

Generalized Forchheimer Equation for Two-Phase Flow Based on Hybrid Mixture Theory

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Abstract

In this paper, we derive a Forchheimer-type equation for two-phase flow through an isotropic porous medium using hybrid mixture theory. Hybrid mixture theory consists of classical mixture theory applied to a multiphase system with volume averaged equations. It applies to media in which the characteristic length of each phase is "small" relative to the extent of the mixture. The derivation of a Forchheimer equation for single phase flow has been obtained elsewhere. These results are extended to include multiphase swelling materials which have non-negligible interfacial thermodynamic properties.

Key words. Porous media, swelling porous media, high velocity flow, non-Darcy flow, two-phase flow, multi-phase flow, mixture theory, Forchheimer equation.

1 Introduction

Darcy-type equations are used to describe the flow of a single-phase fluid through porous media in a number of situations. The classical Darcy equation, first derived experimentally in 1856, states that the flux is proportional to the pressure gradient, i.e.

$$\varepsilon \mathbf{v} = -\mathbf{K}(\nabla p - \rho \mathbf{g}), \quad (1)$$

where \mathbf{v} is the fluid velocity, \mathbf{K} is the permeability of the porous material, ε is the volume fraction of the fluid, p is the pressure in the fluid, ρ is the mass density of the fluid, and \mathbf{g} is the gravity vector. In this simple form, Darcy's equation applies to a fluid flowing at low velocity through such nonswelling media as sandstone or granular media. On the other hand, even if we consider a fluid flowing at a very low velocity through a granular medium, there are situations, such as when the fluid is non-Newtonian, the medium is heterogeneous, or when molecular surface forces or electrochemical reactions are present, where (1) is not valid. For example, deviations due to molecular surface forces are significant for gas flow when the dimensions of the pores are commensurate with, or smaller than, the molecular mean free paths of the flowing gas; while the ones due to electrochemical reactions may arise when ionic exchange occurs [51].

Recently, equation (1) has been extended in a systematic way (non-heuristically) for low velocity flows of a Newtonian fluid in a swelling medium. If there are strong interactions between the fluid phase and solid, such as occurs in montmorillonite clays, the above form of Darcy's equation no longer applies [38, 39, 22]. Hybrid mixture theory can be used to derive an alternative form [2, 42], where an interactive potential term appears. In terms of the Helmholtz potential, A , this generalized Darcy's equation is given by

$$\mathbf{v} = -\mathbf{K}(\varepsilon) \left(\varepsilon \nabla p - \varepsilon \rho \mathbf{g} + \varepsilon \rho \frac{\partial A}{\partial \varepsilon} \nabla \varepsilon \right), \quad (2)$$

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where the added term may be significant for fluids whose behavior is affected by the proximity of the solid phase, e.g. when the swelling pressure is significant. Note that when $\mathbf{K}(\varepsilon) = \mathbf{K}/\varepsilon^2$ we recover equation (1) exactly.

Most recently, it has been suggested that at low moisture content, the behavior of fluid in a swelling medium can also be influenced by the *macroscopic strain* of the solid phase, whose precise definition is given in Section 2. We have in mind a medium composed of disjoint solid minerals such as montmorillonite or bentonite which are composed of thin flat platelets separated by less than five layers of water. In this approach, it is hypothesized that interstitial fluid can support a shearing force, i.e. that the energy of the fluid is affected by the shearing of the solid phase [11, 18]. As defined by hybrid mixture theory, the *macroscale* solid strain tensor is not the average of individual solid strains, but it reflects the geometry of the solid phase. Thus, for example, in the case of montmorillonite clay, shearing one platelet relative to the other changes the microscopic strain of each platelet very little, but it changes the macroscale solid strain tensor significantly. Assuming the energy of the absorbed water is a function of the solid strain tensor, \mathbf{E} , automatically incorporates the above mentioned case, where the fluid energy is a function of only the volume fraction (for a direct calculation of the reduction to the simpler case, see [43]). An application of hybrid mixture theory to the case just mentioned yields the following Darcy equation [11, 18]:

$$\mathbf{v} = -\mathbf{K}(\mathbf{E}) (\varepsilon \nabla p - \varepsilon \rho \mathbf{g} + \varepsilon \rho \frac{\partial A}{\partial \mathbf{E}} : \nabla \mathbf{E}). \quad (3)$$

Note that a gradient of the solid strain matrix induces flow, indicating that fluid will flow in such a manner as to reduce the internal energy (strain) of the solid matrix.

We are interested in studying how the form of the Darcy type equations presented above should be modified to cover the higher velocity case. Experimental evidence shows that when the velocity of the fluid is high enough, a nonlinear relationship develops between the flux and pressure gradient, even for a Newtonian fluid flowing through a granular medium. To be more specific about when these non-linear effects occur, consider the four distinct regions of flow as described by Wright ([63]) who used air and well-rounded gravel, and water and well-rounded coarse sand for his experiments: (1) *Laminar* regime, where viscous forces predominate, the micro-velocity is stationary, and the "head loss" is directly proportional to the velocity. (2) *Steady inertial* flow regime, where the micro-velocity is still stationary but the "head loss" ceases to vary linearly with the velocity. In this regime the viscous forces and inertial actions both play a role "causing the faster moving filaments to fan out near grain walls and create secondary motions". (3) *Turbulent transition* regime, where the inertial phenomena are prevalent and vortices are shed at regular intervals from individual grains. The micro-velocity fluctuates at any experimental sample point in the pores of the bed, with increasing but regular frequency. And (4) *turbulent* regime, which is an unsteady and chaotic flow regime qualitatively resembling turbulent flow, where the micro-velocity fluctuates randomly about a mean. With reference to the above "classification", we are interested in the first change of flow, where the velocity is high enough to produce nonlinearity but is still stationary. We will refer to this regime as nonlinear laminar flow.

To model single-phase flow in this nonlinear laminar regime, Dupuit (1863, [20]) and Forchheimer (1901, [23]) suggested modifying Darcy's equation by adding a quadratic term in the bulk velocity with the coefficient depending on the geometry of the pores. This modification, known as the Forchheimer equation, has been the focus of many investigations, in which authors have been trying to justify or derive its use using a variety of approaches.

Among the multitude of works presenting experimental evidence for the use of Forchheimer equation, we mention the works of Tek [55], Ward [57], Beavers and Sparrow [9], and Dullien and Azzam [7]. In addition, computational experiments have been performed by numerically solving the flow equations at a pore scale. Of interest is the paper by Coulaud, Morel and Caltagirone (1988, [16]). In this work, the authors modeled a porous medium by using "cylinders of either equal or unequal diameters arranged in a regular pattern with a square or triangular base", and used the unsteady Navier-Stokes equations to describe the microscopic flow in the porous spaces. For a given flow the

pressure drop was evaluated numerically. The numerical computations showed three distinct zones of flow. A first zone of linear flow, a second transition zone where the inertial terms are no longer negligible, and a third zone where a Forchheimer-type equation was recovered.

Theoretical derivations of Forchheimer equation have been attempted using several different methods. Among them we recall dimensional analysis [57], averaging [3, 49], homogenization [62, 41, 25], and hybrid mixture theory [31]. In general, for a single-phase fluid these theoretical methods recover generalized nonlinear equations of which the classical Forchheimer equation is a quadratic approximation. It should be noted, however, that in [62, 41] the authors recover a classical Forchheimer-type equation only for anisotropic material; in the isotropic case they find a cubic correction to Darcy's equation. An important part of these theoretical studies is to understand the origin of the nonlinearity. There is no general agreement, but turbulence, *microscopic* inertial forces and increased microscopic drag forces are the three most commonly suggested causes. Experimental evidence indicates that nonlinear dependence appears much before the onset of real turbulence, so that most authors agree that nonlinearity is not initiated by turbulence [9, 52, 51]. In addition, it is almost universally recognized that the *macroscopic* inertial forces are not the cause of the first change of flow, even if they can be approximated by a quadratic term [31, 49]. Using both the hybrid mixture theory [31] and averaging [49], it has been shown that the important nonlinear term arises from terms involving integrals over the solid-liquid interface. Hassanizadeh and Gray support the theory that this term is due to increased microscopic drag forces, and Ma and Ruth conclude that it is due to microscopic inertial forces which indirectly distort the area integral term by affecting the pressure and velocity fields. However, up-scaling techniques such as hybrid mixture theory, averaging, and homogenization all indicate that the source of the Forchheimer nonlinearity is due to the portion of microscopic inertial forces which do not manifest themselves in macroscopic inertial terms.

Although a lot is known about nonlinear flow for a single-phase fluid system, it is not clear which equations govern the flow in the non-linear, laminar flow regime when two immiscible fluids are involved. Even in the Darcy regime, most theoretical and experimental works in two-phase flow are based on a heuristic extension of Darcy's law where the corresponding permeabilities, denoted *relative permeabilities*, are functions of saturation and porosity (see e.g., Bear [8]). Criticism of these formulations, especially of the one introduced by Leverett [40] and Buckley and Leverett [14], centers about the applicability of the concept of relative permeabilities (see e.g. Jones and Roszelle [35]). Furthermore, experimental validation of these various formulations has been hindered by limitations of the quantities which are measurable. For example, it has been experimentally and theoretically documented that in the Darcy regime, cross terms, which account for the interaction between two fluids, can be important (Raats and Klute [46], Rose [47, 48], de Gennes [24], de la Cruz and Spanos [17], Auriault [4], Auriault et al. [5], Kalaydjian [36, 37]). But as Avraam and Payatakes [6] point out, "At present there is no well-established method that can determine the cross-terms directly from data taken during two-phase flow in porous media. For this reason, the experimental data are still being analyzed in terms of the conventional permeabilities". These limitations can be partially overcome by performing numerical simulations, where, for example, the effects of interfaces are being considered (see e.g. Gray et al. [26]). Since the modeling of two-phase flow in the Darcy regime is still in its infancy, it is not surprising that little or no work has been performed in the higher velocity regime (non-linear, laminar flow). It is hoped that this work will provide a blue print for future experiments.

The aim of our work is to employ hybrid mixture theory in order to derive a thermodynamically admissible form of Forchheimer's equation for two-phase flow. The difference between averaging and homogenization, and hybrid mixture theory is that in the averaging approach of Ma and Ruth [49] or Whitaker [58, 59, 60, 45] or of homogenization [50, 12, 19], the microscale constitutive equations are used (and averaged or homogenized) to obtain macroscale equations, whereas in hybrid mixture theory, constitutive equations are formulated directly at the macroscale, subject to the second law of thermodynamics. Thus, hybrid mixture theory produces equations with macroscopic variables which are necessarily thermodynamically admissible. Since we assume that measurements needed to obtain the values of the coefficients for the Forchheimer type flow will be made at the macroscale, we have chosen to use hybrid mixture theory.

Hybrid mixture theory is a macroscale theory, first developed by Hassanizadeh and Gray in 1979 [28, 29, 30], which combines the averaging approach of Whitaker [58, 59] and the mixture theory approach of Bowen [13]. The technique involves volume averaging the field equations (i.e. conservation of mass, momentum balance, energy balance, and entropy balance) to obtain macroscale field equations, where each variable is defined precisely in terms of its microscale counterpart. Thus at the macroscale, a three-phase porous medium (e.g. solid, liquid, air) is viewed as three overlaying continua, and each variable of each phase is defined spatially everywhere. As previously mentioned, constitutive equations at the microscale are not upscaled, resulting in many more unknowns than equations. To close the system consistently with classical thermodynamics, the entropy inequality is exploited in the sense of Coleman and Noll [15]. With the exception that the macroscale field variables are defined precisely in terms of their microscale counterparts, the resulting system of equations is consistent with those derived using classical mixture theory. Hence the terminology, hybrid mixture theory. The first application of hybrid mixture theory was to model single-phase flow through a deformable, elastic porous medium. In this case Darcy's equation was recovered [30]. The theory was then extended to multiphase flow in porous media with interfacial effects [33], which in turn led to a generalized Darcy's equation and a nonequilibrium capillary pressure relation [34]. The generalized Darcy's equation derived involves a coupling between the velocities of each fluid phase and the velocity of each interface relative to the solid phase velocity. This result is consistent with those obtained by others [56, 61, 36]. An extension of this work by Achanta et. al [2], to a multiphase system with interfaces, in which the medium is capable of swelling, produced a Darcy's equation with the above mentioned coupling terms, but where the interaction potential involves gradients of volume fractions instead of gradients of saturation.

In Section 2, we review the work of Hassanizadeh and Gray [31], where the Forchheimer equation for single-phase flow is derived via hybrid mixture theory. In Section 3, we use this technique to obtain a Forchheimer-type equation for a porous medium which contains two immiscible fluids, and then extend the result to include interfacial effects as well as media which are capable of swelling. We conclude with a summary.

2 Review of Derivation of Forchheimer Equation for Single Phase Flow

In [31], Hassanizadeh and Gray derive the Forchheimer equation for single-phase flow using hybrid mixture theory, and discuss the origin of the onset of such nonlinearity. In hybrid mixture theory the Forchheimer equation, as well as Darcy's equation, originates from the macroscale balance of momentum equation, derived by averaging the microscopic momentum equation over a representative elementary volume (REV),

$$\varepsilon \rho \frac{D\mathbf{v}}{Dt} - \nabla \cdot (\varepsilon \mathbf{t}) - \varepsilon \rho \mathbf{g} = \widehat{\mathbf{T}}. \quad (4)$$

In the above, ρ is the volume averaged mass density of the fluid, ε the volume fraction of the fluid phase, \mathbf{v} the mass averaged fluid velocity, \mathbf{t} the macroscale stress tensor, \mathbf{g} the body force (gravity), $\widehat{\mathbf{T}}$ the momentum gained by the liquid phase due to the presence of the solid phase, and D/Dt the material time derivative with respect to the solid phase. In terms of microscale variables, ρ , \mathbf{v} , and \mathbf{t} are given by

$$\rho = \langle \rho' \rangle \quad \mathbf{v} = \frac{\langle \rho' \mathbf{v}' \rangle}{\rho} \quad \mathbf{t} = \langle \mathbf{t}' \rangle - \langle \rho' \widetilde{\mathbf{v}} \widetilde{\mathbf{v}} \rangle, \quad (5)$$

where prime denotes the microscale quantities, $\langle \cdot \rangle$ denotes the volume averaged quantities (over the fluid phase), and $\widetilde{\mathbf{v}}$ is the velocity fluctuation given by the difference between the microscale velocity and \mathbf{v} . The second term on the right hand side of (5) is the portion of the macroscopic stress tensor

due to microscopic inertial forces, illustrating one of the strengths of hybrid mixture theory in that it gives the relationship between the microscopic and macroscopic variables. The exchange of momentum term is

$$\widehat{\mathbf{T}} = \frac{1}{|\delta V|} \int_{\partial A_{l_s}} [\mathbf{t}' + \rho' \tilde{\mathbf{v}}(\mathbf{w}^{l_s} - \mathbf{v}')] \cdot \mathbf{n}^l da, \quad (6)$$

where \mathbf{w}^{l_s} is the velocity of the interface, δV is the volume of the REV, and ∂A_{l_s} and \mathbf{n}^l are the interfacial area bounding the liquid phase within the REV and its unit outward normal, respectively.

Up to this point no assumptions concerning the medium have been made. We now consider the porous medium to be composed of an elastic solid saturated with a Newtonian fluid. As such, it is assumed that the behavior of the medium at any point is dictated by the following list of variables, referred to as independent variables [21]:

$$\rho, \varepsilon, \nabla \varepsilon, \mathbf{E}, \mathbf{d}, \theta, \nabla \theta, \mathbf{v}^{l,s}. \quad (7)$$

Here the macroscopic strain tensor of the solid phase is given by $\mathbf{E} = \frac{1}{2} [(\nabla \mathbf{F}^s)^T \cdot \nabla \mathbf{F}^s - \mathbf{I}]$ in which \mathbf{F}^s is the function mapping the spatial coordinate to the averaged solid material coordinate [21, 28]. Further, \mathbf{d} is the deformation rate tensor for the fluid phase, θ is temperature, and $\mathbf{v}^{l,s}$ is the velocity of the fluid relative to the solid phase. By the term independent, we mean that all constitutive variables (or dependent variables) are assumed to be functions of the above listed ones (7). In this formulation the temperature is assumed to be the same for each phase, i.e. $\theta^s = \theta^l$. Thus temperature can still vary spatially but we assume that the time scale is such that differences in phase temperatures are negligible.

Under these assumptions the Coleman and Noll method of exploiting the entropy inequality yields [30]

$$\mathbf{t} = -p\mathbf{I} + \boldsymbol{\sigma} \quad (8)$$

$$\widehat{\mathbf{T}} = p\nabla \varepsilon + \widehat{\boldsymbol{\tau}}, \quad (9)$$

where p represents the pressure of the liquid, and $\boldsymbol{\sigma}$ and $\widehat{\boldsymbol{\tau}}$ represent the dissipative parts of the liquid stress tensor and the momentum exchange term, respectively. Physically, the latter represents the drag force of the solid phase acting on the fluid phase. These two terms must vanish at equilibrium, and for an isotropic medium, they must be objective functions (invariant under orthogonal transformations) of the list of variables given in (7).

Combining (4), (8) and (9) we obtain

$$\varepsilon \rho \frac{D\mathbf{v}}{Dt} - \nabla \cdot (\varepsilon \boldsymbol{\sigma}) = -\varepsilon \nabla p + \varepsilon \rho \mathbf{g} + \widehat{\boldsymbol{\tau}}.$$

We remark here that if $\boldsymbol{\sigma}$ is approximated as being linear in the deformation rate tensor, \mathbf{d} , then the Brinkman correction term appears [10].

If the macroscopic length scale is much larger than the microscale, the order of magnitude analysis presented in [31], to which we refer the reader for the details, shows that the terms on the left-hand-side are negligible when compared with the terms on the right-hand-side for Reynolds number (defined to be $\rho \varepsilon |\mathbf{v}| l / \mu$, where l is the pore length and μ is the viscosity of the fluid) on the order of 10. Hence, we consider

$$\varepsilon \nabla p - \varepsilon \rho \mathbf{g} = \widehat{\boldsymbol{\tau}}. \quad (10)$$

Next, we assume that the deformation rate tensor and the solid strain have negligible effects on the interfacial drag. Then, according to isotropic function theory [54, 44] an objective vector function of three scalars ($\rho, \varepsilon, \theta$) and three vectors ($\mathbf{v}^{l,s}, \nabla \varepsilon, \nabla \theta$), must be of the form

$$\widehat{\boldsymbol{\tau}} = \alpha_1 \mathbf{v}^{l,s} + \alpha_2 \nabla \theta + \alpha_3 \nabla \varepsilon$$

where the coefficients α_1 , α_2 , and α_3 are functions of the scalar independent variables and the inner products of the vectorial independent variables. Linearly approximating these coefficients and substituting into (10) yields

$$\begin{aligned} \varepsilon \nabla p - \varepsilon \rho \mathbf{g} &= (a_1 + a_2 |\mathbf{v}^{l,s}| + a_3 |\nabla \theta| + a_4 |\nabla \varepsilon|) \mathbf{v}^{l,s} \\ &\quad (b_1 + b_2 |\mathbf{v}^{l,s}| + b_3 |\nabla \theta| + b_4 |\nabla \varepsilon|) \nabla \theta \\ &\quad (c_1 + c_2 |\mathbf{v}^{l,s}| + c_3 |\nabla \theta| + c_4 |\nabla \varepsilon|) \nabla \varepsilon, \end{aligned} \quad (11)$$

where all coefficients are functions of the three scalars. Equation (11) may be simplified somewhat by recalling that $\widehat{\boldsymbol{\tau}}$ must vanish at equilibrium. In fact, defining equilibrium to be the state where \mathbf{d} , $\nabla \theta$, and $\mathbf{v}^{l,s}$ are zero, we obtain $c_1 = c_4 = 0$.

Analogously, if $\widehat{\boldsymbol{\tau}}$ is only a function of the vector $\mathbf{v}^{l,s}$, (11) simplifies to a more familiar form of the Forchheimer equation, i.e.

$$\varepsilon \nabla p - \varepsilon \rho \mathbf{g} = (a_1 + a_2 |\mathbf{v}^{l,s}|) \mathbf{v}^{l,s},$$

where again a_1 and a_2 may be a function of the fluid density, the volume fraction, and temperature. As expected, if the velocity is small enough, Darcy's equation is recovered. In [32] Hassanizadeh and Gray pointed out that the approximation of the coefficients α_i is not unique, and that the coefficients a_1 and a_2 cannot be determined theoretically and must be measured, i.e. they are medium dependent. Likewise, the coefficients of equation (11) must also be determined experimentally, several of which may be negligible.

3 Multi-Phase Flow

We will use the technique illustrated in the previous section to derive a Forchheimer-type equation for a system composed of a solid, s , and two immiscible fluids, denoted by w and n for wetting and nonwetting, respectively. We start by considering the momentum equation for each phase:

$$\varepsilon^\alpha \rho^\alpha \frac{D\mathbf{v}^\alpha}{Dt} - \nabla \cdot (\varepsilon^\alpha \mathbf{t}^\alpha) - \varepsilon \rho^\alpha \mathbf{g} = \sum_{\beta=n,w,s, \beta \neq \alpha} \widehat{\mathbf{T}}_\beta^\alpha \quad \alpha = w, n, \quad (12)$$

with ε^α the volume fraction of each phase ($\varepsilon^w + \varepsilon^n + \varepsilon^s = 1$), and ρ^α the volume averaged mass density of phase α . The stress tensors, \mathbf{t}^α , are defined as in equation (5), and $\widehat{\mathbf{T}}_\beta^\alpha$, which accounts for the gain of momentum of the α -phase due to interactions with the β -phase, is defined as in (6) with ls replaced by $\alpha\beta$.

For the first case, we assume the behavior of the medium is dictated by the following list of variables:

$$\rho^\alpha, \rho^s, \varepsilon^w, \nabla \varepsilon^w, \mathbf{E}, \mathbf{d}^\alpha, \mathbf{v}^{\alpha,s} \quad \alpha = w, n, \quad (13)$$

where \mathbf{E} is the macroscopic solid strain tensor defined in Section 2, \mathbf{d}^α is the deformation rate tensor of the α -phase, and $\mathbf{v}^{\alpha,s}$ is the velocity of phase α relative to the solid phase. The volume fraction of the non-wetting phase is not included in the list of variables since it can be determined by knowing the strain tensor, the density of the solid phase, and the volume fraction of the wetting fluid [21, 11]. In this system we neglect thermal and interfacial effects, which may not be realistic for media with interfacial tension (e.g. oil and water, or air and water), but we assume the medium capable of swelling by including the term $\nabla \varepsilon^w$. If the solid phase is incompressible, then the volume fraction can be directly eliminated in favor of the saturation as an independent variable. However in swelling media, it is possible to have a fully saturated porous medium at various void space volume fractions. To be

consistent with the more general formulation which follows, we chose the volume fraction, instead of saturation, as an independent variable, which is more appropriate for swelling porous media.

Making the same assumptions as in Section 2, we get the following form of the momentum equation [2, 10]:

$$\varepsilon^\alpha \nabla p^\alpha + \varepsilon^\alpha \rho^\alpha \frac{\partial A^\alpha}{\partial \varepsilon^w} \nabla \varepsilon^w - \varepsilon^\alpha \rho^\alpha \mathbf{g} = \sum_{\beta=n,w,s, \beta \neq \alpha} \widehat{\boldsymbol{\tau}}_\beta^\alpha \quad \alpha = w, n. \quad (14)$$

The momentum exchange term, $\widehat{\boldsymbol{\tau}}_\beta^\alpha$, represents the non-equilibrium portion of the gain of momentum of the α -phase due to interaction with the β -phase, which by definition must be zero at equilibrium and may be a function of all the independent variables given in (13). Physically, it represents interfacial drag. As mentioned in the introduction, the middle term on the left hand side is an interaction term which comes from the assumption that the energy of the fluid is affected by the volume fraction of the same phase. This assumption is based on the experimental observation that in swelling porous media the behavior of the fluid is strongly affected by the relative amount of fluid in the system. For example, in clay soils the dryer the system, the more viscous the fluid [27]. This is not assumed in [31]. This interaction potential is relevant if there are any repulsive or attractive forces which depend upon the separation distance (e.g. hydration forces) [2, 1, 42]. But again, since we have assumed negligible interfacial effects, it does not include, for example, energy loss or gain due to interfacial tensions.

Following the derivation of Section 2, we assume the interfacial drag is not a function of the deformation rate tensor nor the solid phase stress tensor. So for an isotropic medium, the right hand side of (14) is an objective vector, which may be a function of four scalars (ρ^α , ε^α) and three vectors ($\mathbf{v}^{w,s}$, $\mathbf{v}^{n,s}$, $\nabla \varepsilon^w$). By isotropic function theory [54, 44], for $\alpha = w$, we have

$$\sum_{\beta=n,s} \widehat{\boldsymbol{\tau}}_\beta^w = -\alpha_1^w \mathbf{v}^{w,s} - \alpha_2^w \mathbf{v}^{n,s} - \alpha_3^w \nabla \varepsilon^w, \quad (15)$$

where α_i are functions of all pairwise scalar products of the above mentioned vectors (e.g. $\mathbf{v}^{w,s} \cdot \mathbf{v}^{w,s}$, $\mathbf{v}^{w,s} \cdot \mathbf{v}^{n,s}$, $\mathbf{v}^{n,s} \cdot \mathbf{v}^{n,s}$, etc.), as well as the four scalars mentioned above. Since, $\alpha_3^w \nabla \varepsilon^w$ can be incorporated into the left hand side of (14), it is dropped from consideration in the expansion of the interfacial drag. The negative signs in (15) are introduced as a reminder that in order to satisfy the entropy inequality for all physical processes, the matrix, \mathbf{R} , whose coefficients are given by $R_{11} = \alpha_1^w$, $R_{12} = \alpha_2^w$, $R_{21} = \alpha_1^n$, $R_{22} = \alpha_2^n$, where α_1^n and α_2^n are defined as in (15) with w replaced by n , must be positive definite. Using a Taylor series expansion to approximate the functions α_1^w and α_2^w , and retaining only first order terms, gives the following Forchheimer type approximation for the momentum equation (14):

$$\begin{aligned} & \varepsilon^\alpha \nabla p^\alpha + f^\alpha \nabla \varepsilon^\alpha - \varepsilon^\alpha \rho^\alpha \mathbf{g} \\ &= - \left[a_0^\alpha + a_1^\alpha |\mathbf{v}^{\alpha,s}| + a_2^\alpha \sqrt{|\mathbf{v}^{\alpha,s} \cdot \mathbf{v}^{\beta,s}|} + a_3^\alpha |\mathbf{v}^{\beta,s}| + a_4^\alpha \sqrt{|\mathbf{v}^{\beta,s} \cdot \nabla \varepsilon^w|} \right. \\ & \quad \left. + a_5^\alpha |\nabla \varepsilon^w| + a_6^\alpha \sqrt{|\mathbf{v}^{\alpha,s} \cdot \nabla \varepsilon^w|} \right] \mathbf{v}^{\alpha,s} \\ & - \left[b_0^\alpha + b_1^\alpha |\mathbf{v}^{\alpha,s}| + b_2^\alpha \sqrt{|\mathbf{v}^{\alpha,s} \cdot \mathbf{v}^{\beta,s}|} + b_3^\alpha |\mathbf{v}^{\beta,s}| + b_4^\alpha \sqrt{|\mathbf{v}^{\beta,s} \cdot \nabla \varepsilon^w|} \right. \\ & \quad \left. + b_5^\alpha |\nabla \varepsilon^w| + b_6^\alpha \sqrt{|\mathbf{v}^{\alpha,s} \cdot \nabla \varepsilon^w|} \right] \mathbf{v}^{\beta,s} \quad \alpha = w, n, \quad \beta = w, n \neq \alpha. \end{aligned} \quad (16)$$

Here a_i^α and b_i^α , $i = 1, \dots, 6$, are functions of ρ^α and ε^α , and $f^\alpha = \varepsilon^\alpha \rho^\alpha \frac{\partial A^\alpha}{\partial \varepsilon^\alpha} + \alpha_3^w$ is a function of all independent variables listed in (13). The functional expressions for these coefficients must be experimentally determined and the expressions may depend on the type of materials involved. However, since these coefficients are velocity independent, a_0^α and b_0^α may be determined at low velocities and this same expression may be used at higher velocities.

The nonlinear nature of equation (16) indicates that it may be quite difficult to solve for the velocities of the two fluids, even if the geometry, pressure gradient, gravity field, and all the coefficients on the right hand side are known.

Note that the terms involving the square root, which we label cross terms, did not appear in (11). We believe that the cross terms involving the two velocities (i.e. the coefficients of a_2^α and b_2^α) may be as significant as the non-cross terms involving velocities, due to its incorporation of relative direction. If the velocities of the two liquid phases are perpendicular to each other, then these cross terms will have no impact. However, if they run parallel to each other, then the magnitude of these terms will be magnified. Intuitively, we have that if the two fluids are flowing in parallel, then the flow of one maximally affects the flow of the other, whereas a perpendicular flow pattern inhibits flow.

Next, consider a swelling system with non-negligible interfacial effects. In this system, an important independent variable is the interfacial area density, denoted by $\varepsilon^{\alpha\beta}$, which is defined to be the total surface area of the $\alpha\beta$ interface within the REV divided by the volume of the REV. In addition, we consider the most general case discussed in the introduction for porous media, i.e. the formulation where the Helmholtz free energy of the fluid phases is a function of the macroscale solid strain. Consequently, we choose the following list of independent variables:

$$\rho^\alpha, \rho^s, \varepsilon^w, \nabla\varepsilon^w, \varepsilon^{\alpha\beta}, \nabla\varepsilon^{\alpha\beta}, \mathbf{E}, \nabla\mathbf{E}, \mathbf{d}^\alpha, \theta, \nabla\theta, \mathbf{v}^{\alpha,s}, \mathbf{v}^{\alpha\beta,s}, \quad (17)$$

with $\alpha = w, n$, $\alpha\beta = ws, ns, wn$, and $\mathbf{v}^{\alpha\beta,s}$ the mass averaged velocity of interface $\alpha\beta$ relative to the solid phase. By including $\nabla\theta$, thermal effects are incorporated. Implicitly we have assumed that the temperature of the individual phases are identical. Other variables, such as interfacial strains, interfacial excess densities, and concentrations of constituents (as well as the gradients of these terms) may also be included (see e.g. [2, 10, 33]).

For this more general formulation, we assume that the interfacial drag of the wetting fluid is negligibly affected by \mathbf{d}^α , the ns -interfacial area density, and the velocity of the ns interface. If in addition we assume a first order dependence on the gradient of the strain tensor we have the objective vector, $\hat{\boldsymbol{\tau}}^w$, must be of the form [53]

$$\begin{aligned} \sum_{\beta=n,s, \beta \neq \alpha} \hat{\boldsymbol{\tau}}_\beta^w &= \alpha_1 \nabla\varepsilon^w + \alpha_2 \nabla(\text{tr}\mathbf{E}) + \alpha_3 \nabla \cdot \mathbf{E} + \alpha_4 \nabla\varepsilon^{ws} + \alpha_5 \nabla\varepsilon^{nw} \\ &+ \alpha_6 \nabla\theta + \alpha_7 \mathbf{v}^{n,s} + \alpha_8 \mathbf{v}^{w,s} + \alpha_9 \mathbf{v}^{ws,s} + \alpha_{10} \mathbf{v}^{nw,s}, \end{aligned} \quad (18)$$

where $\text{tr}\mathbf{E}$ represents the trace of the strain tensor, \mathbf{E} .

In a more general setting, besides the first invariant of \mathbf{E} , $\text{tr}\mathbf{E}$, the right-hand-side of (18) would contain the effects of the other two invariants of \mathbf{E} , i.e. $\text{tr}(\mathbf{E})^2$ and $\text{tr}(\mathbf{E})^3$, as well as contractions of the strain tensor (and gradient of the strain tensor) with the vectors given in (17). However, we will neglect these as higher order corrections [53]. Note that as expected, there is a symmetry with the volume fraction, ε^w , and the trace of the solid phase strain tensor due to the fact that the volume fraction of the solid phase is related to the strain tensor through the relation [21]

$$\frac{\varepsilon_o^s \rho_o^s}{\varepsilon^s \rho^s} = \sqrt{\det(2\mathbf{E} + \mathbf{I})} \quad (19)$$

where subscript o denotes the initial value. For small strains the determinant of the strain can be approximated linearly in terms of the trace of the strain [21].

Let V denote the set of vectors: $\{\nabla\varepsilon^w, \nabla\varepsilon^{\alpha\beta}, \nabla\theta\}$. If \mathbf{v} and \mathbf{w} are vectors in V , then the coefficients of α_i , $i = 1, \dots, 10$ in (18) are functions of all scalars listed in (17), $\text{tr}\mathbf{E}$, $\text{tr}\mathbf{E}^2$, $\text{tr}\mathbf{E}^3$, and all quantities of the form $\mathbf{v} \cdot \mathbf{w}$, $\mathbf{v} \cdot \mathbf{E}\mathbf{w}$, and $\mathbf{v} \cdot \mathbf{E}^2\mathbf{w}$. Many of these quantities can be neglected in a second order approximation. Incorporating the terms involving α_1 , α_2 , α_4 , and α_5 into the left hand side of the momentum equation, results in the following form for a Forchheimer type equation:

$$\begin{aligned} \varepsilon^w \nabla p^w + \mathbf{B} : \nabla\mathbf{E} + f_w \nabla\varepsilon^w + f_{ws} \nabla\varepsilon^{ws} + f_{nw} \nabla\varepsilon^{nw} = \\ -\beta_1^w \nabla \cdot \mathbf{E} - \beta_2^w \nabla\theta - \beta_3^w \mathbf{v}^{n,s} - \beta_4^w \mathbf{v}^{w,s} - \beta_5^w \mathbf{v}^{ws,s} - \beta_6^w \mathbf{v}^{nw,s}, \end{aligned} \quad (20)$$

where \mathbf{B} is in general a second order tensor which contracts with the gradient of the strain tensor in the following manner: $B_{ij}E_{ij,k}$. The coefficients, β_i , $i = 1, \dots, 6$ can be linearly approximated as

$$\beta_i^w = a_{0i} + a_{1i}|\nabla\varepsilon^w| + a_{2i}|\nabla\varepsilon^{ws}| + a_{3i}|\nabla\varepsilon^{nw}| + a_{4i}|\nabla\theta| + a_{5i}|\mathbf{v}^{n,s}| + a_{6i}|\mathbf{v}^{w,s}| + a_{7i}|\mathbf{v}^{ws,s}| + a_{8i}|\mathbf{v}^{nw,s}|. \quad (21)$$

We have neglected the cross terms for ease of exposition. The coefficients \mathbf{B} , f_w , f_{ws} , and f_{nw} are functions of all independent variables listed in (17), while a_{ni} $n = 1, \dots, 8$, $i = 1, \dots, 6$ are functions only of ε^w , ε^{ws} , ε^{ns} , ρ^α , ρ^s , $\text{tr}\mathbf{E}$, $\text{tr}(\mathbf{E}^2)$, and $\text{tr}(\mathbf{E}^3)$. Darcy-versions of multi-phase flow for swelling porous media with interfacial effects have been given in [2, 10], but can also be deduced from the above expression by eliminating terms which are quadratic in velocity.

The above equation may seem needlessly complicated. But it is the authors' opinion that it is more prudent to start with a methodically derived general formulation, and simplify as is appropriate for specific problems, than to heuristically generalize a simple equation. For example, if the thermodynamic properties of the wetting fluid are affected by the shearing of the solid phase, it is unlikely that the velocity of this fluid would be high enough to warrant retaining the second order velocity terms of the wetting fluid on the right hand side of equation (20). Conversely, if the properties of the wetting fluid are unaffected by the shearing of the solid phase, then the term involving $\nabla\mathbf{E}$ in equation (20) should be replaced by $\nabla\varepsilon^s$. Similar arguments could be applied to the non-wetting fluid. Thus by knowing more details about the media, one can simplify equation (20) appropriately.

4 Conclusions

In this paper we have theoretically derived a Forchheimer-type equation for two-phase flow through porous media. We presented two versions of the equation, one for isotropic swelling porous media with no thermal or interfacial effects, and one for isotropic swelling porous media with interfacial and thermal effects and for which one of the liquids can support shear due to interaction with the solid phase. The assumptions needed in this derivation are minimal. The most critical assumptions include the existence of an REV which must be the same size for all time and space and for all field variables, and the choice of independent variables. The independent variables are the variables which dictate the behavior of the medium; leaving out an independent variable may lead to an incomplete model of the system, but leaving in too many variables can needlessly complicate the resulting set of equations. Hybrid mixture theory was used to methodically derive the form of the momentum equation based on these assumptions. Classical results from isotropic function theory and a Taylor series expansion up to second order terms were used to approximate the nonequilibrium part of the exchange of momentum terms giving a Forchheimer-type equation. We have indicated that in general it is necessary to include terms which measure the impact of the relative direction of the velocities of the two fluids.

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