RIM SENSITIVITY ANALYSIS FROM AN INTERIOR SOLUTION*

HARVEY J. GREENBERG†

Abstract. This establishes theorems about the simultaneous variation of right-hand sides and cost coefficients in a linear program from an interior solution. Some results are extensions of those that have been proven for varying the right-hand side of the primal or the dual, but not both; other results are new. In addition, changes in the optimal partition and what that means in economic terms are related to the basis-driven approach, notably to the Theory of Compatibility. In addition to new theorems about this relation, the transition graph is extended to provide another visualization of the underlying economics.

Key words. Linear programming, sensitivity analysis, computational economics, interior point methods, parametric programming, optimal partition.

1. Introduction.

Consider the primal-dual pair of linear programs:

$$P : Min\{cx : x \geq 0, Ax \geq b\} \qquad D : Max\{\pi b : \pi \geq 0, \pi A \leq c\},$$

where c is a row vector in R^n, called *objective coefficients*; x is a column vector in R^n, called *levels*, b is a column vector in R^m, called *right-hand sides*; π is a row vector in R^m called *prices*; and A is an $m \times n$ matrix.

This paper concerns the simultaneous variation of right-hand sides and objective coefficients, which we call *rim data*: $r = (b, c)$. The change is of the form θh, where $\theta > 0$ and h is a nonzero *direction vector*. We suppose we have an interior solution, and we are interested in the following questions:

- Must the optimal partition change for any positive value of θ? If so, what is the new optimal partition? If not, for what range does this partition remain optimal?
- How does this relate to basic ranges, in particular to compatibility theory?
- How does this relate to the differential Lagrangian?
- How does the optimal objective value change as a function of θ?

Previous results[1, 6, 8] answered most of these questions when b or c change separately, but some of those proofs do not have natural extentions to deal with their simultaneous variation.

The rest of this paper is organized as follows. In the next section, we briefly give the terms and concepts needed for the main results. Then, we consider the first set of questions concerning the optimal partition, both when it does not change and when it does. In doing so, we shall relate this to the differential Lagrangian, and we shall derive the piece-wise quadratic form of the objective value from a new vantage. Finally, we relate the optimal partition change (if any) to basis-driven sensitivity analysis, notably to the *Theory of Compatible Bases*[3].

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† University of Colorado at Denver, Mathematics Department, PO Box 173364, Denver, CO 80217-3364, e-mail: hgreenbe@carbon.cudenver.edu, http://www-math.cudenver.edu/~hgreenbe

2. Terms and Concepts. Let $P(r)$ and $D(r)$ denote the primal and dual polyhedra, respectively. For $(x, \pi) \in P(r) \times D(r)$, we associate *surplus variables*, $s = Ax - b$ and *reduced costs*, $d = c - \pi A$. Let $P^*(r)$ and $D^*(r)$ denote the primal and dual optimality regions, respectively , which we suppose are not empty. The *support set* of a non-negative vector, v, is denoted: $\sigma(v) = \{k : v_k > 0\}$. Then, primal-dual optimality can be represented by *complementary slackness*: $\sigma(x) \cap \sigma(d) = \emptyset$ and $\sigma(\pi) \cap \sigma(s) = \emptyset$. As shown by Goldman and Tucker[2], there must exist a *strictly complementary* solution, whereby the support sets span the rows and columns: $\sigma(\pi) \cup \sigma(s) = \{1, \ldots, m\}$, and $\sigma(x) \cup \sigma(d) = \{1, \ldots, n\}$. This defines the (unique) *optimal partition*, obtained from any strictly complementary (i.e., interior) solution.

Partition A according to the optimal partition:

$$A = \begin{bmatrix} B & N \\ B^* & N^* \end{bmatrix} \begin{array}{l} \leftarrow \sigma(\pi) = \text{rows active in some optimal solution} \\ \leftarrow \sigma(s) = \text{rows inactive in all optimal solutions} \end{array}$$

$$\begin{array}{ll} \uparrow & \uparrow \\ \sigma(x) & \sigma(d) = \text{columns inactive in all optimal solutions} \\ = \text{columns active in some optimal solution.} & \end{array}$$

We refer to B as the *active set* of rows and columns. The rows are active because they never have surplus in any optimal solution (i.e., $s_i = 0 \; \forall i \in \sigma(\pi)$), and for each row we have an optimal solution where its price is positive. The columns are active because they never have a positive reduced cost (i.e., $d_j = 0 \; \forall j \in \sigma(x)$), and for each column we have an optimal solution where its level is positive. The complementary rows and columns are called *inactive*. Those rows never have a positive price, and each inactive row has a positive surplus in at least one optimal solution. The inactive columns never have a positive level, and each inactive column has positive reduced cost in at least one optimal solution.

Partition the rim data vectors conformally: $b = \begin{pmatrix} b_N \\ b_B \end{pmatrix}$ and $c = (c_B \; c_N)$. Also, $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$, $s = \begin{pmatrix} s_N \\ s_B \end{pmatrix}$, $\pi = (\pi_N \; \pi_B)$, and $d = (d_B, d_N)$. So, the original linear programs are equivalent to the following primal-dual pair:

Primal	Dual
$Min\ c_B x_B + c_N x_N :$	$Max\ \pi_N b_N + \pi_B b_B :$
$B\,x_B + N\,x_N - s_N = b_N$	$\pi_N B + \pi_B B^* + d_B = c_B$
$B^* x_B + N^* x_N - s_B = b_B$	$\pi_N N + \pi_B N^* + d_N = c_N$
$x,\ s \geq 0$	$\pi,\ d \geq 0.$

Maintaining the partition conditions, $x_N = 0, s_N = 0, \pi_B = 0$ and $d_B = 0$, we define the following primal and dual *polyhedral conditions*, which we shall use later.

$$\mathcal{P}(b) = \{(x_B, 0) : x_B \geq 0,\ Bx_B = b_N,\ B^* x_B \geq b_B\};$$
$$\mathcal{D}(c) = \{(\pi_N, 0) : \pi_N \geq 0,\ \pi_N B = c_B,\ \pi_N N \leq c_N\}.$$

Their relative interiors are the strictly complementary solutions, given by the following:

$$ri(\mathcal{P}(b)) = \{(x_B, 0) : x_B > 0,\ Bx_B = b_N,\ B^* x_B > b_B\};$$
$$ri(\mathcal{D}(c)) = \{(\pi_N, 0) : \pi_N > 0,\ \pi_N B = c_B,\ \pi_N N < c_N\}.$$

In particular, $\mathcal{P}(b) \times \mathcal{D}(c) = P^*(r) \times D^*(r)$ for $r = (b, c)$.

We say $h = (\delta b, \delta c)$ is *admissible* if the LP has an optimal solution for $r + \theta h$ for some $\theta > 0$. The set of admissible directions, say $\mathcal{H}$, is composed of those h for which the primal and dual feasibility conditions hold:

$$\mathcal{H} = \{(\delta b, \delta c) \in R^{m+n} : \exists \theta > 0, x \geq 0, \pi \geq 0 \ni Ax \geq b + \theta \delta b \text{ and } \pi A \leq c + \theta \delta c\}.$$

A basis, B, is optimal if its associated primal and dual solutions are feasible. For $h \in R^{m+n}$, we say B is *compatible* with h (and h with B) if B is optimal for $r + \theta h$ for some $\theta > 0$. Its *range of compatability* is $\rho(\mathrm{B}; h) = Sup\{\theta :\ \mathrm{B}$ is optimal for $r + \theta h\}$. (Note: B is optimal throughout $[r, r + \rho(\mathrm{B}; h)h]$.) Let $H(\mathrm{B})$ denote the set of directions compatible with B:

$$H(\mathrm{B}) = \{h \in R^{m+n} : \rho(\mathrm{B}; h) > 0\}.$$

One of the fundamental theorems of (basic) compatibility[3] is: $\mathcal{H} = \bigcup_{\mathrm{B}} H(\mathrm{B})$. We shall relate this to a new theory of compatibility in connection with the optimal partition. Also, we denote the *basic spectrum*: $\rho^*(h) = Sup\{\rho(\mathrm{B}; h) :\ \mathrm{B}$ is optimal for $r\}$.

Given $h \in \mathcal{H}$, the objective value is $z(r + \theta h)$, as θ increases from zero. Suppose B is a compatible basis (one must exist) with (x, π) the associated basic solution. Then, since the basis remains optimal in $[0, \rho(\mathrm{B}; h)]$, the optimal value is quadratic:

$$z(r + \theta h) = z(r) + \theta(\delta c_B x_B + \pi_N \delta b_N) + \theta^2(\delta c_B \mathrm{B}^{-1} \delta b_N).$$

We shall prove a similar result holds when the optimal partition does not change.

We say z has *constant functional form* if the coefficients are constant. In particular, z has constant functional form on $[0, \rho^*(h)]$ $\forall h \in \mathcal{H}$. Further, if either $\delta b = 0$ or $\delta c = 0$, the quadratic term is zero, and $z(r + \theta h) - z(r)$ is linear in θ. In this case, we call the range of θ for which z has the constant functional form a *linearity interval*. It has already been proven[1, 6] that the break points of the linearity intervals correspond precisely to where the optimal partition changes (which is not necessarily the same as when the basis must change – see [4] for an example). Here we extend this to the more general rim variation, where the functional form is piece-wise quadratic.

3. The Optimal Partition for the Perturbation. Define the range for which the optimal partition does not change for a given direction (h):

$$\tau(h) \equiv Sup\{\theta :\ \text{the optimal partition does not change throughout } [r,\ r + \theta h]\}.$$

In this definition, the left endpoint of the line segment is closed, so if the partition must change at r (for any $\theta > 0$), $\tau(h) = 0$. If $0 < \tau(h) < \infty$, the optimal partition is invariant on $[r,\ r + \tau(h)h)$, but it could change at $r + \tau(h)h$.

LEMMA 3.1. *Suppose h is an admissible direction. Then, the optimal partition for $r+h$ is the same as the optimal partition for r if, and only if, $ri(\mathcal{P}(r+h)\times\mathcal{D}(r+h))\neq\emptyset$, in which case the optimal partition remains the same throughout the line segment, $[r,\ r+h]$.*

Proof. The first part follows from the uniqueness of the optimal partition, determined by any strictly complementary solution. To show the optimal partition remains invariant on the line segment, $[r,\ r+h]$, let (x^0,π^0) be a strictly complementary solution in $P(r)\times D(r)$, and let (x',π') be a strictly complementary solution in $P(r+h)\times D(r+h)$. Suppose $r''=\alpha r+(1-\alpha)(r+h)$ for some $\alpha\in[0,1]$, and define $(x,\pi)=\alpha(x^0,\pi^0)+(1-\alpha)(x',\pi')$. Since the optimal partition for r and $r+h$ is the same, we have:

$$\begin{aligned} x_B &= \alpha x_B^0+(1-\alpha)x'_B>0 \quad \text{and} \quad & x_N &= \alpha x_N^0+(1-\alpha)x'_N=0;\\ \pi_N &= \alpha\pi_N^0+(1-\alpha)\pi'_N>0 \quad \text{and} \quad & \pi_B &= \alpha\pi_B^0+(1-\alpha)\pi'_B=0. \end{aligned}$$

Thus, $\sigma(x)=\sigma(x^0)$ and $\sigma(\pi)=\sigma(\pi^0)$. Further,

$$\begin{aligned} B\,x_B &= B[\alpha x_B^0+(1-\alpha)x'_B] &&= \alpha b_N+(1-\alpha)b'_N &&= b''_N\\ B^*x_B &= B^*[\alpha x_B^0+(1-\alpha)x'_B] &&> \alpha b_B+(1-\alpha)b'_B &&= b''_B\\ \pi_B B &= [\alpha\pi_B^0+(1-\alpha)\pi'_B]B &&= \alpha c_B+(1-\alpha)c'_B &&= c''_B\\ \pi_N N &= [\alpha\pi_N^0+(1-\alpha)\pi'_N]N &&< \alpha c_N+(1-\alpha)c'_N &&= c''_N. \end{aligned}$$

Thus, $\sigma(s)=\sigma(s^0)$ and $\sigma(d)=\sigma(d^0)$, so (x,π) is a strictly complementary solution for the LP defined by r'', and it has the same partition. This must therefore be the optimal partition, since it is unique. □

Suppose $h=(\delta b,\delta c)$ is an admissible direction, so θ^*h is an admissible change for some $\theta^*>0$. If the optimal partition for $r+\theta^*h$ is the same as it is for r, lemma 3.1 establishes that it is the same for $r+\theta h$ $\forall\theta\in[0,\theta^*]$. In that case, the objective value changes with constant functional form. To see this, use the construction in the proof: $(x,\pi)=\alpha(x^0,\pi^0)+(1-\alpha)(x',\pi')$, where (x^0,π^0) is strictly complementary for r, (x',π') is strictly complementary for $r+\theta^*h$, and $\alpha=1-\theta/\theta^*$. Then, since the optimal partition is the same, (x,π) is strictly complementary for $r+\theta h$, and

$$\begin{aligned} z(r+\theta h) &= (c+\theta\delta c)[(1-\theta/\theta^*)x+\theta/\theta^*x']\\ &= z(r)+\theta[c_B(x'_B-x_B^0)/\theta^*+\delta c_B\,x_B^0]+\theta^2\,\delta c_B(x'_B-x_B^0)/\theta^*. \end{aligned}$$

This essentially proves the following generalization of the linear case[1, 6, 8, 9].

THEOREM 3.2 (OPTIMAL VALUE FUNCTION). *The optimal partition changes at r for change direction h if, and only if, z changes functional form.*

Further, lemma 3.1 extends to the following convexity property.

THEOREM 3.3 (OPTIMAL PARTITION CONVEXITY). *If the optimal partition is the same throughout $[r,\ r+h^1]$ as it is throughout $[r,\ r+h^2]$, it is the same throughout $[r,r+\alpha h^1+(1-\alpha)h^2]$ $\forall\alpha\in[0,1]$.*

Proof. Let (x^k, π^k) be a strictly complementary solution for $k = 1, 2$, so they satisfy the primal-dual conditions:

$$\begin{array}{ll} B\,x^k_B = b_N + \delta b^k_N & \pi^k_N B = c_B + \delta c^k_B \\ B^* x^k_B > b_B + \delta b^k_B & \pi^k_N N < c_N + \delta c^k_N \\ x^k_B > 0,\ x^k_N = 0 & \pi^k_N > 0,\ \pi^k_B = 0. \end{array}$$

Define $(x, \pi) = \alpha(x^1, \pi^1) + (1 - \alpha)(x^2, \pi^2)$. Multiply the above by α for $k = 1$ and by $1 - \alpha$ for $k = 2$ to satisfy the following for $h = \alpha h^1 + (1 - \alpha)h^2 = (\delta b, \delta c)$:

$$\begin{array}{ll} B\,x_B = b_N + \delta b_N & \pi_N B = c_B + \delta c_B \\ B^* x_B > b_B + \delta b_B & \pi_N N < c_N + \delta c_N \\ x_B > 0,\ x_N = 0 & \pi_N > 0,\ \pi_B = 0. \end{array}$$

So, (x, π) is a strictly complementary solution for $r + \alpha h^1 + (1 - \alpha)h^2$ with the same partition. It follows from lemma 3.1 that the optimal partition remains the same throughout $[r,\ r + \alpha h^1 + (1 - \alpha)h^2]$. □

In the special case that $h^1 = (\delta b, 0)$ and $h^2 = (0, \delta c)$, the Optimal Partition Convexity Theorem can be strenghthened to the following *decoupling principle*:

COROLLARY 3.4. *The optimal partition does not change in* $[r,\ r + (\delta b, \delta c)]$ *if, and only if, it does not change in* $[r,\ r + (\delta b, 0)] \cup [r,\ r + (0, \delta c)]$.

Proof. If the optimal partition does not change in $[r,\ r + (\delta b, \delta c)]$, the following primal-dual system has a solution:

$$\begin{array}{ll} B\,x_B = b_N + \delta b_N & \pi_N B = c_B + \delta c_B \\ B^* x_B > b_B + \delta b_B & \pi_N N < c_N + \delta c_N \\ x_B > 0 & \pi_N > 0. \end{array}$$

Let (x', π') be a solution, and let (x^0, π^0) be a strictly complementary solution for r. Then, (x', π^0) is a strictly complementary solution for $r + (\delta b, 0)$, and (x^0, π') is a strictly complementary solution for $r + (0, \delta c)$. These imply that the optimal partition does not change in $[r,\ r + (\delta b, 0)] \cup [r,\ r + (0, \delta c)]$.

Conversely, if the optimal partition does not change in $[r,\ r + (\delta b, 0)]$, there exists x' to satisfy the primal conditions, and if the optimal partition does not change in $[r,\ r + (0, \delta c)]$, there exists π' to satisfy the dual conditions. Since the partitions are the same, (x', π') is a strictly complementary solution for $r + (\delta b, \delta c)$, so the partition is the same throughout $[r,\ r + (\delta b, \delta c)]$. □

Let the optimal partition be *compatible* with h (and h with it) if $\tau(h) > 0$. Define the *set of compatible directions*: $H = \{h : \tau(h) > 0\}$. Then, we have the following analogy to the *Basis Compatibility Convexity Theorem*[3].

THEOREM 3.5 (PARTITION COMPATIBILITY). *The following properties hold for* H *and* τ.

(1) H *is a nonempty convex cone.*

(2) τ is quasi-concave on H — i.e., $\tau(\alpha h^1 + (1-\alpha)h^2) \geq Min\{\tau(h^1), \tau(h^2\}$ for $h^1, h^2 \in H$ and $\alpha \in [0,1]$.

(3) H satisfies the decoupling principle — i.e., $(\delta b, \delta c) \in H$ if, and only if, $(\delta b, 0) \in H$ and $(0, \delta c) \in H$.

Proof. (1) Suppose $h^1, h^2 \in H$ and define $\theta^* = Min\{\tau(h^1), \tau(h^2)\} > 0$. Then, for $\theta \in (0, \theta^*)$, $\exists (x^k, \pi^k)$ to satisfy the strictly complementary primal-dual conditions:

$$\begin{aligned} B\,x_B^k &= b_N + \theta\delta b_N^k & \pi_N^k B &= c_B + \theta\delta c_B^k \\ B^* x_B^k &> b_B + \theta\delta b_B^k & \pi_N^k N &< c_N + \theta\delta c_N^k \\ x_B^k &> 0 & \pi_N^k &> 0 \end{aligned}$$

for $k = 1, 2$. Define $(x, \pi) = \frac{1}{2}(x^1, \pi^1) + \frac{1}{2}(x^2, \pi^2)$, then multiply the above by $\frac{1}{2}$ and sum to obtain the following:

$$\begin{aligned} B\,x_B &= b_N + \tfrac{1}{2}\theta\delta b_N & \pi_N B &= c_B + \tfrac{1}{2}\theta\delta c_B \\ B^* x_B &> b_B + \tfrac{1}{2}\theta\delta b_B & \pi_N N &< c_N + \tfrac{1}{2}\theta\delta c_N \\ x_B &> 0 & \pi_N &> 0. \end{aligned}$$

Define $\theta' = \frac{1}{2}\theta$ and $\theta'^* = \frac{1}{2}\theta^*$, and we have the desired result: the optimal partition for $r + \theta'(h^1 + h^2)$ is the same as the optimal partition for r, so $h^1 + h^2 \in H$. To show H is nonempty, let $h = (b, c)$, so $r + \theta h = (1+\theta)r$. Then, by rescaling ($x' = x/(1+\theta)$ and $\pi' = \pi/(1+\theta)$), the strictly complementary solution has the same partition for all $\theta \geq 0$.

(2) Let $\theta^* \equiv Min\{\tau(h^1), \tau(h^2\} > 0$ and $(x, \pi) = \alpha(x^1, \pi^1) + (1-\alpha)(x^2, \pi^2)$. For $\theta \in (0, \theta^*)$, multiply the first system ($k = 1$) by α, the second ($k = 2$) by $1 - \alpha$, and sum to prove that (x, π) is a strictly complementary solution for the partition:

$$\begin{aligned} B\,x_B &= b_N + \theta\delta b_N & \pi_N B &= c_B + \theta\delta c_B \\ B^* x_B &> b_B + \theta\delta b_B & \pi_N N &< c_N + \theta\delta c_N \\ x_B &> 0 & \pi_N &> 0. \end{aligned}$$

Thus, $\tau(\alpha h^1 + (1-\alpha)h^2) \geq Sup\{\theta : \theta < \theta^*\} = \theta^*$.

(3) Let $h = (\delta b, \delta c) \in H$. Then, $\exists \theta^* > 0$ such that for $\theta \in [0, \theta^*)$, the primal-dual conditions have a strictly complementary solution, say (x, π) (with the same partition). Let (x^0, π^0) be a strictly complementary solution for r. Then, since these have the same partition, (x, π^0) is a strictly complementary solution for $r + \theta(\delta b, 0)$, and (x^0, π) is a strictly complementary solution for $r + \theta(0, \delta c)$. Conversely, if (x, π^0) is a strictly complementary solution for $r + \theta(\delta b, 0)$, and (x^0, π) is a strictly complementary solution for $r + \theta(0, \delta c)$, both having the partition defined by B, it follows that (x, π) is a strictly complmentary solution for $r + \theta(\delta b, \delta c)$. □

Now suppose h is an admissible direction, but the optimal partition changes: $ri(\mathcal{P}(r + \theta h) \times \mathcal{D}(r + \theta h)) = \emptyset$ for all $\theta > 0$. The following theorem shows the fundamental relationship the new partition has with the differential linear programs that comprise Mills'[7] differential Lagrangian when A does not change. (Mills' Theorem was extended[5] to apply to any linear program, rather than the special case of a game.)

The following theorem applies generally, even if the optimal partition does not change. The new result is (3), and the proofs[1, 6] of (1) and (2) do not extend (they are included here for self-containment).

THEOREM 3.6 (OPTIMAL PARTITION PERTURBATION). *Suppose* (x^0, π^0) *is a strictly complementary solution for* r *and* $(\delta b, \delta c)$ *is an admissible direction. Define the differential linear programs:*

$$\delta P : Min\{(\delta c)x : x \in P^*(r)\} \qquad \delta D : Max\{\pi(\delta b) : \pi \in D^*(r)\}.$$

Let x^* *and* π^* *be respective interior solutions. There exists* $\theta^* > 0$ *such that the following are true for* $\theta \in (0, \theta^*)$.

(1) The optimal partition for $r + \theta(\delta b, 0)$ *is the same as the optimal partition for* δD, *and* $z(r + \theta(\delta b, 0)) = z(r) + \theta\pi^*_N(\delta b_N)$.

(2) The optimal partition for $r + \theta(0, \delta c)$ *is the same as the optimal partition for* δP, *and* $z(r + \theta(0, \delta c)) = z(r) + \theta(\delta c_B)x^*_B$.

(3) The optimal partition for $r + \theta(\delta b, \delta c)$ *is* $\sigma(x^*)$ *from* δP *and* $\sigma(\pi^*)$ *from* δD. *Further,* $z(r + \theta(\delta b, \delta c)) = z(r) + \theta(\delta c_B\, x^*_B + \pi^*_N\, \delta b_N) + \theta^2(\delta c_B B^+ \delta b_N)$, *where* B^+ *is any generalized inverse of* B.

Proof. (1) The following proof is from Jansen, Roos and Terlaky[6]. The dual of δD is $Min\{c\xi : B\xi_B + N\xi_N \geq \delta b_N, \xi_N \geq 0\}$. Since δD has an optimal solution, there is a strictly complementary optimum, say (ξ, π^*). Consider $x = x^0 + \theta\xi$. Since $x^0_B > 0$, there exists $\theta' > 0$ for which $x_B > 0$ for $\theta \in [0, \theta')$. Further, $x_N = \theta\xi_N \geq 0$, so $x \geq 0$, and we have $[B\ N]x = Bx^0_B + \theta(B\xi_B + N\xi_N) \geq b_N + \theta\delta b_N$. Further, $[B^*\ N^*]x = B^*x^0_B + \theta(B^*\xi_B + N^*\xi_N)$. Since $B^*x^0_B > b_B$, there exists $\theta'' > 0$ such that $[B^*\ N^*]x > b_B + \theta\delta b_B$ for $\theta \in [0, \theta'')$. Let $\theta^* = Min\{\theta', \theta''\} > 0$. We so far have that (x, π^*) satisfies the primal-dual conditions for all $\theta \in [0, \theta^*)$:

$$\begin{array}{ll} B\,x^0_B + \theta(B\,\xi_B + N\,\xi_N) \geq b_N + \theta\delta b_N & \pi^*_N B = c_B \\ B^*x^0_B + \theta(B^*\xi_B + N^*\xi_N) > b_B + \theta\delta b_B & \pi^*_N N \leq c_N \\ x^0_B + \theta\xi_B > 0,\ \theta\xi_N \geq 0 & \pi^*_N \geq 0. \end{array}$$

We now prove (x, π^*) is a strictly complementary solution for $r + \theta(\delta b, 0)$, where $\theta > 0$. Suppose $B_{i\bullet}x_B + N_{i\bullet}x_N = b_i + \theta\delta b_i$. Since $B_{i\bullet}x_B = b_i$, we must have $B_{i\bullet}\xi_B + N_{i\bullet}\xi_N = \delta b_i$. This implies $\pi^*_i > 0$ since (ξ, π^*) is strictly complementary for δD and its dual, so $\sigma(\pi^*) =\sim \sigma(s')$. Also, since (ξ, π^*) is strictly complementary, $\sigma(d^*) =\sim \sigma(\xi_N) \cap \sim \sigma(x^0_B) =\sim (\sigma(\xi_N) \cup \sigma(x^0_B)) =\sim \sigma(x')$. Thus, we have proven (x', π^*) is a strictly complementary solution for $r + \theta(\delta b, 0)$, with the same optimal partition as D, for all $\theta \in (0, \theta^*)$. Further, $z(r + \theta(\delta b, 0)) = cx' = cx + \theta c\xi$. We have $cx = z(r)$ and $c\xi = \pi^*(\delta b)$ (from duality), so $z(r + \theta(\delta b, 0)) = z(r) + \theta\pi^*(\delta b)$. Since $\pi_B = 0$ for all $\pi \in D^*(r)$, we conclude $z(r + \theta(\delta b, 0)) = z(r) + \theta\pi^*_N(\delta b_N)$. The proof of (2) is similar by constructing $\pi = \pi^0 + \theta\xi'$, where ξ' is the vector of variables for the dual of $\delta P : Max\{\xi' b : \xi'_B \geq 0,\ \xi'_N B + \xi'_B B^* \leq \delta c_B\}$. We now prove (3).

From (1), there exists $\theta' > 0$ such that the optimal partition does not change throughout $(r,\ r + \theta'(\delta b, 0))$, and the set of active columns is $\sigma(x^*)$. (Note from the proof of (1) that the set of active rows does not change.) From (2), there exists $\theta'' > 0$ such that the optimal partition does not change throughout $(r,\ r + \theta''(0, \delta c))$, and the set of active rows is $\sigma(\pi^*)$. (By analogy, the proof of (2) shows that the set of active columns does not change.) Let $\theta^* = Min\{\theta', \theta''\} > 0$. Then, the optimal partition does not change throughout $(r,\ r + \theta^*(\delta b, \delta c))$, and its active sets are the rows in $\sigma(\pi^*)$ and the columns in $\sigma(x^*)$.

(Note: we cannot use the solutions in (1) and (2) directly because (x, π) need not be complementary, in which case it is not a solution for $r + \theta h$. This proof can be viewed as first moving to $r + \theta(\delta b, 0)$, where $\theta < \theta^*$, and the optimal partition is defined by $\sigma(x^*)$ and $\sigma(\pi)$, then changing c by $\theta\delta c$ to move to $r + \theta h$, where the optimal partition is defined by $\sigma(x^*)$ and $\sigma(\pi^*)$. Equivalently, we can move first to $r + \theta(0, \delta c)$, with optimal partition defined by $\sigma(x)$ and $\sigma(\pi^*)$, then move to $r + \theta h$ to obtain the same result. This argument is similar to the one used by Roos[9] for a different result.)

Finally, to show that $z(r + \theta h)$ has the asserted quadratic form, we shall use the defining properties of generalized inverses. Let B correspond to the optimal partition throughout $(r,\ r + \theta^* h)$. Then, $x_B(\theta) = B^+(b_N + \theta\delta b_N) + (I - B^+B)v(\theta)$, where B^+ is any generalized inverse of B, and $v(\theta)$ is any vector in $R^{|\sigma(x)|}$. The defining property of B^+ is that $BB^+B = B$, and a fundamental property is that the equation has a solution if, and only if, $BB^+(b_N + \theta\delta b_N) = b_N + \theta\delta b_N$. Since this applies to $\theta = 0$, we must have $BB^+b_N = b_N$, which then implies we must also have $BB^+\delta b_N = \delta b_N$. Similarly, the dual equations are $\pi_N(\theta)B = c_B + \theta\delta c_B$, so we must have $\pi_N(\theta) = (c_B + \theta\delta c_B)B^+ + u(\theta)(I - BB^+)$, where $u(\theta)$ is any vector in $R^{|\sigma(\pi)|}$. Then,

$$\begin{aligned} z(r + \theta h) &= (c_B + \theta\delta c_B)[B^+(b_N + \theta\delta b_N) + (I - B^+B)v(\theta)] \\ &= c_B B^+ b_N + \theta(\delta c_B B^+ b_N + c_B B^+ \delta b_N) + \theta^2(\delta c_B B^+ \delta b_N), \end{aligned}$$

where the terms with $v(\theta)$ are zero because $c_B + \theta\delta c_B = (c_B + \theta\delta c_B)B^+B$, so $(c_B + \theta\delta c_B)(I - B^+B)v(\theta) = (c_B + \theta\delta c_B)B^+B(I - B^+B)v(\theta) = (c_B + \theta\delta c_B)B^+(B - BB^+B)v(\theta) = 0$ (recall $B = BB^+B$). □

Example:

$$Min\ x_1 + 3x_2 + 2x_3 : x \geq 0, x_1 + x_2 \geq 1, x_2 + x_3 \geq 1.$$

An interior optimal solution is $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\pi = (1, 2)$, so the optimal partition has $\sigma(x) = \{1, 2, 3\}$ and $\sigma(\pi) = \{1, 2\}$, which gives the optimality regions:

$$\begin{aligned} P^*(r) &= \{x : x \geq 0,\ x_1 + x_2 = 1,\ x_2 + x_3 = 1\} = \{(1 - x_2, x_2, 1 - x_2) : 0 \leq x_2 \leq 1\}; \\ D^*(r) &= \{\pi : \pi \geq 0,\ \pi_1 = 1,\ \pi_1 + \pi_2 = 3,\ \pi_2 = 2\} = \{(1, 2)\}. \end{aligned}$$

For $\delta b = (-1, 0)$ and $\delta c = (-1, 0, 0)$, the two differential LPs and their duals are as follows:

$$\begin{aligned} &P : Min\{-x_1 : x \in P^*(r)\} && Max\{\xi_1' + \xi_2' : \xi_1' \leq -1, \xi_1' + \xi_2' \leq 0, \xi_2' \leq 0\}; \\ &D : Max\{-\pi_1 : \pi \in D^*(r)\} && Min\{\xi_1 + 3\xi_2 + 2\xi_3 : \xi_1 + \xi_2 \geq -1, \xi_2 + \xi_3 \geq 0\}. \end{aligned}$$

A strictly complementary solution for P and its dual is $x^* = (1, 0, 1)$ and $\xi' = (-1, 0)$, so the optimal partition has $\sigma(x^*) = \{1, 3\}$. A strictly complementary solution for D and its dual is $\pi^* = (1, 2)$ and $\xi = (-2, 1, -1)$, so its optimal partition has $\sigma(\pi^*) = \{1, 2\}$. As given in the Optimal Partition Perturbation Theorem, $\sigma(x) = \sigma(x^*)$ from P, and $\sigma(\pi) = \sigma(\pi^*)$ from D for the optimal partition in $(r,\ r + \theta h)$. Let us verify this.

The perturbed LP is the following primal-dual pair:

$$\begin{array}{cc} Min\ (1-\theta)x_1 + 3x_2 + 2x_3 : x \geq 0, & Max\ \pi_1(1-\theta) + \pi_2 : \pi \geq 0, \\ x_1 + x_2 \geq 1 - \theta,\ x_2 + x_3 \geq 1 & \pi_1 \leq 1 - \theta,\ \pi_1 + \pi_2 \leq 3,\ \pi_2 \leq 2. \end{array}$$

For $\theta \in (0, 1)$ an interior optimal solution is $x = (1-\theta, 0, 1)$ (so $s = 0$) and $\pi = (1-\theta, 2)$ (so $d = (0, \theta, 0)$). Indeed, $\sigma(x) = \{1, 3\}$ and $\sigma(\pi) = \{1, 2\}$.

4. Relation to Basic Compatibility. Now we develop a range theory analogous to the *range of compatibility*[3], except that the optimal partition can change. Let $\Upsilon(h)$ denote the greatest value of θ for which the optimal partition does not change throughout $(r,\ r + \theta h)$ for $h \in \mathcal{H}$. Note the line segment is open, so the optimal partition need not be the same at the endpoints. In particular, the partition might have to change at r (i.e., $\tau(h) = 0$); otherwise, $\Upsilon(h) = \tau(h)$. The Optimal Partition Perturbation Theorem tells us that $\Upsilon(h) > 0$ when h is admissible, in which case $z(r + \theta h)$ has constant functional form for $\theta \in [0, \Upsilon(h)]$. When h is decoupled (i.e., $\delta b = 0$ or $\delta c = 0$), $[0, \Upsilon(h)]$ is a linearity interval of $z(r + \theta h) - z(r)$.

The following lemma says that this bounds each basic range of compatibility, which establishes the Optimal Partition Range Theorem (4.2).

LEMMA 4.1. *Suppose h is an admissible direction for which B is a compatible basis with range $\rho = \rho(B; h)$. Then, the optimal partition does not change throughout $(r,\ r + \rho h)$.*

Proof. From the Optimal Partition Perturbation Theorem (3.6), there exists $\theta > 0$ such that the optimal partition does not change in $(r,\ r + \theta h)$. Let θ^* be the supremum value of θ for which this is true. If $\theta^* \geq \rho$, we are done, so suppose $\theta^* < \rho$. Let (x^0, π^0) be any strictly complementary solution for $r + \frac{1}{2}\theta^* h$, so that $\sigma(x^0)$ and $\sigma(\pi^0)$ determine the optimal partition throughout $(r,\ r + \theta^* h)$. We shall reach a contradiction by constructing (x', π') that is optimal for $r + \theta h$, where $\theta^* < \theta < \rho$, and $\sigma(x') = \sigma(x^0), \sigma(s') = \sigma(s^0), \sigma(\pi') = \sigma(\pi^0)$, and $\sigma(d') = \sigma(d^0)$.

Define $\alpha = (\rho - \theta)/\rho$, so $1 - \alpha = \theta/\rho$, and $\alpha \in (0, 1)$. We shall form a convex combination of the strictly complementary solution and the basic solution response values, which we shall prove is feasible and has the same support sets as the strictly complementary solution. Suppose the basic solution, $(\overline{x}, \overline{\pi})$ changes by $(\Delta x, \Delta \pi)$. Then, define the following convex combination:

$$x = \alpha x^0 + (1 - \alpha)(\overline{x} + \Delta x) \quad \text{and} \quad \pi = \alpha \pi^0 + (1 - \alpha)(\overline{\pi} + \Delta \pi).$$

Clearly, $(x, \pi) \geq 0$. Further, we have $\Delta x_B = \rho \mathrm{B}^{-1} \delta b_N$ and $\Delta x_N = 0$, so the primal equations are given by:

$$\mathrm{B}\, x_B + N\, x_N \ = \ \mathrm{B}[\alpha x_B^0 + (1 - \alpha)(\overline{x}_B + \rho \mathrm{B}^{-1} \delta b_N)] + \alpha N x_N^0$$

$$\begin{aligned} &= \alpha[\mathrm{B}x_B^0 + Nx_N^0] + (1-\alpha)[\mathrm{B}\overline{x}_B + \rho\delta b_N] \\ &\geq \alpha b_N + (1-\alpha)b_N + \theta\delta b_N = b_N + \theta\delta b_N \end{aligned}$$

$$\begin{aligned} B^*x_B + N^*x_N &= B^*[\alpha x_B^0 + (1-\alpha)(\overline{x}_B + \rho\mathrm{B}^{-1}\delta b_N)] + \alpha N^* x_N^0 \\ &= \alpha[B^*x_B^0 + N^*x_N^0] + (1-\alpha)B^*[\overline{x}_B + \Delta x_B] \\ &\geq \alpha b_B + (1-\alpha)(b_B + \rho\delta b_B) = b_B + \theta\delta b_B. \end{aligned}$$

Thus, $Ax \geq b + \theta\delta b$, which proves x is feasible in the primal. Similarly, $\Delta\pi_N = \rho\delta c_B \mathrm{B}^{-1}$ and $\Delta\pi_B = 0$, so the dual equations are given by:

$$\begin{aligned} \pi_N\mathrm{B} + \pi_B B^* &= [\alpha\pi_N^0 + (1-\alpha)(\overline{\pi}_N + \rho\delta c_B\mathrm{B}^{-1})]\mathrm{B} + \alpha\pi_B^0 B^* \\ &= \alpha[\pi_N^0\mathrm{B} + \pi_B^0 B^*] + (1-\alpha)[\overline{\pi}_N\mathrm{B} + \rho\delta c_B] \\ &\leq \alpha c_B + (1-\alpha)(c_B + \rho\delta c_B) = c_B + \theta\delta c_B \end{aligned}$$

$$\begin{aligned} \pi_N N + \pi_B N^* &= [\alpha\pi_N^0 + (1-\alpha)(\overline{\pi}_N + \rho\delta c_B\mathrm{B}^{-1})]N + \alpha\pi_B^0 N^* \\ &= \alpha[\pi_N^0 N + \pi_B^0 N^*] + (1-\alpha)[\overline{\pi}_N + \Delta\pi_N]N \\ &\leq \alpha c_N + (1-\alpha)(c_N + \rho\delta c_N) = c_N + \theta c_N. \end{aligned}$$

Thus, $\pi A \leq c + \theta\delta c$, which proves π is feasible in the dual.

We have proven that (x, π) satisfies the primal-dual conditions for $r + \theta h$. We now prove its support sets are the same as those of (x^0, π^0). Let β_j denote the j-th row of B^{-1}. For a nonbasic activity (j), $x_j = \alpha x_j^0$, so $j \in \sigma(x)$ if, and only if, $j \in \sigma(x^0)$. For a basic activity (j), $x_j = \alpha x_j^0 + (1-\alpha)(\overline{x}_j + \rho\beta_j\delta b_N)$. For $j \in \sigma(x^0)$, we have $x_j > 0$ because $\overline{x}_j + \rho\beta_j\delta b_N \geq 0$, so $\sigma(x^0) \subseteq \sigma(x)$. Now suppose $j \in \sigma(x)$, so

$$0 < x_j = \alpha x_j^0 + (1-\alpha)\overline{x}_j + \theta\beta_j\delta b_N.$$

We shall prove $x_j^0 = 0$ leads to a contradiction. Upon so doing, we then will have proven $\sigma(x) \subseteq \sigma(x^0)$, thus proving $\sigma(x) = \sigma(x^0)$.

The contradiction comes from the meaning of the optimal partition: every optimal solution, say $x^*(\lambda)$, for $r + \lambda h(\lambda \in (0, \theta^*))$, must have $x_j^*(\lambda) = 0 \,\forall j \in \sigma(x^0)$. One such optimal solution is the basic one: $\overline{x}_j + \lambda\beta_j\delta b_N = 0$. Since this must hold for all $\lambda \in (0, \theta^*)$, we must have $\overline{x}_j = 0$ and $\beta_j\delta b_N = 0$, so we reach the contradiction: $x_j = 0$. We have thus considered all cases in proving $\sigma(x) = \sigma(x^0)$. The remaining support set equalities follow in a similar manner. □

The opposite inequality does not hold. The optimal partition can be invariant on $(r,\, r + \theta^* h)$, but the optimal bases at r may have a range far less than θ^*. For example, consider the following LP[1]:

$$min\ x_2 : (x_1, x_2) \geq 0,\ 1 \leq x_1 + x_2 \leq 4,\ -1 \leq x_1 - x_2 \leq 2,\ \theta \leq x_2 \leq 2.$$

[1] The author thanks Tamás Terlaky for pointing this out and Kees Roos for the example.

For $\theta \in [0, 2]$, the interior solution is $(\frac{3}{2}, \theta)$, and the optimal bases correspond to two extreme points, starting with $(1, 0)$ and $(2, 0)$ at $\theta = 0$. No matter which (compatible) basis is used, $\rho^*(h) = 1$, stopped by the turning point at $x_2 = 1$ when $\theta = 1$. Thus, the optimal partition does not change throughout $[r, r + 2h]$, but there is no basis that is optimal at r and at $r + \theta h$ for $\theta > 1$.

THEOREM 4.2 (OPTIMAL PARTITION RANGE). $\Upsilon(h) \geq \rho^*(h)$.

Proof. This is immediate from lemma 4.1. □

The Optimal Partition Range Theorem says that the range of the perturbation for which the (possibly new) partition remains the same is at least as great as the maximum of the ranges of compatibility, taken over all optimal bases. Thus, the associated interval for which $z(r + \theta h) - z(r)$ has constant functional form in θ is the same (linear if h is decoupled).

Using the previous example, there are three optimal bases, as follows, with compatibility conditions following the semi-colon:

$$\begin{array}{llllll} B^1 = [A_{\bullet 1}\ A_{\bullet 2}]: & x^1 = (0,1,0), & \pi^1 = (1,2); & \delta b_1 - \delta b_2 \geq 0, & \delta c_1 - \delta c_2 + \delta c_3 \geq 0. \\ B^2 = [A_{\bullet 1}\ A_{\bullet 3}]: & x^2 = (1,0,1), & \pi^2 = (1,2); & & \delta c_1 - \delta c_2 + \delta c_3 \leq 0. \\ B^3 = [A_{\bullet 2}\ A_{\bullet 3}]: & x^3 = (0,1,0), & \pi^3 = (1,2); & -\delta b_1 + \delta b_2 \geq 0, & \delta c_1 - \delta c_2 + \delta c_3 \geq 0. \end{array}$$

For $\delta b = (-1, 0)$ and $\delta c = (-1, 0, 0)$, only B^2 is compatible, and its range of compatibility is $\rho(B^2; h) = \rho^*(h) = 1$. Thus, the Basic Compatiblity Theorem tells us that $z(r + \theta h) - z(r)$ has constant functional form if we decrease b_1 and c_1, both at unit rate. In particular, we have the following quadratic function for $\theta \in [0, 1]$:

$$\begin{aligned} z(r + \theta h) - z(r) &= (c_B + \theta \delta c_B)[B^2]^{-1}(b_N + \theta \delta b_N) - c_B[B^2]^{-1} b_N \\ &= [1 - \theta,\ 2] \begin{bmatrix} 1 - \theta \\ 1 \end{bmatrix} - 3 \\ &= -2\theta + \theta^2. \end{aligned}$$

The interior point approach gave us the same result, but in a different manner.

From one of the main results of basic compatibilty theory[3], we have the following.

COROLLARY 4.3. *h is admissible if, and only if, $\Upsilon(h) > 0$.*

Proof. This follows from $\mathcal{H} = \{h : \rho^*(h) > 0\}$. □

This corollary says that the set of admissible directions equals the set of directions for which the optimal partition is invariant on the associated open line segment. We now consider another example[3] to help understand economic interpretations, introduce the *Optimal Partition Transition Graph*, and illustrate a form of activity analysis, built on how the optimal partition changes, rather than on how optimal bases change.

There are three fuels from which to generate electricity: coal, oil and uranium. Define six activities, as follows:

PCL... purchase coal	GCL... generate electricity from coal
POL... purchase oil	GOL... generate electricity from oil
PUR... purchase uranium	GUR... generate electricity from uranium

Figure 1 shows the LP. The objective is to minimize cost, shown as the first row, while meeting the required electricity demand, shown as the last row. Rows BCL, BOL and BUR balance the associated fuels: what is purchased must be at least as great as what is used for generation. Row LUR limits the generation from uranium: GUR ≤ 10.

FIG. 1. *Electricity Generation Example*

	Purchase			Generate					
	PCL	POL	PUR	GCL	GOL	GUR			
COST	18	15	20	.8	.6	.4	=	Min	
BCL	1			-1			$\geq$	0	balance coal
BOL		1			-1		$\geq$	0	balance oil
BUR			1			-1	$\geq$	0	balance uranium
LUR						-1	$\geq$	-10	limit nuclear generation
DEL				.33	.3	.4	$\geq$	10	demand electricity

Generation from uranium is the least costly (per unit of electricity generated, including the cost of uranium), so its level is as much as possible, limited to 10 units, which generates 4 units of electricity. The other 6 units are generated from oil, and none is generated from coal. Thus, the levels of PCL and GCL are zero in every optimal solution; however, PCL is in one optimal basis (compatible with increasing the right-hand side), and GCL is in another (compatible with decreasing the right-hand side).

Figure 2 shows the active submatrix, where the optimal partition has $\sigma(x) = \{$POL, PUR, GOL, GUR$\}$ (activities to generate electricity from oil and uranium), and $\sigma(\pi)$ equal to all rows (B^* and N^* are null).

FIG. 2. *Optimal Partition for the Electricity Generation Example*

$$\begin{array}{cc}
 & \sigma(x): \ \text{POL} \quad \text{PUR} \quad \text{GOL} \quad \text{GUR} \\
B = & \left[\begin{array}{cccc} & & & \\ 1 & & -1 & \\ & 1 & & -1 \\ & & & -1 \\ & & .3 & .4 \end{array}\right]
\end{array}
\quad
\begin{array}{c} \sigma(\pi) \\ \text{BCL} \\ \text{BOL} \\ \text{BUR} \\ \text{LUR} \\ \text{DEL} \end{array}
\quad
\begin{array}{cc}
 & \sigma(d): \ \text{PCL} \quad \text{GCL} \\
N = & \left[\begin{array}{cc} 1 & -1 \\ & \\ & \\ & \\ & .33 \end{array}\right]
\end{array}$$

For $\theta > 0$, we increase the right-hand side of the coal balance row (BCL), which corresponds to a *stockpile requirement.* The theory of basic compatibility says that the coal purchase activity (PCL) needs to be in the basis to provide the appropriate reponse:

buy coal. As $\theta < 0$, we are providing free coal, making the cost of electricity generation consist of only the operation and maintenance cost. This is \$.80 per unit of coal, which is \$2.42 per unit of electricity (\$.8 $\div$ .33). Thus, the generation activity (GCL) needs to be in the basis to provide the appropriate response: displace oil-fired generation with coal-fired generation. The displacement continues until all oil-fired generation is displaced, which occurs at $\theta = 18.18$. A view of these is with the *basis transition graph*, shown in figure 3, that is a part of the Theory of Basic Compatibility, which we now extend.

Let $\delta b = -e_1$ (i.e., decrease the right-hand side of row BCL). An interior point approach first considers the differential LP:

$$Max\{-\pi_1 : \pi \in D^*(r)\} = Max\{-\pi_1 : \pi = (p, 15, 20, .4, 52), 16.36 \le p \le 18\} = -16.36.$$

This gives us the new optimal partition for $r - \theta e_1$ with θ sufficiently small. (Our goal is to obtain the greatest value of θ, which defines $\Upsilon(-e_1)$.)

The new optimal partition adds activity GCL to the set of active columns, so the following equations must hold as θ is increased:

$$Bx_B = \begin{bmatrix} & & & & -1 \\ 1 & & -1 & & \\ & 1 & & -1 & \\ & & & -1 & \\ & & .3 & .4 & .33 \end{bmatrix} \begin{bmatrix} x_{POL} \\ x_{PUR} \\ x_{GOL} \\ x_{GUR} \\ x_{GCL} \end{bmatrix} = \begin{bmatrix} -\theta \\ 0 \\ 0 \\ -10 \\ 10 \end{bmatrix}.$$

This gives the following primal conditions that limit θ:

$$\begin{aligned} \Upsilon(-e_1) = \quad & Max\{\, \theta : x \ge 0, \\ & \qquad -x_{GCL} = -\theta \\ & \qquad x_{POL} - x_{GOL} = 0 \\ & \qquad x_{PUR} - x_{GUR} = 0 \\ & \qquad -x_{GUR} = -10 \\ & .33\, x_{GCL} + .3\, x_{GOL} + .4\, x_{GUR} = 10 \,\}. \end{aligned}$$

This reduces to $\Upsilon(-e_1) = Max\{\theta : .33\theta \le 6\} = 18.18$. While this equals the range we obtained from the basis-driven approach, the reasoning is different. At $b_{BCL} = -18.18$, the optimal partition changes again to deactivate oil-fired generation — i.e., exclude activities GOL and POL from the set of active columns. (POL must remain basic in the Theory of Basic Compatibility, even though its level is zero in every optimal solution, in order to have the correct price of oil, π_{BOL}.) Analogous to basic compatibility, the optimal partition changes due to an *event* that makes something change status: from inactive to active, or visa versa.

Whereas figure 3 shows the basis transition graph that was introduced in [3] for varying the amount of coal, figure 4 introduces a *partition transition graph.* Notice that in the basis transition graph, events occur at the threshold, choosing the event that is compatible with the particular variation (left or right transition). By contrast, in the optimal partition transition graph, events occur just on one side of each threshold. At $b_1 = 0$, it is *after* $\theta > 0$ that coal purchases begin (i.e., activity PCL is activated by entering $\sigma(x)$). Similarly, it is *after* $b_1 < 0$ that coal-fired generation begins (i.e., activity GCL is activitated). As we continue to move to the left, the optimal partition remains invariant on the open interval, $(-18.18, 0)$. At the threshold, all of the oil is displaced by coal, so the optimal partition changes at $r - 18.18e_1$. It is just *before* this change that the event occurs: deactivate POL and GOL. Then, the optimal partition is invariant on the half-open interval: $\theta \in [18.18, 30.3) \implies \sigma(x) = \{\text{PUR, GCL, GUR}\}$ for $r - \theta e_1$.

FIG. 3. *Basis Transition Graph*

		prepare for coal surplus $\longleftarrow$		prepare to displace uranium $\longleftarrow$		prepare to displace oil $\longleftarrow$	
b_1:		-30.3		-18.18		0	
		$\swarrow\ \searrow$		$\swarrow\ \searrow$		$\swarrow\ \searrow$	
Basis:	GCL		GCL		GCL	———	PCL
	POL		POL		POL		POL
	PUR		PUR		PUR		PUR
	LUR		LUR	———	GOL		GOL
	BCL	———	GUR		GUR		GUR
		$\longrightarrow$ prepare to generate from uranium		$\longrightarrow$ prepare to generate from oil		$\longrightarrow$ prepare to purchase coal	

This view of events that *activate* or *deactive* activities just before or after the threshold where the optimal partition changes complements the basic view that describes which basis is a compatible one in terms of events that *prepare* for the movement away from the threshold. Of course, phrases like "just before" and "just after" are not mathematical, but the idea is to gain insight from the solution, and this distinction in the transition graphs does provide an added vantage, based on the underlying events.

5. Summary. Here is a summary of the main points:

- The new optimal partition is obtained by solving two differential linear programs, one over the primal optimality region, the other over the dual. The new set of active columns equals that of the primal differential LP; the new set of active rows equals that of the dual differential LP.
- The interval for which the objective value has constant functional form, obtained from the range of the (possibly new) optimal partition, is the same as the interval obtained from the range of compatible bases.

FIG. 4. *Optimal Partition Transition Graph*

		deactivate PUR, GUR to stop nuclear generation (dispaced) ⟵		deactivate POL, GOL to stop oil-fired generation (displaced) ⟵		activate GCL to begin coal-fired generation (displace oil) ⟵		deactivate PCL to stop coal purchases (not req.) ⟵	
b_1:	−30.3		−18.18				0		
$\sigma(x)$:									PCL
					POL		POL		POL
			PUR		PUR		PUR		PUR
	GCL		GCL		GCL				
					GOL		GOL		GOL
			GUR		GUR		GUR		GUR
$\sigma(s)$:	BCL								
		⟶ activate PUR, GUR to begin nuclear generation (displace coal)		⟶ activate POL, GOL to begin oil-fired generation (displace coal)		⟶ deactivate GCL to stop coal-fired generation (displaced)		⟶ activate PCL to begin coal purchases (required)	

- The optimal partition transition graph, which shows threshold events when the optimal partion changes, provides another visualization of the underlying economics.

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