar{B}w \in V$, there exists a $\tilde{w} \in \bar{W}$ such that*

$$\langle \lambda, \bar{B}\tilde{w}\rangle = \langle \lambda, \bar{B}w\rangle \quad \text{and} \quad \|\tilde{w}\|^2_S \le C(1 + \log H/h)^\alpha \|\bar{B}^T \bar{B} w\|^2_S.$$

where $\alpha = 1$, and $\alpha = 0$ if $\frac{1}{2}\bar{B}\bar{B}^T = I$, which happens when there are no nodes shared by more than two subdomains.

Proof. Let us first prove that in the general case we obtain $\alpha \le 1$. Let $w \in \bar{W}$ and $\bar{B}w \in V$. That is $\bar{Z}^T \bar{B}^T \bar{B} w = 0$ and $C^T \bar{B} w = 0$. We define $\tilde{w} = \bar{B}^T(\bar{B}\bar{B}^T)^+ \bar{B} w$. Then, $\bar{B}\tilde{w} = \bar{B}w$. By the triangle inequality, we may write

$$\|\tilde{w}\|_S \le \|\tfrac{1}{2}\bar{B}^T \bar{B} w\|_S + \|\tfrac{1}{2}\bar{B}^T(I - (\tfrac{1}{2}\bar{B}\bar{B}^T)^+)\bar{B}w\|_S. \tag{3.26}$$

Denote $z = \frac{1}{2}\bar{B}^T(I - (\frac{1}{2}\bar{B}\bar{B}^T)^+)\bar{B}w$. From the definition of $\bar{B}$ in (2.19), z is zero at all nodes that belong to at most two subdomains. The remaining nodes lie on subdomain crosspoints. At every such node, $z_i(x)$ is a linear combination of the entries of $\bar{B}^T\bar{B}w$ that correspond to the same node x and the coefficients of the linear combinations are bounded only in terms of the number of subdomains to which the node belongs. In addition, since $C^T \bar{B} w = 0$, the transversal displacement components of z at crosspoints are zero. Using (3.24) and Lemma 3.15 for the subdomain crosspoint vertices, one subdomain at a time, we obtain

$$\|z\|^2_S \le C(1 + \log\frac{H}{h}) \sum_{i=1}^{N_s} \left(\frac{1}{H}\|\nabla I_{HCT} \bar{B}^T \bar{B} w\|^2_{L^2(\partial\Omega_i)} + |\nabla I_{HCT}\bar{B}^T\bar{B}w|^2_{H^{1/2}(\partial\Omega_i)} \right)$$

This together with the Poincaré inequality and (3.26) yields the result. Finally, if $\frac{1}{2}\bar{B}\bar{B}^T = I$, we simply choose $\tilde{w} = \frac{1}{2}\bar{B}^T\bar{B}w = w$. ☐

Here is is the converse bound:

LEMMA 3.21. *For all* $\lambda \in V'$ *and* $w \in \bar{W}, w \perp \text{Ker } S$*, there is a* $\tilde{w} \in \bar{W}$ *such that* $\bar{B}\tilde{w} \in V$,

$$\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle, \quad \textit{and} \quad ||\bar{B}^T\bar{B}\tilde{w}||_S^2 \leq C\left(1 + \log\frac{H}{h}\right)^2 ||w||_S^2.$$

Proof. Let $\lambda \in V'$, $w \in \bar{W}, w \perp \text{Ker } S$, and consider $\bar{B}\tilde{w} = P\bar{B}w + PFPC\alpha$. Since $\lambda \in V'$, it is easily verified that $\langle \lambda, \bar{B}w \rangle = \langle \lambda, \bar{B}\tilde{w} \rangle$. α can be found from the condition $C^T\bar{B}\tilde{w} = 0$:

$$\langle PFPC\alpha, C\tilde{\alpha} \rangle = -\langle P\bar{B}w, C\tilde{\alpha} \rangle \quad \forall\tilde{\alpha}.$$

By definition of F, we can rewrite this as

$$\langle S^+\bar{B}^TPC\alpha, \bar{B}^TPC\tilde{\alpha} \rangle = -\langle w, \bar{B}^TPC\tilde{\alpha} \rangle \quad \forall\tilde{\alpha}.$$

Since S^+ is positive semidefinite, the equation yields α such that

$$\langle {S^+}^{1/2}\bar{B}^TPC\alpha, {S^+}^{1/2}\bar{B}^TPC\alpha \rangle = -\langle S^{1/2}w, {S^+}^{1/2}\bar{B}^TPC\alpha \rangle$$

and from the Cauchy inequality, $||{S^+}^{1/2}\bar{B}^TPC\alpha|| \leq ||S^{1/2}w||$. Therefore,

$$||S^+\bar{B}^TPC\alpha||_S \leq ||w||_S. \tag{3.27}$$

We need to estimate $||\bar{B}^T\bar{B}\tilde{w}||_S$. Let $\bar{B}_{ij}$ be the matrix constructed from $\bar{B}$ by zeroing out all the rows that do not correspond to the interface conditions between $\partial\Omega_i$ and $\partial\Omega_j$. Then,

$$\bar{B}^T\bar{B}\tilde{w} = \sum_{i,j=1,i<j}^{N_s} \bar{B}_{ij}^T\bar{B}\tilde{w},$$

and by triangle inequality and (3.24)

$$||\bar{B}^T\bar{B}\tilde{w}||_S \leq \sum_{i,j=1,i<j}^{N_s} \left(|\nabla I_{HCT}\bar{B}_{ij}^T\bar{B}\tilde{w}|_{H^{1/2}(\partial\Omega_i)} + |\nabla I_{HCT}\bar{B}_{ij}^T\bar{B}\tilde{w}|_{H^{1/2}(\partial\Omega_j)}\right)$$

Since $C^T\bar{B}\tilde{w} = 0$, Lemma 3.17 can be applied to $\nabla I_{HCT}\bar{B}_{ij}^T\bar{B}\tilde{w}, i, j = 1, \ldots, N_s$. It follows that

$$||\bar{B}^T\bar{B}\tilde{w}||_S^2 \leq c_1 \sum_k \left(|\nabla I_{HCT}\bar{B}\tilde{w}|_{H^{1/2}(\Gamma_k)}^2 + \left(1 + \log\frac{H}{h}\right) ||\nabla I_{HCT}\bar{B}\tilde{w}||_{L^\infty(\Gamma_k)}^2\right)$$

where summation is carried out over all edges Γ_k of the intersubdomain interface. The L^∞ norm, by Lemma 3.18, can be bounded as follows

$$||\nabla I_{HCT}\bar{B}\tilde{w}||_{L_\infty(\Gamma_k)}^2 \leq$$
$$c_2\left(1 + \log\frac{H}{h}\right)\left(\frac{1}{H}||\nabla I_{HCT}\bar{B}\tilde{w}||_{L^2(\Gamma_k)}^2 + |\nabla I_{HCT}\bar{B}\tilde{w}|_{H^{1/2}(\Gamma_k)}^2\right).$$

Thus,

$$\|\bar{B}^T\bar{B}\tilde{w}\|_S^2 \leq$$
$$c_1(c_2+1)\left(1+\log\frac{H}{h}\right)^2\sum_k\left(\frac{1}{H}\|\nabla I_{HCT}\bar{B}\tilde{w}\|^2_{L^2(\Gamma_k)}+|\nabla I_{HCT}\bar{B}\tilde{w}|^2_{H^{1/2}(\Gamma_k)}\right).$$

Denote $u = w + S^+\bar{B}^T PC\alpha$. Then, $\bar{B}\tilde{w} = P\bar{B}u$. The triangle inequality and (3.27) yield

$$\|u\|_S \leq \|w\|_S + \|S^+\bar{B}^T PC\alpha\|_S \leq 2\|w\|_S.$$

So we only have left to prove that

$$\sum_k\left(\frac{1}{H}\|\nabla I_{HCT}P\bar{B}u\|^2_{L^2(\Gamma_k)}+|\nabla I_{HCT}P\bar{B}u|^2_{H^{1/2}(\Gamma_k)}\right)\leq c_3\|u\|_S^2.$$

We first use the triangle inequality to get

$$|\nabla I_{HCT}P\bar{B}u|_{H^{1/2}(\Gamma_k)} \leq |\nabla I_{HCT}\bar{B}u|_{H^{1/2}(\Gamma_k)} + |\nabla I_{HCT}(I-P)\bar{B}u|_{H^{1/2}(\Gamma_k)}.$$

From (2.19), $\bar{B}$ reduces to a linear combination with constant coefficients ± 1 along every edge. Since $(I-P)\bar{B}u \in \operatorname{Im} G$, $I_{HCT}(I-P)\bar{B}u$ is a restriction of a linear function on every edge, hence $|\nabla I_{HCT}(I-P)\bar{B}u|_{H^{1/2}(\Gamma_k)} = 0$. Furthermore,

$$\sum_k |\nabla I_{HCT}Bu|^2_{H^{1/2}(\Gamma_k)} \leq 2\sum_{i=1}^{N_s}|\nabla I_{HCT}u|^2_{H^{1/2}(\partial\Omega_i)}. \tag{3.28}$$

To estimate the L^2 terms, we again first use the triangle inequality

$$\|\nabla I_{HCT}P\bar{B}u\|_{L^2(\Gamma_k)} \leq 2\|\nabla I_{HCT}u\|_{L^2(\Gamma_k)} + \|\nabla I_{HCT}(I-P)\bar{B}u\|_{L^2(\Gamma_k)}.$$

Since $\nabla I_{HCT}(I-P)\bar{B}u$ is a linear function on every edge and P is an orthogonal projection, it holds that

$$\sum_k \|\nabla I_{HCT}(I-P)\bar{B}u\|^2_{L^2(\Gamma_k)} \leq c_4 h\|(I-P)\bar{B}u\|^2$$
$$\leq c_5 h\|\bar{B}u\|^2 \leq 2c_5 h\|u\|^2$$

Furthermore, Lemma 3.19 shows that

$$h\|u\|^2 \leq c_6(1+H^2)\sum_{i=1}^{N_s}\|\nabla I_{HCT}u\|^2_{L^2(\partial\Omega_i)}$$

Finally, since $u \perp \operatorname{Ker} S$, the Poincaré inequality and the equivalence of the norms (3.25) conclude the estimate. ☐

3.3. Condition Number Estimate. We have now everything ready to prove the final result.

THEOREM 3.22. *The condition number of the generalized FETI method with the Dirichlet preconditioner (2.18) satisfies*

$$\kappa = \frac{\lambda_{max}(QDPF)}{\lambda_{min}(QDPF)} \leq C\left(1+\log\frac{H}{h}\right)^\gamma$$

with $\gamma = 3$, and $\gamma = 2$ if there are no crosspoints between more than two subdomains.

Proof. Lemmas 3.20 and 3.21 verify the assumptions (i) and (ii), of Theorem 3.12, with $C_1 = C(1+\log H/h)^\alpha$, $\alpha = 0$ or 1, and $C_2 = C(1+\log H/h)^2$, respectively. ☐

TABLE 4.1
Fixed number of subdomains, series of refined meshes

2x2 subdomains, H=$\frac{1}{2}$

$\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{10}$	18	2578	12	7.6
$\frac{1}{20}$	22	30101	15	12.6
$\frac{1}{40}$	26	409987	17	18.6

4x4 subdomains, H=$\frac{1}{4}$

$\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{10}$	61	6795	21	11.5
$\frac{1}{20}$	86	84199	27	17
$\frac{1}{40}$	119	1038120	36	24.4

8x8 subdomains, H=$\frac{1}{8}$

$\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{10}$	172	21707	25	13
$\frac{1}{20}$	247	275004	34	19.4
$\frac{1}{40}$	323	3920613	42	27.6

TABLE 4.2
Fixed global mesh, series of refined mesh partitions

Global mesh 120x120, h=$\frac{1}{120}$

splitting H	splitting $\frac{h}{H}$	FETI Iterations	FETI Cond. Numb.	NEW FETI Iterations	NEW FETI Cond. Numb.
$\frac{1}{2}$	$\frac{1}{60}$	27	2079032	18	23.2
$\frac{1}{3}$	$\frac{1}{40}$	64	839240	29	22.4
$\frac{1}{4}$	$\frac{1}{30}$	104	391470	32	21
$\frac{1}{5}$	$\frac{1}{24}$	135	234504	33	19.9
$\frac{1}{6}$	$\frac{1}{20}$	164	160173	32	18.6
$\frac{1}{8}$	$\frac{1}{15}$	222	94285	31	16.6
$\frac{1}{10}$	$\frac{1}{12}$	255	63896	29	14.9
$\frac{1}{12}$	$\frac{1}{10}$	245	46921	27	13.6

4. Computational Results. In order to illustrate the potential of the new FETI method for the iterative solution of discretized fourth order partial differential equations, we consider the plate bending problem on a unit square, with thickness equal to 10^{-3}. The Young modulus is $E = 10^6$ and the Poisson coefficient is $\nu = 0.3$. The plate is discretized by a uniform mesh of three-noded triangular DKT plate elements and subjected to a uniform pressure load. Several meshes with different parameters h and several mesh partitions with different parameters H are constructed for assessing the performance of the new FETI method described in this paper. In all cases, the FETI and new FETI methods are preconditioned with the Dirichlet

TABLE 4.3
Fixed local mesh, series of refined meshes and mesh partitions

Local mesh 15x15, $\frac{h}{H}=\frac{1}{15}$

splitting		FETI		NEW FETI	
H	h	Iterations	Cond. Numb.	Iterations	Cond. Numb.
$\frac{1}{2}$	$\frac{1}{30}$	20	11088	13	10
$\frac{1}{3}$	$\frac{1}{45}$	49	19004	21	13.4
$\frac{1}{4}$	$\frac{1}{60}$	74	29041	25	14.6
$\frac{1}{5}$	$\frac{1}{75}$	109	40120	28	15.4
$\frac{1}{6}$	$\frac{1}{90}$	145	55068	29	15.9
$\frac{1}{8}$	$\frac{1}{120}$	222	94285	31	16.6
$\frac{1}{10}$	$\frac{1}{150}$	318	144556	32	16.9

TABLE 4.4
Performance results for 960000 d.o.f. and 64 subdomains

FETI			NEW FETI		
Iterations	Total time	Time per iter.	Iterations	Total time	Time per iter.
314	265 s.	0.8 s.	45	105 s.	1.1 s.

preconditioner [12, 15], and the following stopping criterion is used

$$\frac{||z_{k-1}||}{||f||} \leq \varepsilon = 10^{-3} \tag{4.1}$$

where z_{k-1} is the preconditioned residual in Algorithm 2.2. It was shown in [15] that (4.1) is a good estimator for the global error criterion

$$\frac{||Ku - f||}{||f||} \leq \varepsilon,$$

where K is the global stiffness matrix and u is the approximate solution averaged between subdomains, and $||\cdot||$ is the l^2 norm. This stopping criterion is more adequate and more stringent than a local stopping criterion based on the jump of the solution across the subdomain interfaces [15]. It also eliminates the need for constructing the global solution u at every PCG iteration and bypasses the computation of Ku.

Three series of computational experiments are reported. First, the number of subdomains is fixed to 4, 16, and 64, and in each case, three different meshes corresponding to $h = 1/10$, $h = 1/20$, and $h = 1/40$ are generated. The corresponding performance results (number of iterations and condition number of the preconditioned interface problem) of the FETI and new FETI methods are summarized in Table 1. These results confirm that for a given H, the interface problem associated with the original FETI method has a condition number that grows fast with the mesh size h, while that of the new FETI method has a condition number that is much smaller and grows only weakly with h. For large number of subdomains ($H = 1/8$, $N_s = 64$) the new FETI method is reported to converge about 7 times faster than the original one. However, it is also interesting to note that when the mesh size h is decreased, the number of iterations of the original FETI method starts from a value that is higher than that of the new FETI algorithm for plates, but grows only weakly and almost at the same slow rate as in the new FETI method. We believe that this observation is a result of the superconvergence properties of the original FETI method [15]. Namely, a non-optimal preconditioner applied to the dual problem gives a bad condition number but good separation of eigenvalues as long as there are not too many subdomains.

Next, the mesh size is fixed to $h = 1/120$ (28800 elements and 86400 d.o.f.), and H is varied between $H = 1/2$ (4 subdomains) and $H = 1/12$ (144 subdomains). The obtained performance results are depicted in Table 2. In that case, the condition numbers of both FETI methods are shown to decrease with the number of subdomains. This is an expected result because when h is fixed and H is decreased, the size of the coarse problem increases for both algorithms. Both FETI methods are also shown to achieve convergence in a number of iterations that is asymptotically independent of the number of subdomains. However, the new FETI method reaches the asymptotic behavior much faster than the original one, and for large number of subdomains ($H \leq 1/8$), the new FETI method is reported to converge about 8 times faster than the original one.

Finally, the subdomain problem size is fixed to $h/H = 1/15$, and the number of subdomains as well as the size of the global problem are increased. The performance results reported in Table 3 show that in that case too, the new FETI method outperforms significantly the original one.

Since solution time is ultimately the most important criterion for assessing performance, we have also benchmarked both FETI methods for the same plate bending problem described above with 960000 d.o.f. and 64 subdomains. The performance results obtained on a 64-processor IBM SP2 are summarized in Table 4. They show that even though the new FETI method consumes an amount of CPU time (55.5 s.) equivalent to that of 50 of its iterations to set up and preprocess the coarse problem (2.14), and even though each of its iterations is 1.3 times more expensive than an iteration of the original FETI method, the new FETI method is 2.5 times faster than the original one at solving the system of 960000 plate bending equations on a 64-processor IBM SP2.

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