Domination of Graphs with Maximum Degree Three

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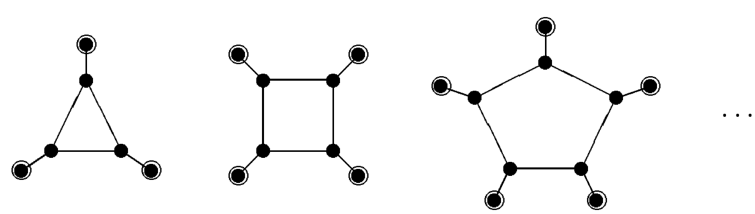
Abstract

We show that a graph with n nodes, e edges, i isolated nodes, and maximum degree 3 or less has domination number at most $\frac{1}{4}(3n - e + i)$. This overlaps a result of Reed that an n-node graph with minimum degree 3 has domination number at most $\frac{3}{8}n$.

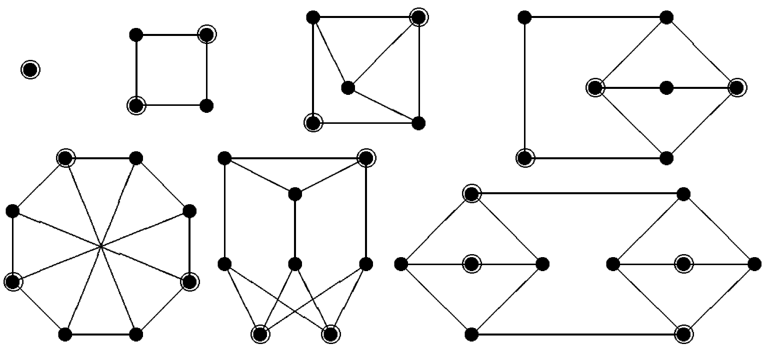
Let G be a graph. Let $N(v)$ be the neighbors of a node v and $N[v] = N(v) \cup \{v\}$. Let $d(v) = |N(v)|$ be the *degree* of v. Let δ and Δ be the minimum and maximum of $d(v)$. If $d(v) = k$, we say "v is a k-node" or "v is a k-neighbor" of a neighboring node. Thus "0-nodes" are isolated nodes.

Let $N[S]$ be the union of the closed neighborhoods of nodes in a set S. A *domination* is a set D with every node in $N[D]$. The *domination number* γ is the minimum size of a domination of G.

We show that if $\Delta \leq 3$, then $\gamma \leq \frac{1}{4}(3n - e + i)$ where n, e and i are the number of nodes, edges, and 0-nodes in G. This is exact for an infinite family of graphs (circles show a minimal domination):



We know seven other connected graphs with $\gamma = \frac{1}{4}(3n - e + i)$.



Any graph whose components are the above graphs also has $\gamma = \frac{1}{4}(3n - e + i)$.

Since cubic graphs ($\delta = \Delta = 3$) have $\frac{3}{2}n$ edges, a corollary is $\gamma \le \frac{3}{8}n = 0.375n$ for cubic graphs. *How does this compare to other bounds?* Arnautov [**1**] analyzed a greedy algorithm to prove that $\gamma \le \left(1 + \frac{1}{2} + \cdots + \frac{1}{\delta+1}\right)\frac{n}{\delta+1}$ which is $\gamma \le \frac{25}{48}n \approx 0.528333n$ for $\delta = 3$. A more precise analysis (from Clark, Fisher, Shekhtman, and Suen [**2**]) showed $\gamma \le \left(1 - [(1 + \frac{1}{\delta})(1 + \frac{1}{2\delta})(1 + \frac{1}{3\delta}) \cdots (1 + \frac{1}{\delta(\delta+1)})]^{-1}\right) n$. For $\delta = 3$, this is $\gamma \le \frac{212}{455}n \approx 0.465934n$. They also give a better bound for regular graphs ($\delta = \Delta$) which gives $\gamma \le \frac{23}{50}n = 0.46n$ for cubic graphs.

Other bounds are based on an argument of Ore [**6**]: *If D is a minimal domination of a graph without 0-nodes, then D^c is also a domination. So $\gamma \le \frac{1}{2}n = 0.5n$ for any graph without 0-nodes.* Flach and Volkmann [**3**] improved this. Let P be the maximum size of a set of nodes with disjoint closed neighborhoods (P is the "packing" or "strong independence" number). They showed that connected graphs, with the exception of the 4-cycle, have $\gamma \le \frac{1}{2}[n - (\delta - 1)P]$. Since $P \ge \frac{1}{10}n$ when $\Delta \le 3$, this gives $\gamma \le \frac{2}{5}n = 0.4n$ for cubic graphs. McCuaig and Shepherd [**5**] showed that if G is a connected graph with $\delta \ge 2$, then $\gamma < \frac{2}{5}n$ unless G is in a finite family. Reed [**7**] proved, using a partition of the nodes into paths, that $\gamma \le \frac{3}{8}n$ for graphs with $\delta \ge 3$.

Our result is similar to a result from Fraughnaugh [4] that the independence number (the maximum size of a set of nonadjacent nodes) of a triangle-free graph with $\Delta \le 3$ is at least $\frac{1}{28}(13n - 2e)$.

Theorem. *A graph G with n nodes, e edges, i 0-nodes, and maximum degree 3 or less has domination number at most $\frac{1}{4}(3n - e + i)$.*

Proof. The proof is by induction. Clearly K_1 satisfies the theorem. Let n', e', and i' be the number of nodes, edges, and 0-nodes of a subgraph G' of G. Let γ' be its domination number. We assume that $\gamma' \leq \frac{1}{4}(3n' - e' + i')$ whenever $n' < n$. Let $\hat{n} = n - n'$, $\hat{e} = e - e'$, $\hat{\imath} = i' - i$, and $\hat{\gamma} = \gamma - \gamma'$. Then

$$\gamma \leq \frac{1}{4}(3n' - e' + i') + \hat{\gamma} = \frac{1}{4}(3n - e + i) + \frac{1}{4}(4\hat{\gamma} - 3\hat{n} + \hat{e} + \hat{\imath}).$$

Thus G satisfies the theorem if $\sigma \equiv 4\hat{\gamma} - 3\hat{n} + \hat{e} + \hat{\imath} \leq 0$ for a smaller graph G'. Note that if $G = G - G'$, then $\sigma \leq 0$ is equivalent to $\gamma \leq \frac{1}{4}(3n - e + i)$.

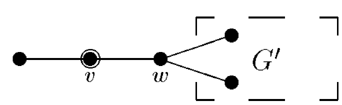
Diagrams show the cases where σ is largest – not all edges into G' necessarily exist or actually go into G'. However, the arguments are sufficiently general to account for all possibilities. Unless stated otherwise, let $G' = G - N[v]$ for node v.

Case 1: $\delta = 0$. If v is a 0-node, then $\sigma = 4 \cdot 1 - 3 \cdot 1 + 0 + (-1) = 0$.

Case 2: $\delta = 1$.

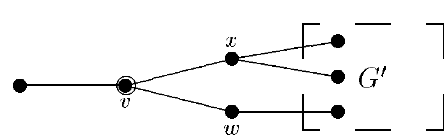
Case 2a: *Two 1-nodes are adjacent.* The 1-nodes form a component K_2. Let $G' = G - K_2$. Then $\sigma = 4 \cdot 1 - 3 \cdot 2 + 1 + 0 = -1$.

Case 2b: *A 1-node has a 2-neighbor.*



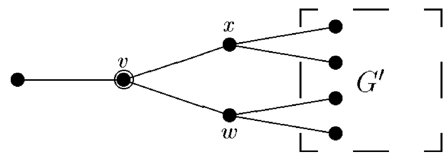
Here $\hat{n} = 3$, $\hat{e} = d(w) + 1$, and $\hat{\gamma} \leq 1$ (because adding v to a minimum domination of G' gives a domination of G). Since the only potential 0-nodes in G' are w's neighbors, $\hat{\imath} \leq d(w) - 1$. So $\sigma \leq 4 \cdot 1 - 3 \cdot 3 + (d(w) + 1) + (d(w) - 1) = 2d(w) - 5$. This is nonpositive unless $d(w) = 3$ and $\hat{\imath} = 2$. In this case, $N[v, w]$ is a component. Then let $G' = G - N[v, w]$, giving $\sigma \leq 4 \cdot 2 - 3 \cdot 5 + 4 + 0 = -3$.

Case 2c: *Every 1-node has a 3-neighbor and some 1-node has a 3-neighbor that does not have two 3-neighbors.*

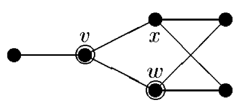


Since w cannot have a 1-neighbor, every 0-node of G' must be a neighbor of x. So $\hat{\imath} \le d(x) - 1$. Thus $\sigma \le 4 \cdot 1 - 3 \cdot 4 + (d(w) + d(x) + 1) + (d(x) - 1) = d(w) + 2d(x) - 8 \le 0$.

Case 2d: *Every 1-node has a 3-neighbor, every 3-node with a 1-neighbor has two 3-neighbors, and some 3-node with a 1-neighbor is adjacent to a 3-node without a 1-neighbor.*



If $\hat{\imath} \le 1$, then $\sigma \le 4 \cdot 1 - 3 \cdot 4 + 7 + 1 = 0$. If $\hat{\imath} = 2$, then either (1) both 0-nodes of G' are adjacent to w and x, or (2) one is adjacent to w and x, and the other is adjacent to only one of w or x (say w).



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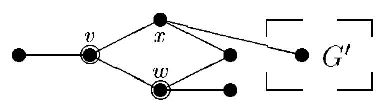
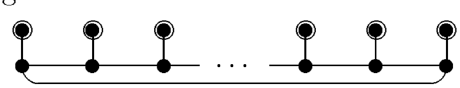


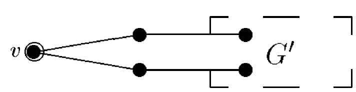
Case 2e: *Every 1-node has a 3-neighbor, and each such 3-node has two 3-neighbors each of which has a 1-neighbor.* Then the induced subgraph on 3-nodes with 1-neighbors is one or more cycles. Let H be the component consisting of a k-cycle of such 3-nodes with their associated 1-neighbors.



Let $G' = G - H$. Then $\sigma = 4k - 3 \cdot 2k + 2k + 0 = 0$.

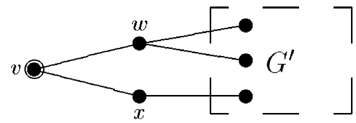
Case 3: $\delta = 2$.

Case 3a: *Some 2-node has two 2-neighbors.*



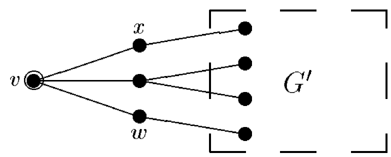
Since the only possible 0-nodes of G' have degree 2 or 3, we have $\hat{\imath} \le 1$. So $\sigma \le 4 \cdot 1 - 3 \cdot 3 + 4 + 1 = 0$.

Case 3b: *No 2-node has two 2-neighbors, but some 2-node has one 2-neighbor.*



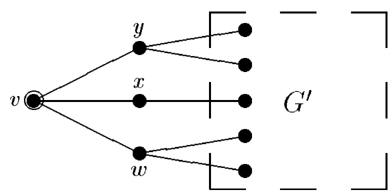
If x has a neighbor in G', it must have degree 3, because a 2-node cannot have two 2-neighbors. So G' has no 0-nodes. Thus $\sigma \leq 4 \cdot 1 - 3 \cdot 3 + 5 + 0 = 0$.

Case 3c: *The 2-nodes are independent, and some 3-node has two or more 2-neighbors.*

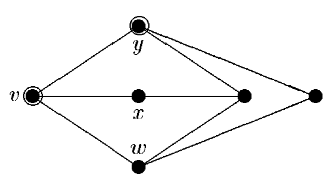


Since the 2-nodes are independent, w and x have only 3-neighbors. Thus G' has at most one 0-node giving $\sigma \leq 4 \cdot 1 - 3 \cdot 4 + 7 + 1 = 0$.

Case 3d: *The 2-nodes are independent, and every 3-node has at most one 2-neighbor.* Let v have one 2-neighbor x and two 3-neighbors w and y.

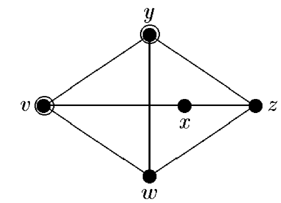


Since at most five edges leave $N[v]$, we have $\hat{\imath} \leq 2$. If $\hat{\imath} = 0$, then $\sigma \leq 4 \cdot 1 - 3 \cdot 4 + 8 + 0 = 0$. If $\hat{\imath} = 2$, then $N[v, y]$ is a component.



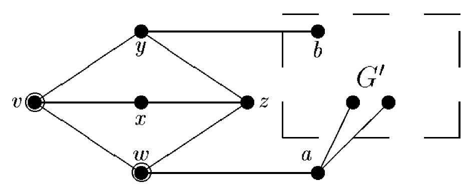
Let $G' = G - N[v, y]$. Then $\sigma = 4 \cdot 2 - 3 \cdot 6 + 8 + 0 = -2$.

Otherwise, there is exactly one 0-node z in $G - N[v]$. First assume z is adjacent to x. Then $d(z) = 3$ as we cannot have adjacent 2-nodes. If yw is an edge, then $N[v, y]$ is a component.

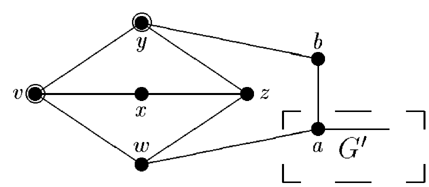


Let $G' = G - N[v, y]$. Then $\sigma = 4 \cdot 2 - 3 \cdot 5 + 7 + 0 = 0$.

If yw is not an edge, then

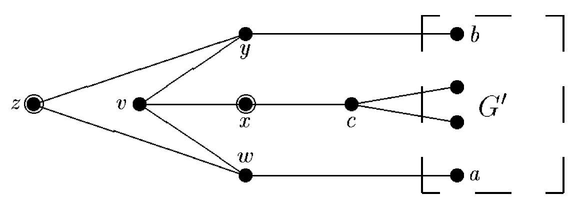


Let $G' = G - N[v, w]$. Then $\sigma \le 4 \cdot 2 - 3 \cdot 6 + 10 + \hat{\imath} = \hat{\imath} \le 1$. If $\hat{\imath} = 1$, then ab is an edge, $d(b) = 2$, and $d(a) = 3$ because 2-nodes cannot be adjacent.

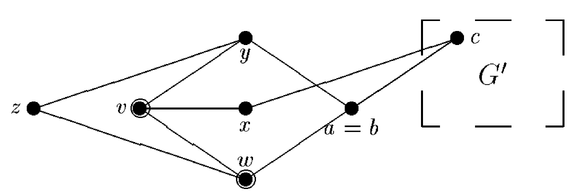


Let $G' = G - N[v, y]$. Then $\sigma \le 4 \cdot 2 - 3 \cdot 6 + 9 + 0 \le 0$.

Otherwise z is not adjacent to x.

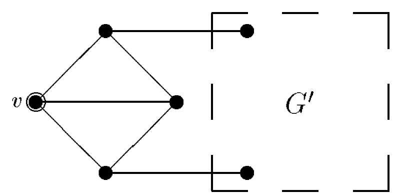


Let $G' = G - N[x, z]$. Then $\sigma \le 4 \cdot 2 - 3 \cdot 6 + 10 + \hat{\imath} = \hat{\imath}$. Since c, y, and w already have 2-neighbors, the only possible 0-node in G' would have degree 3 and be adjacent to w, y, and c.



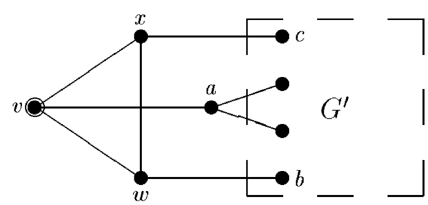
Redefine $G' = G - N[v, w]$. Since $d(c) = 3$, $\hat{\imath}$=0 and $\sigma \leq 4 \cdot 2 - 3 \cdot 6 + 9 + 0 = -1$.

Case 4: *G is a cubic graph with a 3-cycle.* If two 3-cycles share an edge, then



Since $\delta = 3$, there are no 0-nodes in G'. So $\sigma \leq 4{\cdot}1 - 3{\cdot}4 + 7 + 0 = -1$.

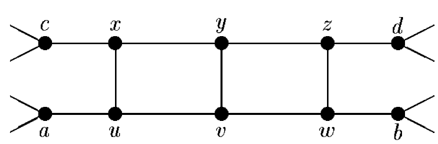
Otherwise all 3-cycles are disjoint.



Since $\delta = 3$ and $b \neq c$ (or else bwx and vwx would be 3-cycles that share an edge), $\hat{\imath} = 0$. So $\sigma \leq 4 \cdot 1 - 3 \cdot 4 + 8 + 0 = 0$.

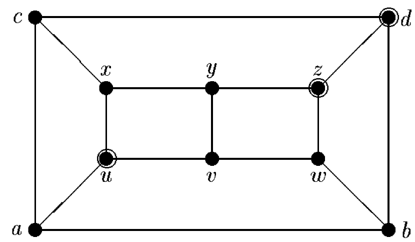
Case 5: *G is a cubic graph without 3-cycles but with a 4-cycle.*

Case 5a: *Two 4-cycles share exactly one edge.*

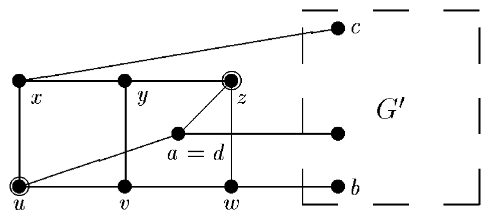


Since G has no 3-cycles, neither uw nor xz are edges. If wx and uz are both edges, then $N[u, x] = K_{3,3}$ is a component. Let $G' = G - N[u, x]$. Then $\sigma = 4 \cdot 2 - 3 \cdot 6 + 9 + 0 = -1$.

Otherwise one of wx or uz is not an edge (say uz is not an edge). Let $G' = G - N[u, z]$. If $a \neq d$, then $\sigma \leq 4 \cdot 2 - 3 \cdot 8 + 15 + \hat{\imath} = -1 + \hat{\imath}$. If $\hat{\imath} \leq 1$, then $\sigma \leq 0$. If $\hat{\imath} = 2$, then (since $\delta = 3$) one 0-node is adjacent to w, a, and d, and the other is adjacent to x, a, and d. So a component H is

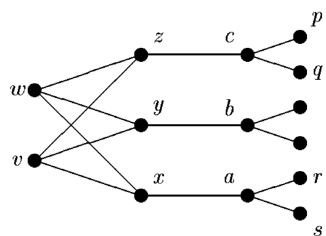


Let $G' = G - H$. Then $\sigma = 4\cdot 3 - 3\cdot 10 + 15 + 0 = -3$. If $a = d$, then



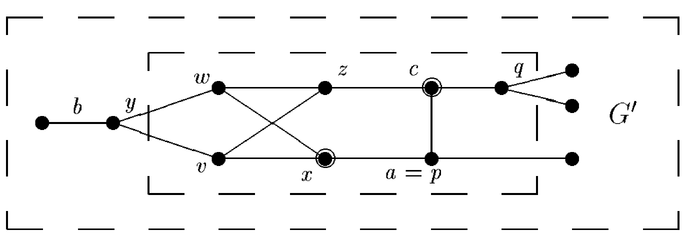
Here $G' = G - N[u, z]$ and $\sigma \leq 4\cdot 2 - 3\cdot 7 + 12 + 1 = 0$.

Case 5b: *No two 4-cycles share exactly one edge, but two 4-cycles share two edges.*

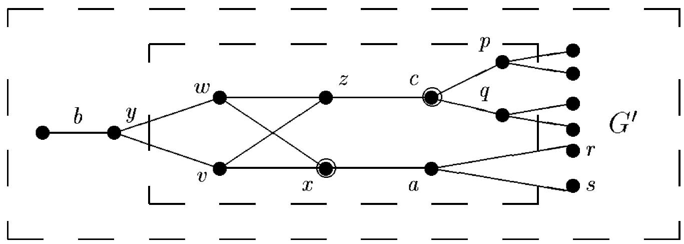


Since G has no 3-cycles, x, y, and z are independent. Also $a \neq b$, else $axvy$ and $vywz$ are 4-cycles that share exactly one edge. Similarly $a \neq c$ and $b \neq c$. Let $G' = G - N[x, c]$.

First suppose $a = p$.

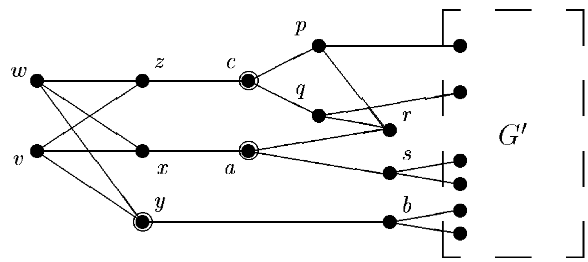


Then $\sigma \leq 4\cdot 2 - 3\cdot 7 + 13 + \hat{\imath} = \hat{\imath}$. Since at most five edges leave $N[x, c]$, we have that $\hat{\imath} \leq 1$. If $\hat{\imath} = 0$, then $\sigma \leq 0$. Otherwise, since $a \neq b$ and G is cubic, y is adjacent to q, i.e., $b = q$. Redefine $G' = G - N[c, y, a]$. Then $\sigma = 4\cdot 3 - 3\cdot 9 + 15 + 0 = 0$. So $a \neq p$. Similarly $a \neq q$. By symmetry $b \neq p$, $b \neq q$, $b \neq r$, and $b \neq s$.



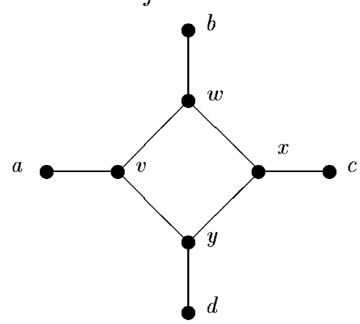
Then $\sigma \leq 4 \cdot 2 - 3 \cdot 8 + 16 + \hat{\imath} = \hat{\imath}$. Since at most eight edges leave $N[x, c]$, we have that $\hat{\imath} \leq 2$. Since $a \neq b$, $b \neq p$, and $b \neq q$, we have that y cannot be a 0-node in G'. If $\hat{\imath} = 2$, then r and s are 0-nodes in G', and both are adjacent to p and q (and a). Then $spcq$ and $spra$ are 4-cycles that share exactly one edge.

If $\hat{\imath} = 1$, but either ap or aq is an edge, then $\hat{e} \leq 15$ and $\sigma \leq 0$. So neither ap nor aq is an edge. Since y is not a 0-node in G', let r be the 0-node in G'. Then $s \neq p$ else there is a 3-cycle. Since $b \neq s$, we have

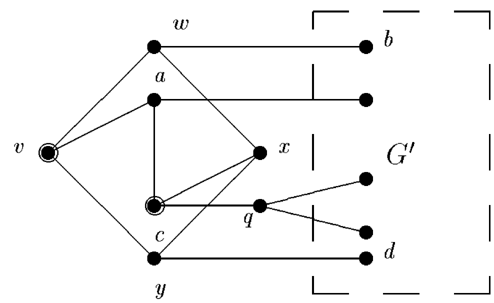


Let $G' = G - N[a, c, y]$. Then since at most six edges leave $N[a, c, y]$, $\hat{\imath} \leq 2$. Thus $\sigma \leq 4 \cdot 3 - 3 \cdot 12 + 21 + 2 = -1$.

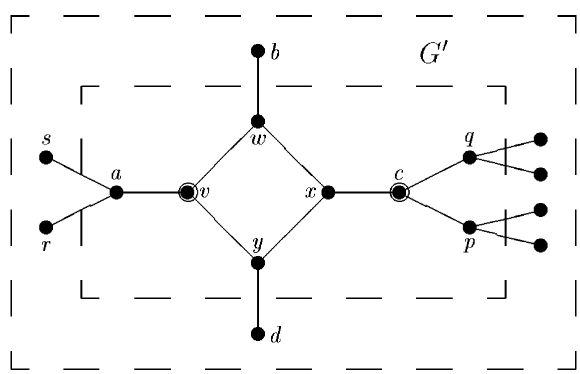
Case 5c: *The 4-cycles are disjoint.*



Since there are no 3-cycles and since no 4-cycle can share an edge with $vwxy$, the labeled nodes are distinct and the only possible edges between the labeled nodes are ac and bd. Let $G' = G - N[v, c]$. If ac exists, then

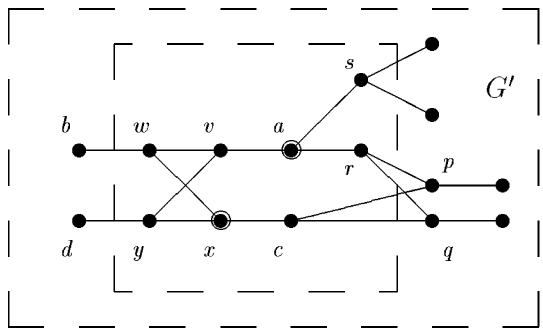


If G' has a 0-node, it must be b or d. However, neither can be a 0-node in G' because $b \neq d$, and ad and ab do not exist. Thus $\sigma \leq 4 \cdot 2 - 3 \cdot 7 + 13 + 0 = 0$. So ac is not an edge. By a symmetric argument, bd is not an edge.



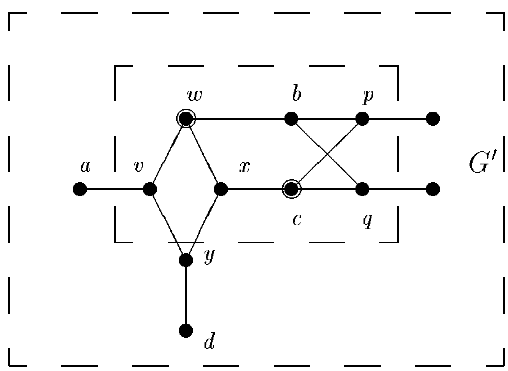
If G' has no 0-nodes, then $\sigma \leq 4 \cdot 2 - 3 \cdot 8 + 16 + 0 = 0$. Suppose G' has at least one 0-node. Then, without loss of generality, either r or b is a 0-node in G'.

If r is a 0-node, then since neither ab nor ad exists, rp and rq are edges. Redefine $G' = G - N[x, a]$.



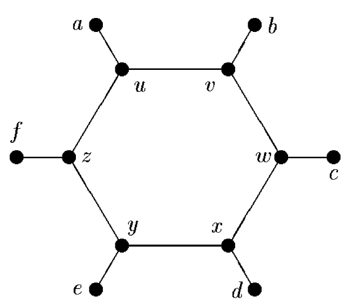
Since b, d, p, and q are all distinct vertices (else there are adjacent 4-cycles), there can be no 0-node in G'. Thus $\sigma \leq 4{\cdot}2-3{\cdot}8+16+0 = 0$.

If b is a 0-node, then bp and bq are edges, because $b \neq d$ and ab is not an edge. Redefine $G' = G - N[w, c]$.

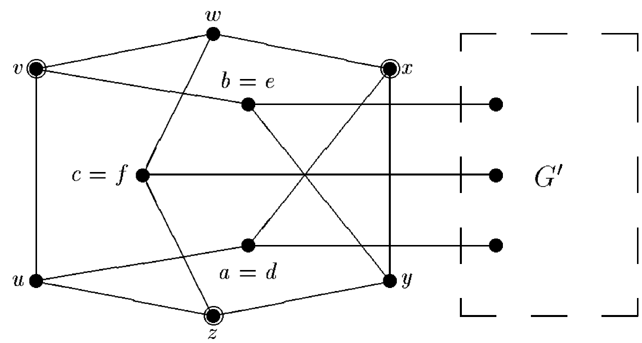


If p and q share a third neighbor, then two 4-cycles share edges. So G' has no 0-node. Then $\sigma \leq 4 \cdot 2 - 3 \cdot 7 + 13 + 0 = 0$.

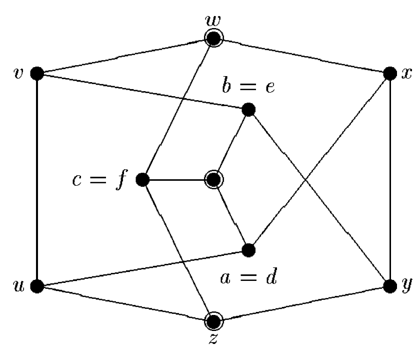
Case 6: *G is a cubic graph without 3-cycles or 4-cycles, but with a 6-cycle.*



If $a = d$, $b = e$, and $c = f$, then

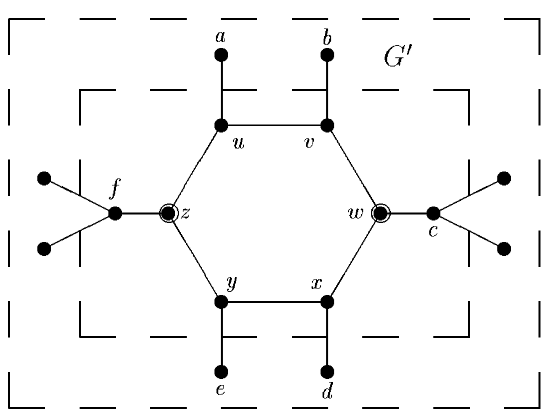


If $\hat{\imath} = 0$, then $\sigma \leq 4 \cdot 3 - 3 \cdot 9 + 15 + 0 = 0$. If $\hat{\imath} = 1$, then a component H is



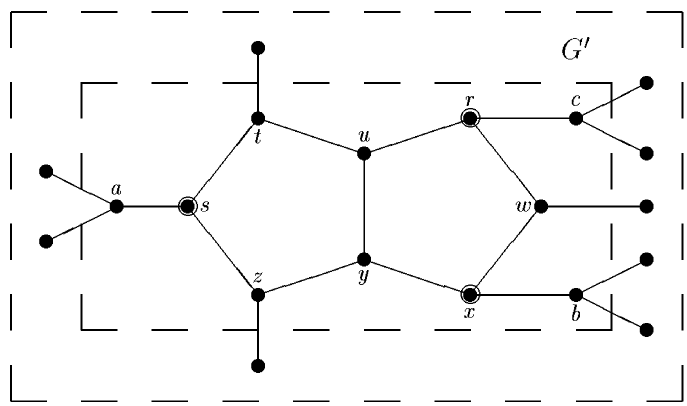
Let $G' = G - H$. Then $\sigma = 4 \cdot 3 - 3 \cdot 10 + 15 + 0 = -3$.

Otherwise $a \neq d$, $b \neq e$, or $c \neq f$. Without loss of generality, assume $c \neq f$. Then



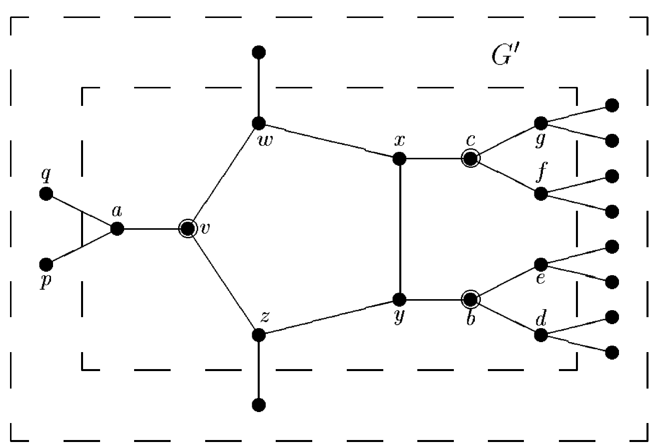
Since G is cubic, every 0-node of G' is adjacent to three nodes of $G - G'$. However, this is impossible without forming a 3-cycle or a 4-cycle. So $\hat{\imath} = 0$ and $\sigma \leq 4 \cdot 2 - 3 \cdot 8 + 16 + 0 = 0$.

Case 7: *G is a cubic graph without 3-cycles, 4-cycles, or 6-cycles, but with a 5-cycle.* If two 5-cycles share three edges, the noncommon edges form a 4-cycle. If two 5-cycles share two edges, the noncommon edges form a 6-cycle. So the only way two 5-cycles can share an edge is if they share only one edge. Let G' be as shown.

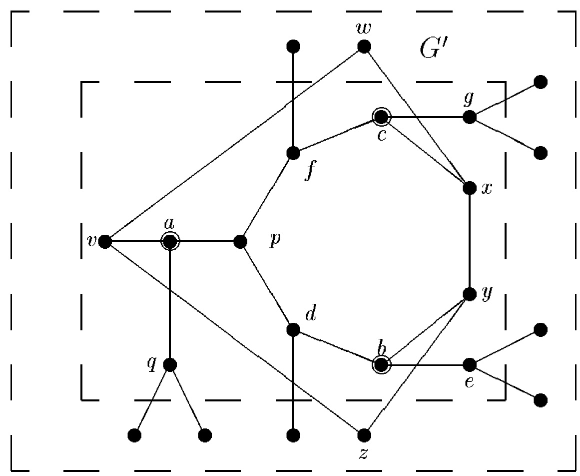


The labeled nodes are distinct, for otherwise we would have a 3-cycle, 4-cycle, or 6-cycle. Further, any node of G can be a neighbor of only one of c, w, or b, and only one of z, a, or t. Since G is cubic, G' does not have 0-nodes. So $\sigma \leq 4 \cdot 3 - 3 \cdot 11 + 21 + 0 = 0$.

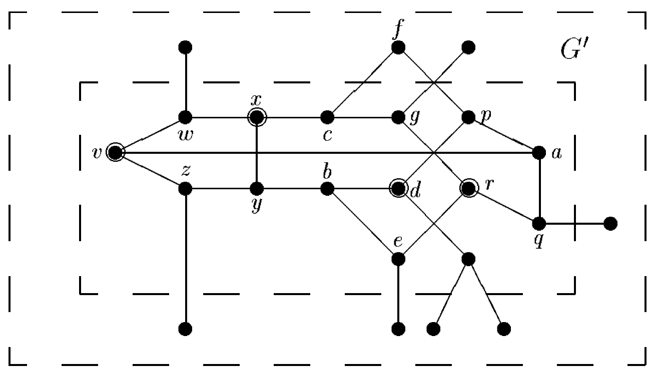
Otherwise, the 5-cycles are disjoint. Let G' be as shown.



The labeled nodes are distinct, for otherwise there would be a 3-cycle, 4-cycle, 6-cycle, or adjacent 5-cycles. If $\hat{\imath} = 0$, then $\sigma \leq 4 \cdot 3 - 3 \cdot 12 + 24 + 0 = 0$. Otherwise, the only way for there to be a 0-node in G' without forming a 3-cycle, 4-cycle, or 6-cycle is for that node to be adjacent to a, either d or e, and either f or g. Without loss of generality, assume p is adjacent to a, d and f. Redefine G' as follows.

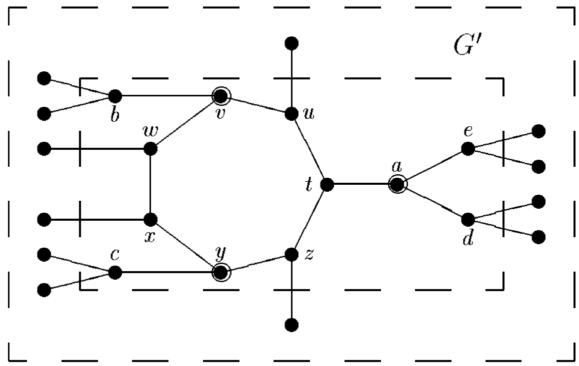


If $\hat{\imath} = 0$, then $\sigma \leq 4 \cdot 3 - 3 \cdot 12 + 24 + 0 = 0$. If $\hat{\imath} > 0$, the only way for G' to have a 0-node r (without making a 3-cycle, 4-cycle, 6-cycle, or adjacent 5-cycles) is for r to be adjacent to e, g and q. So again redefine G'.

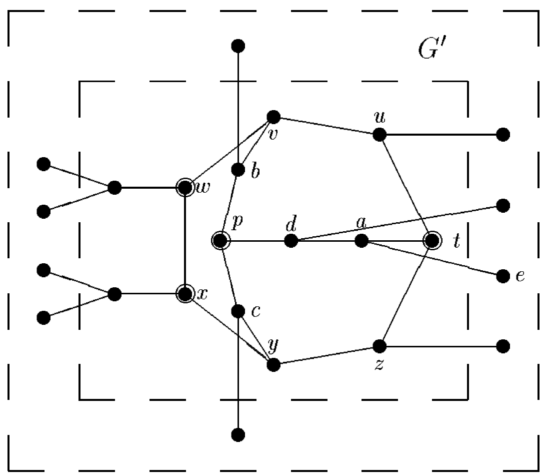


Since G' cannot have a 0-node without forming a 3-cycle, 4-cycle or 6-cycle, $\sigma \leq 4 \cdot 4 - 3 \cdot 15 + 27 + 0 = -2$.

Case 8: *G is a cubic graph without 3-cycles, 4-cycles, 5-cycles, or 6-cycles but with a 7-cycle.*

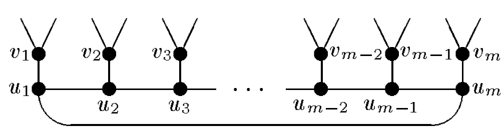


If $\hat{\imath} = 0$, then $\sigma \leq 4 \cdot 3 - 3 \cdot 12 + 24 + 0 = 0$. If $\hat{\imath} > 0$, then the only way (without forming a k-cycle with $k < 7$) for G' to have a 0-node p is for p to be adjacent to b, c and either d or e (say it is adjacent to d). Then



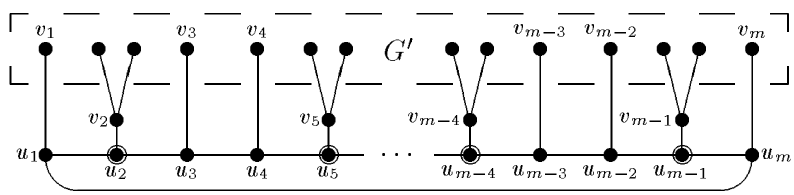
No 0-nodes can occur in G' without forming a 3-cycle, 4-cycle, 5-cycle, or 6-cycle. So $\hat{\imath} = 0$ and $\sigma \leq 4 \cdot 4 - 3 \cdot 14 + 26 + 0 = 0$.

Case 9: *G is a cubic graph with no cycles on fewer than 8 nodes.* Let $u_1 u_2 u_3 \cdots u_m$ be a minimum length cycle in G. For $1 \leq i \leq m$, let v_i be the neighbor of u_i that is not in the cycle.

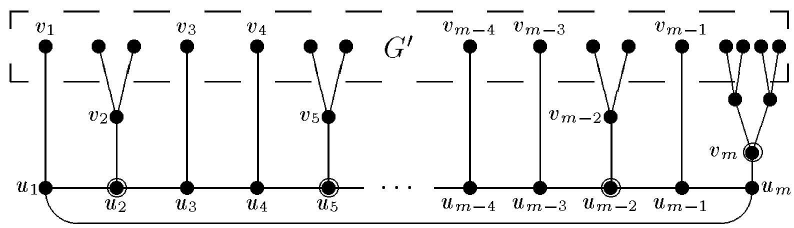


The v_i's are distinct and independent for otherwise there would be a cycle of length at most $\frac{m}{2}+3$ which is less than m for $m \geq 8$.

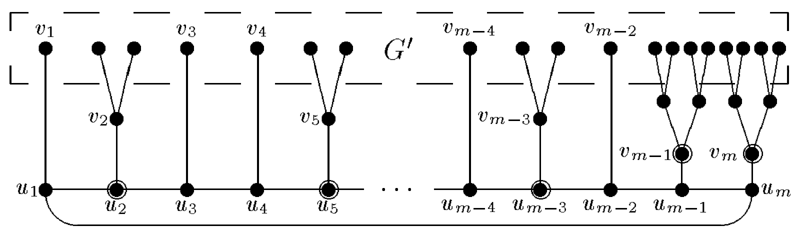
If $m \equiv 0 \pmod 3$, let $G' = G - N[u_2, u_5, \ldots, u_{m-1}]$.



If $m \equiv 1 \pmod 3$, let $G' = G - N[u_2, u_5, \ldots, u_{m-2}, v_m]$.



If $m \equiv 2 \pmod 3$, let $G' = G - N[u_2, u_5, \ldots, u_{m-3}, v_{m-1}, v_m]$.



In any of these cases, suppose G' has a 0-node. Then it is adjacent to three nodes of $G - G'$ of which only one (unless we form a 4-cycle or 7-cycle) can be distance two from the m-cycle. So the lengths of the three cycles that include 0-nodes sum to at most $m+2\cdot3+2\cdot2+2\cdot2 = m+14$. Thus one of these cycles has length at most $\frac{m+14}{3}$ which is less than m. So $\hat{\imath} = 0$. Then

$$\sigma \leq \left\{ \begin{array}{ll} 4\frac{m}{3} - 3\frac{4m}{3} + \frac{8m}{3} + 0 & \text{if } m \equiv 0 \pmod 3 \\ 4\frac{m+2}{3} - 3\frac{4(m+2)}{3} + \frac{8(m+2)}{3} + 0 & \text{if } m \equiv 1 \pmod 3 \\ 4\frac{m+4}{3} - 3\frac{4(m+4)}{3} + \frac{8(m+4)}{3} + 0 & \text{if } m \equiv 2 \pmod 3 \end{array} \right\} = 0. \ \square$$

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