Approximation of the Stokes Problem by Residual-Free Macro Bubbles

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Abstract

Linear-constant velocity-pressure elements are enriched with residual-free macro bubbles. Static condensation prompts a stabilized method for this element, where mesh-dependent jumps of the normal stress are added across internal edges of the underlying macroelements. This procedure renders the SIMPLEST element stable.

1 Introduction

The Stokes problem models creeping flows, which are flows where inertia can be neglected with respect to diffusion and source terms in the momentum equilibrium equation. Applications of this model to some subsurface flows, using variable viscosity coefficients, has gained new impetus with the increased concern in environmental conservation issues. In addition to these applications, the Stokes operator is a major component of the Navier-Stokes equations, and therefore a building block to understanding the different flow regimes in a more complex fluid dynamics problem.

Approximation of the Stokes problem using finite element methods produces a *mixed* variational formulation, where the velocity and pressure variables are approximated independently. A mixed variational formulation is a common framework for problems with constraints and it can be viewed as emanating from a saddle-point formulation. Approximating such formulations requires satisfaction of key stability conditions (a.k.a. inf-sup conditions) that need to be verified for each discretization of the velocity-pressure pair [2]. This mathematical setup is well studied and various authors have suggested "elements" and families of interpolations that pass the inf-sup condition, and thus proven convergent (see [4, 15] for a review and many examples).

Approximately a decade ago it became apparent that obtaining stable finite element formulations for the Stokes problem can be achieved by adding to the mixed formulation mesh-dependent terms (see [7, 11, 18, 19, 20] and references therein). These terms enhance the coercivity of the formulation and, when properly designed, recover the optimal rates of convergence for the velocity interpolated with finite element polynomials of degree one order higher than the pressure. Galerkin-least-squares or stabilized methods, as these methods are known, have broken the rules of the mathematical analysis of mixed methods, in that the new formulations were proven convergent for velocity-pressure pairs (such as low equal-order interpolations) dismissed as unstable because they fail the inf-sup condition. However, the amount of perturbation or the precise value of the stability constant has remained as an open problem.

In the late 80's, Pierre [22] pointed out a relationship between the mixed variational formulation for the Stokes problem using the MINI element [1] and the stabilized method using equal-order linear interpolation. A bridge was born in that remark linking the standard Galerkin method and stabilized methods. The MINI element is based on the choice of equal-order continuous piecewise-linear interpolations for velocity and pressure, with the velocity variables enriched (in 2D) with cubic bubbles, i.e. third-degree polynomials vanishing on element boundaries. Because the bubble functions are zero on each element boundary, then they can be eliminated, or statically condensed, before forming the stiffness matrix for the velocity and pressure unknowns at the vertex of each triangle. In this way static condensation leaves behind a matrix formulation based on the reduced space of equal-order linear interpolations. This matrix formulation is form-identical to the matrix obtained by using stabilized methods with equal-order linears. The difference is that once we fix the bubble function, then the additional perturbation term surfacing from static condensation has no free stability constants to be chosen. But now the problem of choosing the stability constant is transfered to the selection of an "optimal" bubble.

This dilemma was apparent when this relationship was first investigated for advective-diffusive equations in [3]. Using the standard Galerkin method with piecewise linears enriched with bubble functions, it was shown that the SUPG method [9] can be recovered and the stability constant of SUPG in this case depends on the shape of the bubble function employed. To reproduce the appropriate choice of the stability parameter in the advective-dominated regime, it was clear that the bubble function had to have a non standard shape, different from polynomials which were shown to give the appropriate stability coefficient only in the diffusive dominated flow regimes.

The quest was opened to find either the ideal bubble functions for the standard Galerkin method or to design systematically stability constants for Galerkin-least-squares type methods. Recently, *residual-free bubbles* were introduced [5, 8, 12, 13, 14, 17] to address this pursuit. If we split the approximate solution into a piecewise polynomial and bubble functions defined on each element interior, then the idea is to select bubble functions that will solve exactly the governing differential equations in each element up to the piecewise polynomial support of the solution, subject to Dirichlet homogeneous boundary condition on each element boundary. This choice leads to nodally exact approximations in one-dimensional linear problems and it can be related to a variety of successful discretizations (see [12, 13, 14]). For multi-dimensional problems this choice faces the limitations of having to find the exact solution of a problem that may be as complex as the original one. For simple element geometry, such as rectangles, the linear PDE can be solved by separation of variables technique and infinite series may be found for the basis functions for these residual-free bubbles (see [10] for an application to the Helmholtz equation). However, in general, approximations of this idea need to be used to handle unstructured meshes and complex geometry. To partially address this question, a pseudo-residual-free bubble scheme was devised for advective-diffusive problems [6], which has none of the aforementioned drawbacks.

Back to the Stokes problem we were wondering how we can profit from these ideas to obtain stable formulations for the SIMPLEST element. The SIMPLEST element consists in taking piecewise linears for velocity and piecewise constants for pressure. This combination has optimal order of convergence according to the mixed method theory, provided it passes the inf-sup condition. However, this combination is well known to fail the inf-sup condition. Attempts to recover this combination have been made using stabilized methods. In [18] it is proposed to add mesh-dependent pressure jump terms across element boundaries. These additional terms suffice to prove convergence using the SIMPLEST element. In a variation of this idea Silvester and Kechkar [21, 27] suggest to add the pressure jump terms only within macroelements. In this context, macroelements provide the minimal configuration of a patch of elements for which the SIMPLEST element is stable, provided stabilization is addressed within the macroelement. Silvester and Kechkar note that having no jump terms across macroelements does not hurt the stability of the SIMPLEST element.

Herein we introduce a residual-free macro bubble problem for each macroelement in our mesh for the SIMPLEST element. Macroelements do not overlap and their union reproduces the domain. We will show that solving the problem for each macro-bubble is equivalent to adding jump terms within the macroelement. Contrasting with [21, 27] we are able to *derive* these jump terms, without stability constants and here with a distinct feature, namely, the jump is taken over the normal component of the stress tensor. Thus, macro-bubbles hint that the pressure jump terms considered heretofore were just part of what the standard Galerkin method suggests.

The paper is organized as follows: in Section 2 we review how the MINI element and residual-free bubbles for equal-order linears relate to stabilized methods. In Section 3 we take on the discretization using the SIMPLEST element enriched with residual-free *macro* bubbles and show how this method leads to a stabilized scheme with the normal of the stress tensor jumps within each macroelement. In Section 4 we draw conclusions.

2 The MINI Element

In this Section we will revisit the MINI element introduced in [1] and show how it relates to stabilized methods. Let Ω be a polygonal plane domain with boundary Γ and $\boldsymbol{f} \in [L^2(\Omega)]^2$ a source term. The Stokes problem with Dirichlet homogeneous boundary conditions can be written as follows:

$$\begin{cases} -2\mu\, \boldsymbol{A}\boldsymbol{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \Gamma \end{cases} \tag{1}$$

where $\boldsymbol{u}$ and p are the unknown velocity and pressure variables, μ is the viscosity, $\boldsymbol{A}\boldsymbol{u} = \mathbf{d}iv\, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u})$, and $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u})$ is the linearized stress tensor whose components are defined by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{2}$$

$\boldsymbol{A}\boldsymbol{u}$ is given in components by

$$\boldsymbol{A}\boldsymbol{u} = \begin{pmatrix} \dfrac{\partial^2 u_1}{\partial x_1^2} + \dfrac{1}{2}\dfrac{\partial}{\partial x_2}\left\{\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right\} \\ \dfrac{\partial^2 u_2}{\partial x_2^2} + \dfrac{1}{2}\dfrac{\partial}{\partial x_1}\left\{\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right\} \end{pmatrix}. \tag{3}$$

Using the equation $\operatorname{div} \boldsymbol{u} = 0$, the first equation of (1) can also be rewritten as

$$-\mu\, \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \tag{4}$$

but we will adopt the momentum equation as in (1) since it accommodates more general boundary conditions than the equation with the Laplace operator (see [16] for an interesting discussion about this point).

A variational formulation for problem (1) can be written as follows:

$$\begin{cases} \text{find } \boldsymbol{u} \in \boldsymbol{V} \text{ and } p \in Q \text{ such that} \\ 2\mu \displaystyle\int_\Omega \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) - \int_\Omega p \operatorname{div} \boldsymbol{v} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \quad \text{for all } \boldsymbol{v} \in \boldsymbol{V} \\ \displaystyle\int_\Omega q \operatorname{div} \boldsymbol{u} = 0 \quad \text{for all } q \in Q \end{cases} \tag{5}$$

where $\boldsymbol{V} = [H_0^1(\Omega)]^2$ and

$$Q = L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \text{ such that } \int_\Omega q = 0 \right\}. \tag{6}$$

It is well-known that problem (5) has a unique solution. Let us introduce the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on $\boldsymbol{V} \times \boldsymbol{V}$ and $\boldsymbol{V} \times Q$ respectively:

$$a(\boldsymbol{u}, \boldsymbol{v}) = 2\mu \int_\Omega \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}), \quad b(\boldsymbol{v}, q) = -\int_\Omega q \operatorname{div} \boldsymbol{v} \tag{7}$$

and the linear form on $\boldsymbol{V}$

$$F(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \tag{8}$$

so that equation (5) becomes:

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = F(\boldsymbol{v}) & \text{for all } \boldsymbol{v} \in \boldsymbol{V} \\ -b(\boldsymbol{u}, q) = 0 & \text{for all } q \in Q. \end{cases} \tag{9}$$

A Galerkin approximation for problem (9) consists in taking finite dimensional subspaces $\boldsymbol{V}_h$ and Q_h of $\boldsymbol{V}$ and Q, respectively, and then solving problem (9) in $\boldsymbol{V}_h \times Q_h$:

$$\begin{cases} \text{find } \boldsymbol{u}_h \in \boldsymbol{V}_h \text{ and } p_h \in Q_h \text{ such that} \\ \quad a(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) = F(\boldsymbol{v}_h) \quad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h \\ -b(\boldsymbol{u}_h, q_h) = 0 \quad \text{for all } q_h \in Q_h. \end{cases} \tag{10}$$

As it is well–known, the spaces $\boldsymbol{V}_h$ and Q_h must satisfy a compatibility condition in order to lead to a stable formulation [2].

Let $\mathcal{T}_h = \{K\}$ be a regular triangulation of Ω. We define for each K the linear space

$$P_k(K) = \{\text{polynomials on } K \text{ of degree} \leq k\} \tag{11}$$

and

$$\begin{aligned} \mathcal{P}_k &= \left\{ w \in H_0^1(\Omega) \text{ such that } w_{|K} \in P_k(K) \right\} \text{ for } k \geq 1 \\ \mathcal{P}_0 &= \left\{ w \in L_0^2(\Omega) \text{ such that } w_{|K} \in P_0(K) \right\}. \end{aligned} \tag{12}$$

We shall often use in our constructions *bubble functions*. For an element K a bubble function is a function vanishing on ∂K. In general, we denote by $B_k(K)$ the space of bubbles that are polynomials of degree less or equal to k on K and by $\mathcal{B}_k$ the subspace $\oplus_K B_k(K)$ of $H_0^1(\Omega)$. In dimension 2, the space $B_3(K)$ has dimension 1 and is generated for instance by the polynomial given by the product of the equations of the edges.

The MINI element consists in taking "continuous, piecewise linears plus cubic bubbles" for velocities and "continuous, piecewise linears" for pressure:

$$\boldsymbol{V}_h = [\mathcal{P}_1 \oplus \mathcal{B}_3]^2, \quad Q_h = \mathcal{P}_1. \tag{13}$$

The residual-free bubbles approach consists in substituting $\mathcal{B}_3$ in (13) with the space

$$\mathcal{B}_{\mathrm{RF}} = \oplus_K H_0^1(K). \tag{14}$$

We proceed now to "statically condensate" the bubble part of the Galerkin solution. Any function $\boldsymbol{v}_h$ in $\boldsymbol{V}_h = [\mathcal{P}_1 \oplus \mathcal{B}_{\mathrm{RF}}]^2$ can be split in a unique way as

$$\boldsymbol{v}_h = \boldsymbol{v}_{\mathrm{L}} + \boldsymbol{v}_{\mathrm{B}} \tag{15}$$

where $\boldsymbol{v}_{\mathrm{L}} \in [\mathcal{P}_1]^2$ and $\boldsymbol{v}_{\mathrm{B}} \in [\mathcal{B}_{\mathrm{RF}}]^2$. In turn, $\boldsymbol{v}_{\mathrm{B}}$ can be uniquely distributed among the K's:

$$\boldsymbol{v}_{\mathrm{B}} = \sum_K \boldsymbol{v}_{\mathrm{B,K}}, \quad \boldsymbol{v}_{\mathrm{B,K}} \in \left[H_0^1(K)\right]^2. \tag{16}$$

Hence, we can split the variational formulation (10) in the following way:

$$\begin{cases} \text{find } \boldsymbol{u}_{\mathrm{L}} \in [\mathcal{P}_1]^2, \ \boldsymbol{u}_{\mathrm{B,K}} \in \left[H_0^1(K)\right]^2 \text{ and } p_{\mathrm{L}} \in \mathcal{P}_1 \text{ such that} \\ a\left(\boldsymbol{u}_{\mathrm{L}} + \sum_K \boldsymbol{u}_{\mathrm{B,K}}, \boldsymbol{v}_{\mathrm{L}}\right) + b\left(\boldsymbol{v}_{\mathrm{L}}, p_{\mathrm{L}}\right) = F\left(\boldsymbol{v}_{\mathrm{L}}\right) \quad \text{for all } \boldsymbol{v}_{\mathrm{L}} \in [\mathcal{P}_1]^2 \\ a\left(\boldsymbol{u}_{\mathrm{L}} + \boldsymbol{u}_{\mathrm{B,K}}, \boldsymbol{v}_{\mathrm{B,K}}\right)_K + b\left(\boldsymbol{v}_{\mathrm{B,K}}, p_{\mathrm{L}}\right)_K = F\left(\boldsymbol{v}_{\mathrm{B,K}}\right)_K \quad \text{for all } \boldsymbol{v}_{\mathrm{B,K}} \in [H_0^1(K)]^2 \\ -b\left(\boldsymbol{u}_{\mathrm{L}} + \sum_K \boldsymbol{u}_{\mathrm{B,K}}, q_{\mathrm{L}}\right) = 0 \quad \text{for all } q_{\mathrm{L}} \in \mathcal{P}_1 \end{cases} \tag{17}$$

where the subscript K in the second equation above means that all integrals are restricted to the element K.

Let us now consider the second equation of (17). Recall the following integration by parts formula, which holds for any Lipschitz domain D:

$$\int_D \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) = -\int_D \mathbf{div}\, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) \cdot \boldsymbol{v} + \int_{\partial D} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u})\boldsymbol{n} \cdot \boldsymbol{v} \tag{18}$$

where $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u})\boldsymbol{n}$ is the vector whose ith component is $\sum_j \varepsilon(\boldsymbol{u})_{ij} n_j$. Since $\mathbf{div}\, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) = 0$ in K and $\boldsymbol{v}_{\mathrm{B,K}} = 0$ on ∂K, by (18) we have

$$a\left(\boldsymbol{u}_{\mathrm{L}}, \boldsymbol{v}_{\mathrm{B,K}}\right)_K = 2\mu \int_K \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{B,K}}) = 0 \tag{19}$$

and, observing that ∇p_{L} is constant in K, by integrating by parts

$$b\left(\boldsymbol{v}_{\mathrm{B,K}}, p_{\mathrm{L}}\right)_K = -\int_K p_{\mathrm{L}} \operatorname{div} \boldsymbol{v}_{\mathrm{B,K}} = [\,\nabla p_{\mathrm{L}}\,]_{|K} \cdot \int_K \boldsymbol{v}_{\mathrm{B,K}}. \tag{20}$$

Thus, if we assume that $\boldsymbol{f}$ is piecewise-constant on $\mathcal{T}_h$, the equation for $\boldsymbol{u}_{\mathrm{B,K}}$ reduces to

$$2\mu \int_K \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{B,K}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{B,K}}) = [\,\boldsymbol{f} - \nabla p_{\mathrm{L}}\,]_{|K} \cdot \int_K \boldsymbol{v}_{\mathrm{B,K}} \quad \text{for all } \boldsymbol{v}_{\mathrm{B,K}} \in \left[H_0^1(K)\right]^2 \tag{21}$$

or, in strong form,

$$\begin{cases} -2\mu\, \boldsymbol{A}\boldsymbol{u}_{\mathrm{B,K}} = \boldsymbol{f} - \nabla p_{\mathrm{L}} = \text{constant} & \text{in } K \\ \boldsymbol{u}_{\mathrm{B,K}} = 0 & \text{on } \partial K. \end{cases} \tag{22}$$

Hence, the residual-free bubble $\boldsymbol{u}_{\mathrm{B,K}}$ for this problem can be written as

$$\boldsymbol{u}_{\mathrm{B,K}} = \underline{\boldsymbol{b}}_{\mathrm{K}} \left[\boldsymbol{f} - \nabla p_h \right]_{|K} \tag{23}$$

where the bubble matrix $\underline{\boldsymbol{b}}_{\mathrm{K}}$ solves the problem

$$\begin{cases} -2\mu\, \boldsymbol{A}\underline{\boldsymbol{b}}_{\mathrm{K}} = \underline{\mathbf{I}} & \text{in } K \\ \qquad\quad \underline{\boldsymbol{b}}_{\mathrm{K}} = \underline{\mathbf{0}} & \text{on } \partial K, \end{cases} \tag{24}$$

and $\underline{\mathbf{I}}$ is the identity matrix, $I_{ij} = \delta_{ij}$. In other words, if $\boldsymbol{b}_{\mathrm{K}}^{(1)}, \boldsymbol{b}_{\mathrm{K}}^{(2)}$ are the two columns of the bubble matrix $\underline{\boldsymbol{b}}_{\mathrm{K}}$, we have the equations

$$-2\mu\, \boldsymbol{A}\boldsymbol{b}_{\mathrm{K}}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{25}$$

and

$$-2\mu\, \boldsymbol{A}\boldsymbol{b}_{\mathrm{K}}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{26}$$

respectively, and

$$\boldsymbol{u}_{\mathrm{B,K}} = \left[f_1 - \frac{\partial p_h}{\partial x_1} \right]_{|K} \boldsymbol{b}_{\mathrm{K}}^{(1)} + \left[f_2 - \frac{\partial p_h}{\partial x_2} \right]_{|K} \boldsymbol{b}_{\mathrm{K}}^{(2)}. \tag{27}$$

Now we substitute the expression (23) for $\boldsymbol{u}_{\mathrm{B,K}}$ in the first and in the third equation of (17). Since by (19) $a\left(\boldsymbol{u}_{\mathrm{B,K}}, \boldsymbol{v}_{\mathrm{L}}\right)_K = 0$, the first equation of (17) is unchanged with respect to the bubble functions. Considering the third equation, by integration by parts we have

$$\begin{aligned} b\left(\boldsymbol{u}_{\mathrm{B,K}}, q_{\mathrm{L}}\right)_K &= -\int_K q_{\mathrm{L}} \operatorname{div} \boldsymbol{u}_{\mathrm{B,K}} = \\ &\left[\nabla q_{\mathrm{L}} \right]_{|K} \cdot \int_K \boldsymbol{u}_{\mathrm{B,K}} = \left[\nabla q_{\mathrm{L}} \right]_{|K} \cdot \left[\int_K \underline{\boldsymbol{b}}_{\mathrm{K}} \right] \left[\boldsymbol{f} - \nabla p_h \right]_{|K}. \end{aligned} \tag{28}$$

In order to effectively compare with the MINI element, we rewrite the last term as

$$b\left(\boldsymbol{u}_{\mathrm{B,K}}, q_{\mathrm{L}}\right)_K = \int_K \left(\nabla q_{\mathrm{L}}\right) \cdot \left[\frac{1}{|K|} \int_K \underline{\boldsymbol{b}}_{\mathrm{K}} \right] \left(\boldsymbol{f} - \nabla p_h\right). \tag{29}$$

If the triangulation is regular, it can be shown by a scaling argument (see [26, 25, 24] for details) that

$$\frac{1}{|K|} \int_K \underline{\boldsymbol{b}}_{\mathrm{K}} \approx C \frac{h_{\mathrm{K}}^2}{\mu} \underline{\mathbf{I}}, \tag{30}$$

where $h_{\mathrm{K}} = \operatorname{diam} K$, so that the residual-free bubble term takes the more familiar form

$$b\left(\boldsymbol{u}_{\mathrm{B,K}}, q_{\mathrm{L}}\right)_K = C \frac{h_{\mathrm{K}}^2}{\mu} \int_K \left(\boldsymbol{f} - \nabla p_h\right) \cdot \left(\nabla q_{\mathrm{L}}\right). \tag{31}$$

We can then write problem (10) on $[\mathcal{P}_1]^2 \times \mathcal{P}_1$ after the static condensation as

$$\begin{cases} \text{find } \boldsymbol{u}_{\mathrm{L}} \in [\mathcal{P}_1]^2 \text{ and } p_{\mathrm{L}} \in \mathcal{P}_1 \text{ such that} \\ 2\mu \int_\Omega \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) - \int_\Omega p_{\mathrm{L}} \operatorname{div} \boldsymbol{v}_{\mathrm{L}} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_{\mathrm{L}} \quad \text{for all } \boldsymbol{v}_{\mathrm{L}} \in [\mathcal{P}_1]^2 \\ \int_\Omega q_{\mathrm{L}} \operatorname{div} \boldsymbol{u}_{\mathrm{L}} - C \sum_K \frac{h_{\mathrm{K}}^2}{\mu} \int_K (\boldsymbol{f} - \nabla p_{\mathrm{L}}) \cdot (\nabla q_{\mathrm{L}}) = 0 \quad \text{for all } q_{\mathrm{L}} \in \mathcal{P}_1 \end{cases} \tag{32}$$

which, in essence, coincides with the stabilization given by the MINI element.

Remark 1 If the triangulation $\mathcal{T}_h$ is distorted, then the approximation (30) is no longer valid. We plan to address this problem in future work.

Remark 2 The residual-free bubble, as we define it, coincides with the "optimal" choice of the bubble for the MINI spaces proposed by Pierre [23].

Remark 3 By considering the momentum equation as in equation (1) and not as in equation (4), we are led to bubble basis functions that are part of a matrix as indicated by equation (24). In the Laplacian case, the bubble basis functions uncouple, in that it suffices to solve a scalar Poisson equation with right-hand-side equal to one. The latter was considered in [23] and [25].

3 The Residual-Free Macro Bubbles

In this Section we will consider the residual-free macro bubbles. To this end, we assume to have a triangulation $\mathcal{T}_h$ structured with macroelements, i.e. in such a way that $\mathcal{T}_h$ can be partitioned in disjoint groups of four triangles which form a macroelement M. For simplicity, we will assume that the macroelements themselves are triangles. The triangles forming a macroelement M will be denoted by K_0, K_1, K_2, K_3, ordered counterclockwise and with K_0 in the center; we will denote by e_i, $i = 1, 2, 3$, the common edge of K_0 and K_i (see Fig. 1). The approximation space for velocities consists of continuous, piecewise linears plus the residual-free bubbles on macroelements, i.e.

$$\boldsymbol{V}_h = [\mathcal{P}_1 \oplus \mathcal{M}_{\mathrm{RF}}]^2 \tag{33}$$

where

$$\mathcal{M}_{\mathrm{RF}} = \oplus_M H_0^1(M) \tag{34}$$

and the sum is on all macroelements. For pressure, we take as Q_h the space $\mathcal{P}_0$ of piecewise constant functions with respect to $\mathcal{T}_h = \{K\}$. It is simple to show that this couple of spaces satisfies the inf-sup condition.

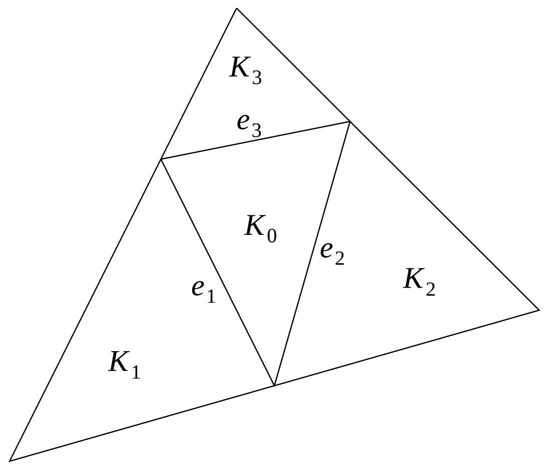

Figure 1: The macroelement M

As seen in the previous Section, any function $\boldsymbol{v}_h \in \boldsymbol{V}_h$ can be split in a unique way as $\boldsymbol{v}_h = \boldsymbol{v}_{\text{L}} + \boldsymbol{v}_{\text{B}}$, with $\boldsymbol{v}_{\text{L}} \in [\mathcal{P}_1]^2$ and $\boldsymbol{v}_{\text{B}} \in [\mathcal{M}_{\text{RF}}]^2$. In turn, $\boldsymbol{v}_{\text{B}}$ can be uniquely split into the macroelements:

$$\boldsymbol{v}_{\text{B}} = \sum_M \boldsymbol{v}_{\text{B,M}}, \quad \boldsymbol{v}_{\text{B,M}} \in \left[H^1_0(M)\right]^2. \tag{35}$$

As a consequence, the variational problem (10) on $\boldsymbol{V}_h \times Q_h$ can be split in the following way:

$$\begin{cases} \text{find } \boldsymbol{u}_{\text{L}} \in [\mathcal{P}_1]^2,\ \boldsymbol{u}_{\text{B,M}} \in \left[H^1_0(M)\right]^2 \text{ and } p_0 \in \mathcal{P}_0 \text{ such that} \\ a\left(\boldsymbol{u}_{\text{L}} + \sum_M \boldsymbol{u}_{\text{B,M}}, \boldsymbol{v}_{\text{L}}\right) + b\left(\boldsymbol{v}_{\text{L}}, p_0\right) = F\left(\boldsymbol{v}_{\text{L}}\right) \quad \text{for all } \boldsymbol{v}_{\text{L}} \in [\mathcal{P}_1]^2 \\ a\left(\boldsymbol{u}_{\text{L}} + \boldsymbol{u}_{\text{B,M}}, \boldsymbol{v}_{\text{B,M}}\right)_M + b\left(\boldsymbol{v}_{\text{B,M}}, p_0\right)_M = F\left(\boldsymbol{v}_{\text{B,M}}\right)_M \quad \text{for all } \boldsymbol{v}_{\text{B,M}} \in [H^1_0(M)]^2 \\ -b\left(\boldsymbol{u}_{\text{L}} + \sum_M \boldsymbol{u}_{\text{B,M}}, q_0\right) = 0 \quad \text{for all } q_0 \in \mathcal{P}_0 \end{cases} \tag{36}$$

where the subscript M in the second equation means that all integrals are restricted to the macroelement M.

Let us now consider the second equation of (36). We have

$$a\left(\boldsymbol{u}_{\text{L}}, \boldsymbol{v}_{\text{B,M}}\right)_M = 2\mu \int_M \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\text{B,M}}) = 2\mu \sum_{i=0}^{3} \int_{K_i} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\text{B,M}}). \tag{37}$$

Using (18) on K_i yields

$$\int_{K_i} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\text{B,M}}) = -\int_{K_i} \mathbf{div}\, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}}) \cdot \boldsymbol{v}_{\text{B,M}} + \int_{\partial K_i} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}})\boldsymbol{n} \cdot \boldsymbol{v}_{\text{B,M}} \tag{38}$$

and since $\mathbf{div}\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}}) = 0$ on K_i, $\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}})$ is constant on e_i and $\boldsymbol{v}_{\text{B,M}} = 0$ on ∂M, we get for $i = 1, 2, 3$

$$\int_{K_i} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\text{B,M}}) = [\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}})\boldsymbol{n}_i\,]_{|e_i} \cdot \int_{e_i} \boldsymbol{v}_{\text{B,M}} \tag{39}$$

and

$$\int_{K_0} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\text{B,M}}) = -\sum_{i=1}^{3} [\,\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}})\boldsymbol{n}_i\,]_{|e_i} \cdot \int_{e_i} \boldsymbol{v}_{\text{B,M}} \tag{40}$$

where we have denoted by $\boldsymbol{n}_i$ the normal unit vector on e_i with respect to K_i (see Fig. 2). Summing all contributions, we have

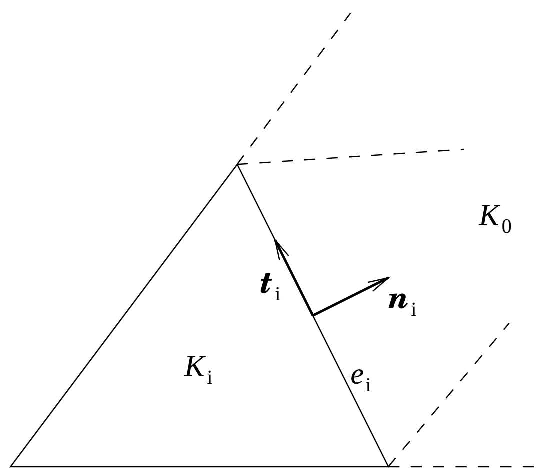

Figure 2: The normal and tangent vectors

$$a\,(\boldsymbol{u}_{\text{L}}, \boldsymbol{v}_{\text{B,M}})_M = 2\mu \sum_{i=1}^{3} [\![\, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\text{L}})\,]\!]_{e_i} \cdot \int_{e_i} \boldsymbol{v}_{\text{B,M}} \tag{41}$$

where $[\![\,\cdot\,]\!]_\Gamma$ is the jump of $(\cdot)$ across Γ and is defined as $(\cdot)^+\boldsymbol{n}^+ + (\cdot)^-\boldsymbol{n}^-$ where $+$ and $-$ indicate quantities at the two sides of Γ. Furthermore,

$$b\,(\boldsymbol{v}_{\text{B,M}}, p_0)_M = -\int_M p_0 \,\mathrm{div}\, \boldsymbol{v}_{\text{B,M}} = -\sum_{i=0}^{3} \int_{K_i} p_0 \,\mathrm{div}\, \boldsymbol{v}_{\text{B,M}} \tag{42}$$

and since p_0 is piecewise constant, for $i = 1, 2, 3$ we have

$$\int_{K_i} p_0 \,\mathrm{div}\, \boldsymbol{v}_{\text{B,M}} = [\,p_0\,]_{|K_i} \int_{K_i} \mathrm{div}\, \boldsymbol{v}_{\text{B,M}} = [\,p_0\,]_{|K_i} \int_{e_i} \boldsymbol{v}_{\text{B,M}} \cdot \boldsymbol{n}_i. \tag{43}$$

As before, the integral on K_0 contains the same contributions with opposite signs for all edges e_i so that

$$b\,(\boldsymbol{v}_{\text{B,M}}, p_0)_M = -\sum_{i=1}^{3} [\![\, p_0\,]\!]_{e_i} \cdot \int_{e_i} \boldsymbol{v}_{\text{B,M}} = -\sum_{i=1}^{3} [\![\, p_0 \underline{\mathbf{I}}\,]\!]_{e_i} \cdot \int_{e_i} \boldsymbol{v}_{\text{B,M}}. \tag{44}$$

Putting all together we have the following equation for the bubble part $\boldsymbol{u}_{\mathrm{B,M}}$ of the discrete solution $\boldsymbol{u}_h$:

$$a\left(\boldsymbol{u}_{\mathrm{B,M}}, \boldsymbol{v}_{\mathrm{B,M}}\right)_M = -\sum_{i=1}^{3} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}}\,]\!]_{e_i} \cdot \int_{e_i} \boldsymbol{v}_{\mathrm{B,M}} + \int_M \boldsymbol{f} \cdot \boldsymbol{v}_{\mathrm{B,M}} \tag{45}$$

for all $\boldsymbol{v}_{\mathrm{B,M}} \in [H_0^1(M)]^2$. The tensor $[\,2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}}]$ is the *Cauchy stress tensor* for the Stokes problem. We can split $\boldsymbol{u}_{\mathrm{B,M}}$ in the following way:

$$\boldsymbol{u}_{\mathrm{B,M}} = \sum_{i=1}^{3} \boldsymbol{u}_{\mathrm{B,M},i} + \boldsymbol{u}_{\mathrm{B,M},f} \tag{46}$$

where $\boldsymbol{u}_{\mathrm{B,M},i}$, $i = 1, 2, 3$, and $\boldsymbol{u}_{\mathrm{B,M},f}$ solve

$$a\left(\boldsymbol{u}_{\mathrm{B,M},i}, \boldsymbol{v}_{\mathrm{B,M}}\right)_M = -[\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}}\,]\!]_{e_i} \cdot \int_{e_i} \boldsymbol{v}_{\mathrm{B,M}} \tag{47}$$

and

$$a\left(\boldsymbol{u}_{\mathrm{B,M},f}, \boldsymbol{v}_{\mathrm{B,M}}\right)_M = \int_M \boldsymbol{f} \cdot \boldsymbol{v}_{\mathrm{B,M}} \tag{48}$$

respectively; in strong form (47) and (48) read as

$$\begin{cases} -2\mu\, \boldsymbol{A}\boldsymbol{u}_{\mathrm{B,M},i} = \underline{\mathbf{0}} & \text{in } M \\ [\![\, \nabla \boldsymbol{u}_{\mathrm{B,M},i}\,]\!]_{e_i} = -[\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}}\,]\!]_{e_i} & \text{on } e_i \\ \boldsymbol{u}_{\mathrm{B,M},i} = \underline{\mathbf{0}} & \text{on } \partial M \end{cases} \tag{49}$$

and

$$\begin{cases} -2\mu\, \boldsymbol{A}\boldsymbol{u}_{\mathrm{B,M},f} = \boldsymbol{f} & \text{in } M \\ \boldsymbol{u}_{\mathrm{B,M},f} = \underline{\mathbf{0}} & \text{on } \partial M. \end{cases} \tag{50}$$

Hence, if for simplicity we assume that $\boldsymbol{f}$ is constant on the macroelement M, $\boldsymbol{u}_{\mathrm{B,M}}$ can be written as

$$\boldsymbol{u}_{\mathrm{B,M}} = -\sum_{i=1}^{3} \underline{\boldsymbol{b}}_{\mathrm{M},i} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}}\,]\!]_{e_i} + \underline{\boldsymbol{b}}_{\mathrm{M},f}\, [\,\boldsymbol{f}\,]_{|M} \tag{51}$$

where the bubble matrices $\underline{\boldsymbol{b}}_{\mathrm{M},i}$, $i = 1, 2, 3$, and $\underline{\boldsymbol{b}}_{\mathrm{M},f}$ solve the equations

$$\begin{cases} -2\mu\, \boldsymbol{A}\underline{\boldsymbol{b}}_{\mathrm{M},i} = \underline{\mathbf{0}} & \text{in } M \\ [\![\, \nabla \underline{\boldsymbol{b}}_{\mathrm{M},i}\,]\!]_{e_i} = \underline{\mathbf{I}} & \text{on } e_i \\ \underline{\boldsymbol{b}}_{\mathrm{M},i} = \underline{\mathbf{0}} & \text{on } \partial M \end{cases} \tag{52}$$

and

$$\begin{cases} -2\mu\, \boldsymbol{A}\underline{\boldsymbol{b}}_{\mathrm{M},f} = \underline{\mathbf{I}} & \text{in } M \\ \underline{\boldsymbol{b}}_{\mathrm{M},f} = \underline{\mathbf{0}} & \text{on } \partial M. \end{cases} \tag{53}$$

respectively. We now go back to the first and third equations of (36). We have by (41):

$$a\left(\boldsymbol{u}_{\mathrm{B,M}}, \boldsymbol{v}_{\mathrm{L}}\right)_M = 2\mu \int_M \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{B,M}}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) = 2\mu \sum_{k=1}^{3} [\![\, \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) \,]\!]_{e_k} \cdot \int_{e_k} \boldsymbol{u}_{\mathrm{B,M}} \tag{54}$$

and by substituting (51) in (54):

$$\begin{aligned} a\left(\boldsymbol{u}_{\mathrm{B,M}}, \boldsymbol{v}_{\mathrm{L}}\right)_M = & - \sum_{i,k=1}^{3} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) \,]\!]_{e_k} \cdot \left[\int_{e_k} \underline{\boldsymbol{b}}_{\mathrm{M},i}\right] [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}} \,]\!]_{e_i} \\ & + \sum_{k=1}^{3} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) \,]\!]_{e_k} \cdot \left[\int_{e_k} \underline{\boldsymbol{b}}_{\mathrm{M},f}\right] [\, \boldsymbol{f} \,]_{|M} \,. \end{aligned} \tag{55}$$

On the other hand, by (44)

$$b\left(\boldsymbol{u}_{\mathrm{B,M}}, q_0\right)_M = - \sum_{k=1}^{3} [\![\, q_0\underline{\mathbf{I}} \,]\!]_{e_k} \cdot \int_{e_k} \boldsymbol{u}_{\mathrm{B,M}} \tag{56}$$

and by (51)

$$\begin{aligned} b\left(\boldsymbol{u}_{\mathrm{B,M}}, q_0\right)_M = & - \sum_{i,k=1}^{3} [\![\, -q_0\underline{\mathbf{I}} \,]\!]_{e_k} \cdot \left[\int_{e_k} \underline{\boldsymbol{b}}_{\mathrm{M},i}\right] [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}} \,]\!]_{e_i} \\ & + \sum_{k=1}^{3} [\![\, -q_0\underline{\mathbf{I}} \,]\!]_{e_k} \cdot \left[\int_{e_k} \underline{\boldsymbol{b}}_{\mathrm{M},f}\right] [\, \boldsymbol{f} \,]_{|M} \,. \end{aligned} \tag{57}$$

If we sum together $a\left(\boldsymbol{u}_{\mathrm{B,M}}, \boldsymbol{v}_{\mathrm{L}}\right)_M$ and $b\left(\boldsymbol{u}_{\mathrm{B,M}}, q_0\right)_M$, we have the stabilization terms produced by the residual-free macro bubbles on the macroelement M:

$$\begin{aligned} & a\left(\boldsymbol{u}_{\mathrm{B,M}}, \boldsymbol{v}_{\mathrm{L}}\right)_M + b\left(\boldsymbol{u}_{\mathrm{B,M}}, q_0\right)_M = \\ & \quad - \sum_{i,k=1}^{3} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) - q_0\underline{\mathbf{I}} \,]\!]_{e_k} \cdot \underline{\boldsymbol{\mathcal{C}}}_{ki} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}} \,]\!]_{e_i} \\ & \quad + \sum_{k=1}^{3} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) - q_0\underline{\mathbf{I}} \,]\!]_{e_k} \cdot \underline{\boldsymbol{\mathcal{D}}}_k \, [\, \boldsymbol{f} \,]_{|M} \end{aligned} \tag{58}$$

where the bubble stabilization constant matrices $\underline{\boldsymbol{\mathcal{C}}}_{ki}$ and $\underline{\boldsymbol{\mathcal{D}}}_k$ are defined by

$$\underline{\boldsymbol{\mathcal{C}}}_{ki} = \int_{e_k} \underline{\boldsymbol{b}}_{\mathrm{M},i}, \quad \underline{\boldsymbol{\mathcal{D}}}_k = \int_{e_k} \underline{\boldsymbol{b}}_{\mathrm{M},f}. \tag{59}$$

By considering the variational formulation of equation (52), it is readily shown that in general

$$\underline{\boldsymbol{\mathcal{C}}}_{ki} = \underline{\boldsymbol{\mathcal{C}}}_{ik} = 2\mu \int_M \underline{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{b}}_{\mathrm{M},k}) : \underline{\boldsymbol{\varepsilon}}(\underline{\boldsymbol{b}}_{\mathrm{M},i}), \quad \text{for } i,k = 1,2,3. \tag{60}$$

Let $h_{\mathrm{M}} = \operatorname{diam} M$. If the triangulation is almost uniform, it can be shown by a scaling argument that

$$\underline{\boldsymbol{\mathcal{C}}}_{ki} \approx C_{ki}\frac{h_{\mathrm{M}}^2}{\mu}\underline{\mathbf{I}}, \quad \underline{\boldsymbol{\mathcal{D}}}_k \approx C''\frac{h_{\mathrm{M}}^3}{\mu}\underline{\mathbf{I}}, \tag{61}$$

with $C'' > 0$. Furthermore, for symmetry reasons, $C_{ki} \approx C'\delta_{ki} + \widehat{C}'(1-\delta_{ki})$ with $0 < \widehat{C}' < C'$. Hence, if we neglect the off-diagonal entries, the stabilization term takes the simplified form

$$\begin{aligned} & a\left(\boldsymbol{u}_{\mathrm{B,M}}, \boldsymbol{v}_{\mathrm{L}}\right)_M + b\left(\boldsymbol{u}_{\mathrm{B,M}}, q_0\right)_M = \\ & \quad -C'\,\frac{h_{\mathrm{M}}^2}{\mu}\,\sum_{k=1}^{3} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) - q_0\underline{\mathbf{I}}\,]\!]_{e_k} \cdot [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}}\,]\!]_{e_k} \\ & \quad +C''\,\frac{h_{\mathrm{M}}^3}{\mu}\,\sum_{k=1}^{3} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) - q_0\underline{\mathbf{I}}\,]\!]_{e_k} \cdot [\,\boldsymbol{f}\,]_{|M}\,, \end{aligned} \tag{62}$$

which can also be written as

$$\begin{aligned} & a\left(\boldsymbol{u}_{\mathrm{B,M}}, \boldsymbol{v}_{\mathrm{L}}\right)_M + b\left(\boldsymbol{u}_{\mathrm{B,M}}, q_0\right)_M = \\ & \quad -\widetilde{C}'\,\frac{h_{\mathrm{M}}}{\mu}\,\sum_{k=1}^{3}\int_{e_k} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) - q_0\underline{\mathbf{I}}\,]\!]_{e_k} \cdot [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_{\mathrm{L}}) - p_0\underline{\mathbf{I}}\,]\!]_{e_k} \\ & \quad +\widetilde{C}''\,\frac{1}{\mu}\,\sum_{k=1}^{3}\int_{e_k} [\![\, 2\mu\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}_{\mathrm{L}}) - q_0\underline{\mathbf{I}}\,]\!]_{e_k} \cdot \int_M \boldsymbol{f}. \end{aligned} \tag{63}$$

We wish to point out that one of the aims of the residual-free bubbles approach is to suggest new stabilization methods whose validity can then be confirmed by a direct analysis. We plan to study in the future, both theoretically and experimentally, the two stabilization strategies emerging from this analysis, i.e. (62) and the more complex (58), and also to examine the case of stretched meshes, for which approximation (61) does not hold true.

Remark 4 The macro bubble when statically condensed gives rise to stabilization via jump terms (cf. (63)). The jumps are taken over the normal of the Cauchy stress tensor and there is an additional term to take into account non-zero source terms. This is the first time, as far as we are aware of, that stabilization *via jumps* take into account the entire stress tensor *and* a modification to the right-hand-side. By artificially simplifying the additional terms given in (63) to consider pressure jumps only, we have in each macroelement

$$b\left(\boldsymbol{u}_{\mathrm{B,M}}, q_0\right)_M = -\widetilde{C}'\,\frac{h_{\mathrm{M}}}{\mu}\,\sum_{k=1}^{3}\int_{e_k} [\![\, -q_0\underline{\mathbf{I}}\,]\!]_{e_k} \cdot [\![\, -p_0\underline{\mathbf{I}}\,]\!]_{e_k} \tag{64}$$

so that the stabilized equation reads as

$$\int_\Omega q_0 \operatorname{div} \boldsymbol{u}_{\mathrm{L}} + \sum_M \widetilde{C}'\,\frac{h_{\mathrm{M}}}{\mu}\,\sum_{k=1}^{3}\int_{e_k} [\![\, q_0\,]\!]_{e_k} \cdot [\![\, p_0\,]\!]_{e_k} = 0, \tag{65}$$

which is form-identical to the modification proposed in [21].

4 Conclusions

First we review the equal-order linear element enriched with bubbles (MINI-element) and how it yields a stabilized formulation consisting of the standard Galerkin method for linears, plus mesh-dependent terms that are integrals on element *interiors* of the Laplace operator for pressure and a right-hand-side correction. In particular, we point out that the residual-free bubbles stabilize similarly to cubic bubbles, and the former is optimal in the sense of Pierre [23].

Next we consider adding macro bubbles to the SIMPLEST element (linear-constant velocity-pressure pair). We consider residual-free macro bubbles defined in patches (or macroelements) of 4 elements (see Fig. 1). By statically condensing these macro bubbles, a stabilized method is unveiled, consisting of the standard Galerkin method for the SIMPLEST element plus mesh-dependent *jumps* across element boundaries in the interior of each macroelement. These jumps are taken with respect to the normal component of the Cauchy-stress tensor, a feature unique to this work. Previous stabilizations for the SIMPLEST [18, 21, 27] consider addition of jumps on pressure only. However, residual-free macro bubbles and the Galerkin method suggest that the SIMPLEST element should be stabilized with additions of terms dependent upon the jumps on the normal component of the Cauchy stress tensor. Another distinct feature of this work is the correction to the right-hand-side due to macro bubbles, implying the need of other terms to insure consistency in stabilized methods for the SIMPLEST element. We plan to report some numerical experiments using this formulation in contrast to others in a forthcoming paper.

Finally, stabilization due to macro bubbles produces jump terms across element boundaries in the interior of each macroelement, terms that are absent when we restrict our attention to bubbles added to the interior of each element only, *vis-à-vis* stabilization of the MINI element. If we consider high order interpolation for pressure, then not only we will have jump stabilization terms across elements, but also least-squares type terms defined in the interior of each element. The standard Galerkin method enriched with macro bubbles sheds some light on the origins of jump stabilization terms and suggest how to improve them.

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