On the Dimension of the Set of Rim Perturbations for Optimal Partition Invariance

H.J. Greenberg†, A.G. Holder†, C. Roos‡, and T. Terlaky‡

December 2, 1996

Abstract

Two new dimension results are presented. For linear programs, it is shown that the sum of the dimension of the optimal set and the dimension of the set of objective perturbations for which the optimal partition is invariant equals the number of variables. A decoupling principle shows that the primal and dual results are additive. The main result is then extended to convex quadratic programs, but the dimension relationships are no longer dependent only on problem size. Further, although the decoupling principle does not extend completely, the dimensions are additive, as in the linear case. Futhermore, if a strictly complementary solution exists, all the results are completely analogous to the linear case.

Keywords: linear programming, optimal partition, polyhedron, polyhedral combinatorics, quadratic programming, computational economics.

†Mathematics Department, University of Colorado at Denver, Denver, Colorado.

‡Faculty of Technical Mathematics and Informatics, Delft University of Technology, Delft, The Netherlands.

1 Introduction and Background

Consider the primal-dual linear programs:

$$\min\{cx : x \geq 0,\ Ax = b\} \qquad \max\{yb : s \geq 0,\ yA + s = c\},$$

where c is a row vector in $\mathbb{R}^n$, called *objective coefficients*; x is a column vector in $\mathbb{R}^n$, called *levels*, b is a column vector in $\mathbb{R}^m$, called *right-hand sides*; y is a row vector in $\mathbb{R}^m$ called *prices*; and A is an $m \times n$ matrix with rank m.

Let P and D denote the primal and dual polyhedra, respectively, and let P^* and D^* denote their optimality regions, which we assume to be non-empty. Let (x^*, y^*, s^*) be a strictly complementary optimal solution, and let the optimal partition be denoted by $(B|N)$, where

$$B = \sigma(x) \equiv \{j : x^*_j > 0\} \quad \text{and} \quad N = \sigma(s) \equiv \{j : s^*_j > 0\}.$$

(For background, see [6].)

This paper first presents a result about the dimension of P^* (D^*) in connection with the set of direction vectors in $\mathbb{R}^n$ (resp., $\mathbb{R}^m$) for which the optimal partition does not change when the objective coefficients (resp., right-hand sides) are perturbed in that direction. After establishing fundamental relations for linear programs, we consider extensions to convex quadratic programs. The technical terms used throughout this paper are defined in the *Mathematical Programming Glossary* [2].

2 Linear Programs

Following Greenberg [3], let $r = (b, c)$ denote the *rim data*, and let H denote the set of rim direction vectors, $h = (\delta b, \delta c)$, for which the optimal partition does not change on the interval, $[r, r + \theta h]$ for some $\theta > 0$. This is characterized by maintaining the following interior optimality conditions.

$$\begin{array}{ll} Ax = b + \theta\,\delta b & yA + s = c + \theta\,\delta c \\ x_B > 0 \text{ and } x_N = 0 & s_B = 0 \text{ and } s_N > 0. \end{array}$$

Here we follow the notation in [1, 6], where a subscript on a vector means it is the subvector restricted to the indexes in the subscript. For example, x_B is the vector of positive levels. This notation extends to matrices: A partitions into $[A_B\, A_N]$.

Let H_c denote the projection of H onto $\mathbb{R}^n$ for changing only c:

$$H_c \equiv \{\delta c : (0, \delta c) \in H\}.$$

Similarly, let H_b denote the projection of H onto $\mathbb{R}^m$ for changing only b:

$$H_b \equiv \{\delta b : (\delta b, 0) \in H\}.$$

Greenberg [3] showed that H is a convex cone that satisfies a *decoupling principle*: $H = H_b \times H_c$.

To help build intuition, notice first that if the dimension of the primal optimality region, $\dim(P^*)$, is zero, this means it is an extreme point. In that case, every vector in $\mathbb{R}^n$ can be used to change c without changing the optimal partition, so $\dim(H_c) = n$. At the other extreme, suppose $\dim(P^*) = n - m$, such as when $c = 0$, so every feasible solution is optimal in the primal LP. In that case, H_c consists of change vectors that maintain equal net effects among the positive variables, so $\dim(H_c) = m$. This latter case can be illustrated with the following.

Example: $\min\{\sum_j 0x_j : \sum_j x_j = 1, x \geq 0\}$.

> In this case, $B = \{1, \ldots, n\}$. In order for this partition not to change for the LP: $\min\{\sum_j \delta c_j x_j : \sum_j x_j = 1, x \geq 0\}$, it is necessary and sufficient that $\delta c_j = \delta c_1$ for all j. Thus, $\dim(H_c) = 1$. □

In both cases, we see that $\dim(P^*) + \dim(H_c) = n$. This is what we shall prove in general along with related results.

Theorem 2.1 *The following equations hold for any linear program whose primal and dual sets have nonempty strict interiors.*

1. $\dim(P^*) + \dim(H_c) = n.$
2. $\dim(D^*) + \dim(H_b) = m.$
3. $\dim(P^* \times D^*) + \dim(H) = n + m.$

Proof: From Lemma IV.44 in [6], we have $\dim(P^*) = |B| - \operatorname{rank}(A_B)$. The conditions for $\delta c \in H_c$ are:

$$yA_B = c_B + \theta\, \delta c_B \text{ and } yA_N < c_N + \theta\, \delta c_N$$

for some $\theta > 0$. Thus, δc_N can be arbitrary, so

$$\begin{aligned} \dim(H_c) &= |N| + \dim(\{\delta c_B : \exists \delta y \in \mathbb{R}^m \ni \delta y A_B = \delta c_B\}) \\ &= |N| + \operatorname{rank}(A_B). \end{aligned}$$

This implies $\dim(P^*) + \dim(H_c) = |B| + |N| = n$.

The second statement has a similar argument. From Lemma IV.44 in [6], $\dim(D^*) = m - \operatorname{rank}(A_B)$. The conditions for $\delta b \in H_b$ are:

$$A_B x_B = b + \theta\ \delta b \text{ and } x_B > 0.$$

Thus, $\dim(H_b) = \operatorname{rank}(A_B)$, so $\dim(D^*) + \dim(H_b) = m$. The last statement follows from the decoupling principle, upon adding the first two equations: $H = H_b \times H_c \Rightarrow \dim(H) = \dim(H_b) + \dim(H_c)$. □

We now consider some corollaries whose proofs are immediate from the theorem, but whose meanings lend insight into how perturbation relates to the dimensions of the primal and dual optimality regions.

The dimension of a set is sometimes called the *degrees of freedom*. If there are n variables and no constraints on their values, the set has the full degrees of freedom, which is n — i.e., each variable can vary independently. When the set is defined by a system of m independent equations, as in our case, we sometimes refer to m as the *degrees of freedom lost*. Because we assume there exists a strict interior solution ($x > 0$), there are no implied equalities among the nonnegativity constraints, so $\dim(P) = n - m$. Thus, the feasibility region has m degrees of freedom lost due to the equations that relate the variables.

A meaningful special case is when there is an excess number of columns, say $|B| = m + k$, and there is enough linear independence retained in the columns that $\operatorname{rank}(A_B) = m$ (recall we assume $\operatorname{rank}(A) = m$). Then, $\dim(P^*) = k$, so $\dim(H_c) = n - k$. In words, the degrees of freedom lost in varying objective coefficients equals the number of excess columns over those of a basic optimal solution. Further, $\text{rank}(A_B) = m$ is equivalent to $\dim(D^*) = 0$ (i.e., unique dual solution), so we can say the following.

Corollary 2.1 *The following are equivalent.*

1. *The dual solution is unique.*
2. $\dim(H_c) = n + m - |B|$.
3. $\dim(H_b) = m$. □

Another special case arises when the LP is a conversion from the inequality constraints, $A'x \geq b$, where A' is $m \times n'$, and $\text{rank}(A') = m$. In that case, $A = [A' \; -I]$, and $n = n' + m$. Suppose $x^* > 0$, so B includes all of the structural variables and some of the surplus variables, say $|B| = n' + k$. Then, $\dim(P^*) = k$, and Theorem 2.1 implies $\dim(H_c) = n' + m - k$. Since we do not allow the costs of the surplus variables to be nonzero, we can reduce this by m, giving $\dim(H_{c'}) = n' - k$. In words, this says that the degrees of freedom lost in varying (structural) cost coefficients equals the number of positive surplus variables.

A similar result follows for the primal. The next corollary says, in part, that $\dim(P^*) = 0$ if, and only if, $\dim(H_c) = n$. In words, this says that the primal solution is unique if, and only if, every objective coefficient can be perturbed independently without changing the optimal partition. The last equivalence includes the special case of a nondegenerate basic solution, in which case $|B| = m$, so every right-hand side can be perturbed without changing the optimal partition.

Corollary 2.2 *The following are equivalent.*

1. *The primal solution is unique.*
2. $\dim(H_c) = n$.
3. $\dim(H_b) = |B|$. □

These corollaries combine to the following, which is the familiar case of a unique strictly complementary optimum (which is basic).

Corollary 2.3 *The following are equivalent.*

1. *The primal-dual solution is unique.*
2. $\dim(H_c) = n$ *and* $\dim(H_b) = m$.
3. $\dim(H) = m + n$. $\square$

The following corollary says $\dim(H_c) \geq m$, and it follows from the main theorem since the maximum dimension of P^* is $n-m$. (The analogous bound for $\dim(H_b)$ is merely that it is nonnegative since the maximum dimension of D^* is m.)

Corollary 2.4 *There are at least m degrees of freedom to vary the objective coefficients without changing the optimal partition.*

In the next section, we extend Theorem 2.1 to convex quadratic programs, and note that care must be taken when specializing it to a linear program.

3 Quadratic Programs

We now extend Theorem 2.1 to the convex quadratic program:

$$\min\{cx + \tfrac{1}{2}x^T Qx : Ax = b,\ x \geq 0\},$$

where Q is symmetric and positive semi-definite. We use the Wolfe dual [2]:

$$\max\{yb - \tfrac{1}{2}u^T Qu : yA + s - u^T Q = c,\ s \geq 0\}.$$

Let QP and QD denote primal and dual feasibility regions, respectively, and let QP^* and QD^* denote their optimality regions, except that we define QD^* exclusive of the u-variables. We shall explain this shortly; for now let $QD_u{}^*$ denote the full dual optimality region to distinguish it from QD^* .

Following Jansen [4] and Berkelaar, Roos and Terlaky [1], an optimal partition is defined by three sets: $(B|T|N)$, where

$$\begin{aligned} B &= \{j : x_j > 0 \text{ for some } x \in QP^*\}, \\ N &= \{j : s_j > 0 \text{ for some } (y,s) \in QD^*\}, \text{ and} \\ T &= \{1, \ldots, n\} \setminus (B \cup N). \end{aligned}$$

We assume the solution obtained is *maximal* [1]:

$$x_j > 0 \Longleftrightarrow j \in B \quad \text{and} \quad s_j > 0 \Longleftrightarrow j \in N.$$

Güler and Ye [5] show that many interior point algorithms converge to a solution whose support sets comprise the maximal partition: $B = \sigma(x)$, $N = \sigma(s)$, and $T = \{1, \ldots n\} \setminus (B \cup N)$.

Unlike linear programming, there is no guarantee of a strictly complementary optimal solution, so T need not be empty. For this and other reasons, there are some important differences (see [1, 4] for details) that affect our extension of Theorem 2.1. In particular, the decoupling principle does not apply since a change in c affects both primal and dual optimality conditions.

We begin our extension with the following lemma. In the proof we use the following notation:

$$\begin{aligned} \mathrm{col}(G) &= \text{column space of } G &&= \{u : u = Gx \text{ for some } x \in \mathbb{R}^n\} \\ \mathcal{N}(G) &= \text{null space of } G &&= \{x : Gx = 0\}. \end{aligned}$$

Lemma 3.1 *Let F and G be $m \times n$ and $g \times n$ matrices, respectively, and consider the set: $S = \{v : v = Fu$ for some $u \ni Gu = 0\}$. Then,* $\dim(S) = \mathrm{rank}\begin{pmatrix} F \\ G \end{pmatrix} - \mathrm{rank}(G)$.

Proof: Without loss in generality assume G has full row rank, and let $\{u_1, \ldots, u_g\}$ be a basis for $\mathrm{col}(G)$. Let $\{v_1, \ldots, v_s\}$ be a basis for S (where $\dim(S) = s$), and consider the following set in $\mathrm{col}\begin{pmatrix} F \\ G \end{pmatrix}$:

$$\left\{ \begin{pmatrix} w_1 \\ u_1 \end{pmatrix} \cdots \begin{pmatrix} w_g \\ u_g \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \cdots \begin{pmatrix} v_s \\ 0 \end{pmatrix} \right\},$$

where $w_i \equiv FG^T[GG^T]^{-1}u_i$. Once we prove this is a basis for $\mathrm{col}\begin{pmatrix} F \\ G \end{pmatrix}$, we have that $g+s = \text{rank}\begin{pmatrix} F \\ G \end{pmatrix}$, which implies the desired result.

First, we shall prove these vectors are linearly independent. Suppose

$$\sum_i \alpha_i \begin{pmatrix} w_i \\ u_i \end{pmatrix} + \sum_j \beta_j \begin{pmatrix} v_j \\ 0 \end{pmatrix} = 0.$$

Since $\{u_1, \ldots, u_g\}$ is a basis, $\alpha = 0$, which then implies $\beta = 0$ because $\{v_1, \ldots, v_s\}$ are also linearly independent.

Second, we shall prove these vectors span $\text{col}\begin{pmatrix} F \\ G \end{pmatrix}$. Let $\begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} x$ for some $x \in \mathbb{R}^n$. Decompose $x = y + z$, where $y \in \text{col}(G^T)$ and $z \in \mathcal{N}(G)$. Then, $Gx = Gy = GG^T\lambda$, where $y = G^T\lambda$, and $Fx = Fy + Fz$. Since $Fz \in S, Fx = FG^T\lambda + \sum_j \beta_j v_j$. We thus have $u = Gx = GG^T\lambda$, but since $u \in \text{col}(G)$, $GG^T\lambda = \sum_i \alpha_i u_i$. This implies $\lambda = \sum_i \alpha_i [GG^T]^{-1} u_i$, so

$$\begin{aligned} Fx &= FG^T \textstyle\sum_i \alpha_i [GG^T]^{-1} u_i + \sum_j \beta_j v_j \\ &= \textstyle\sum_i \alpha_i FG^T [GG^T]^{-1} u_i + \sum_j \beta_j v_j. \end{aligned}$$

By definition of w, we have derived α, β such that:

$$\begin{pmatrix} v \\ u \end{pmatrix} = \textstyle\sum_i \alpha_i \begin{pmatrix} w_i \\ u_i \end{pmatrix} + \sum_j \beta_j \begin{pmatrix} v_j \\ 0 \end{pmatrix}. \quad \square$$

To prove the main theorem, we use the following dimension results of Berkelaar, Roos and Terlaky [1]:

$$\dim(QP) \;=\; |B| - \text{rank}\begin{pmatrix} A_B \\ Q_{BB} \end{pmatrix}. \tag{1}$$

$$\dim(QD_u{}^*) \;=\; m - \text{rank}(A_B\ A_T) + n - \text{rank}(Q). \tag{2}$$

The last portion, $n - \text{rank}(Q)$, accounts for the u-variables because the dual conditions can use x^TQ in place of u^TQ, leaving u to appear only in the equation $Qu = Qx$. For our purposes it is not necessary (or desirable) to include this, so we define the dual optimality region exclusive of the u-variables:

$$QD^* = \{(y,s) : (y,s,x) \in QD_u{}^* \text{ for some } x \in QP^*\}.$$

Then, (2) yields the dimension of the dual optimality region that we shall use:

$$\dim(QD^*) \;=\; m - \text{rank}(A_B\ A_T). \tag{3}$$

As in the linear case, $s_N > 0$ implies each component of c_N can vary independently, so $\dim(H)$ is the sum of $|N|$ and the dimension of the set of other possible changes. Keeping $x_{N\cup T} = 0$ and $s_{B\cup T} = 0$, the partition does not change if, and only if, there exists $(\delta y, \delta u, \delta x_B)$ to satisfy the following primal-dual conditions:

$$\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & 0 & A_B \\ 0 & -Q & Q_{\bullet B} \end{pmatrix} \begin{pmatrix} \delta y \\ \delta u \\ \delta x_B \end{pmatrix} = \begin{pmatrix} \delta c_B \\ \delta c_T \\ \delta b \\ 0 \end{pmatrix}. \tag{4}$$

Here we follow the notation in [1]:

$Q_{I\bullet}$ = rows of Q associated with index set I.
$Q_{\bullet J}$ = columns of Q associated with index set J.
Q_{IJ} = submatrix of Q associated with index sets I and J.

The quadratic extensions rely on the fact that the rank of the matrix in (4) is related to the rank of the matrices found in statements (1) and (3). These relations are formalized in the following.

Lemma 3.2 *The following relations hold for Q positive semi-definite.*

$$\operatorname{rank}(A_B\ A_T) + \operatorname{rank}\begin{pmatrix} A_B \\ Q_{BB} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & 0 & A_B \\ 0 & -Q & Q_{\bullet B} \end{pmatrix} - \operatorname{rank}(Q) \quad (5)$$

$$= \operatorname{rank}\begin{pmatrix} A_B & A_T \\ Q_{BB} & Q_{BT} \end{pmatrix} + \operatorname{rank}(A_B). \quad (6)$$

Proof: To prove (5), performing elementary row and column operations on the large matrix (first on right) produces the following matrix of the same rank:

$$\begin{pmatrix} A_B^T & 0 & -Q_{BB} \\ A_T^T & 0 & -Q_{TB} \\ 0 & 0 & A_B \\ 0 & Q & 0 \end{pmatrix}.$$

So,

$$\operatorname{rank}\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & 0 & A_B \\ 0 & -Q & Q_{\bullet B} \end{pmatrix} = \operatorname{rank}\begin{pmatrix} A_B^T & -Q_{BB} \\ A_T^T & -Q_{TB} \\ 0 & A_B \end{pmatrix} + \operatorname{rank}(Q).$$

The positive semidefiniteness of Q implies that Q_{TB} is linearly dependent on Q_{BB} [1]. Hence,

$$
\begin{pmatrix} A_B^T & -Q_{BB} \\ A_T^T & -Q_{TB} \\ 0 & A_B \end{pmatrix} \Rightarrow \begin{pmatrix} \tilde{A}_B^T & * & 0 & 0 \\ 0 & 0 & 0 & \tilde{Q}_{BB} \\ 0 & \tilde{A}_T^T & 0 & 0 \\ 0 & 0 & \tilde{A}_B & * \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\Rightarrow \begin{pmatrix} \tilde{A}_B^T & * & 0 & 0 \\ 0 & \tilde{A}_T^T & 0 & 0 \\ 0 & 0 & \tilde{A}_B & * \\ 0 & 0 & 0 & \tilde{Q}_{BB} \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

where $\Rightarrow$ is used to represent a series of row and column operations that preserve rank, the resulted submatrices are indicated by $\tilde{}$ and $*$ represents an arbitrary matrix of appropriate size. Hence,

$$
\begin{aligned}
\operatorname{rank}\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & 0 & A_B \\ 0 & -Q & Q_{\bullet B} \end{pmatrix} &= \operatorname{rank}(Q) + \\
& \operatorname{rank}\begin{pmatrix} \tilde{A}_B^T & * \\ 0 & \tilde{A}_T^T \end{pmatrix} + \operatorname{rank}\begin{pmatrix} \tilde{A}_B & * \\ 0 & \tilde{Q}_{BB} \end{pmatrix} \\
&= \operatorname{rank}(Q) + \\
& \operatorname{rank}\begin{pmatrix} A_B^T \\ A_T^T \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_B \\ Q_{BB} \end{pmatrix},
\end{aligned}
$$

which yields the result.

The proof of (6) is similar, using the positive semi-definiteness property of Q in reducing the large matrix to row echelon form. □

We now have what we need to prove the following extension of Theorem 2.1.

Theorem 3.1 *The following equations hold for any convex quadratic program whose primal and dual sets are not empty.*

1. $\dim(QP^*) + \dim(H_c) = n - |T| + \operatorname{rank}(A_B\ A_T) - \operatorname{rank}(A_B)$.
2. $\dim(QD^*) + \dim(H_b) = m - \operatorname{rank}(A_B\ A_T) + \operatorname{rank}(A_B)$.
3. $\dim(QP^* \times QD^*) + \dim(H) = n + m - |T|$.

Proof: To prove 1, we set $\delta b = 0$ in 4, and apply Lemmas 1 and 2 to produce the following:

$$\dim(H_c) = |N| + \operatorname{rank}\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & 0 & A_B \\ 0 & -Q & Q_{\bullet B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} 0 & -Q & Q_{\bullet B} \\ 0 & 0 & A_B \end{pmatrix}$$

$$= |N| + \operatorname{rank}(A_B\ A_T) + \operatorname{rank}\begin{pmatrix} A_B \\ Q_{BB} \end{pmatrix}$$

$$+ \operatorname{rank}(Q) - \operatorname{rank}(Q) - \operatorname{rank}(A_B)$$

$$= |N| + \operatorname{rank}(A_B\ A_T) - \operatorname{rank}\begin{pmatrix} A_B \\ Q_{BB} \end{pmatrix} - \operatorname{rank}(A_B).$$

Adding (1) to the last statement and substituting $n - |T| = |B| + |N|$ gives the first result. Similarly, to prove 2, set $\delta c_B = 0$ and $\delta c_T = 0$ in (4). Then, Lemma 1 implies the equation:

$$\dim(H_b) = \operatorname{rank}\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & 0 & A_B \\ 0 & -Q & Q_{\bullet B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & -Q & Q_{\bullet B} \end{pmatrix}.$$

Using row and column operations on the matrix in the last term together with Lemma 2 we obtain the dimension of H_b:

$$\begin{aligned}
\dim(H_b) &= \operatorname{rank}\begin{pmatrix} A_B^T \\ \\ A_T^T \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_B \\ \\ Q_{BB} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} A_B^T & Q_{BB} \\ \\ A_T^T & Q_{TB} \end{pmatrix} \\
&= \operatorname{rank}(A_B),
\end{aligned}$$

where the last equation follows from (6). Adding this to (3) yields the second result.

The third result does not follow from a decoupling principle, as in the linear case (where $H = H_b \times H_c$). Rather, it needs a development similar to the first two parts just obtained. Using Lemmas 1 and 2 yields the following equations:

$$\begin{aligned}
\dim(H) &= |N| + \operatorname{rank}\begin{pmatrix} A_B^T & -Q_{B\bullet} & 0 \\ A_T^T & -Q_{T\bullet} & 0 \\ 0 & 0 & A_B \\ 0 & -Q & Q_{\bullet B} \end{pmatrix} - \operatorname{rank}\begin{pmatrix} 0 & -Q & Q_{\bullet B} \end{pmatrix} \\
&= |N| + \operatorname{rank}\begin{pmatrix} A_B^T \\ \\ A_T^T \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_B \\ \\ Q_{BB} \end{pmatrix}.
\end{aligned}$$

The sum of the last statement with (1) and (3) plus substituting $n - |T| = |B| + |N|$ imply the third result. □

Notice that the statements in Theorem 3.1 reduce to the corresponding statements in Theorem 2.1 when $T = \emptyset$ and $Q = 0$, which is the case for a linear program. (This reduction occurs because we eliminated the u-variables.) In fact, the statements in the theorem imply each of the following.

$$\begin{array}{rcll} \dim(QP^*) + \dim(H_c) & \leq & n & \text{with equality if } T = \emptyset. \\ \dim(QD^*) + \dim(H_b) & \leq & m & \text{with equality if } T = \emptyset. \\ \dim(QP^* \times QD^*) + \dim(H) & \leq & m + n & \text{with equality if } T = \emptyset. \end{array}$$

The reduction of $QD_u{}^*$ also enables us to have the following extension of Corollary 2.1.

Corollary 3.1 *The following are equivalent.*

1. *The dual solution is unique.*
2. $\dim(H_c) = n + m - |T| - \operatorname{rank}(A_B) + \operatorname{rank}(A_B^T\, Q_{BB})$.
3. $\dim(H_b) = m + \operatorname{rank}(A_B) - \operatorname{rank}(A_B\ A_T)$.

The above cases reduce to the corresponding LP cases in Corollary 2.1, where $Q = 0$ and $T = \emptyset$, as does the following extension of Corollary 2.2.

Corollary 3.2 *The following are equivalent.*

1. *The primal solution is unique.*
2. $\dim(H_c) = n - |T| + \operatorname{rank}(A_B\ A_T) - \operatorname{rank}(A_B)$.
3. $\dim(H_b) = |B| - \operatorname{rank}(A_B^T\, Q_{BB}) + \operatorname{rank}(A_B)$.

Combining these, despite the absence of a decoupling principle, the dimensions are additive, so we also obtain the following extension of Corollary 2.3.

Corollary 3.3 *The following are equivalent.*

1. *The primal-dual solution is unique.*
2. $\dim(H_c) = n - |T|$ *and* $\dim(H_b) = m$.
3. $\dim(H) = m + n - |T|$.

Unlike the LP case, this shows that we lose $|T|$ degrees of freedom in varying the cost coefficients. For example, if $\delta c_j > 0$ for $j \in T$, the partition immediately changes since $s_j = \delta c_j$ is optimal for the perturbed quadratic program. This loss appears in the last extension, which follows.

Corollary 3.4 *There are at least $m - |T| + \text{rank}(A_B\ A_T) - \text{rank}(A_B)$ degrees of freedom to vary the rim data without a change in the optimal partition.*

This lower bound on $\dim(H_c)$ follows in the same way as in Corollary 2.4, and it is m when $T = \emptyset$. More generally, we see that the bound is at most m, which reflects the fact that we can lose some degrees of freedom by lacking strict complementarity.

4 Concluding Comments

For linear programs the dimension of the cone of rim direction vectors for which the optimal partition does not change has an Eulerian property with the dimension of the optimality region: they sum to the number of variables and equations. This decouples into Eulerian properties for varying the primal and dual right-hand sides separately: cost coefficients change with lost degrees of freedom equal to the dimension of primal space; right-hand sides change with lost degrees of freedom equal to the dimension of dual space. The comparable equation for quadratic programs is not Eulerian in that the sum of dimensions depends on the partition — notably on the number of complementary coordinate pairs that are both zero.

Acknowledgement

The authors thank Stanley Payne for his advice on lemma 3.1.

References

[1] A. Berkelaar, C. Roos, and T. Terlaky, *Chapter 6: The optimal set and optimal partition approach to linear and quadratic programming,*

in Recent Advances in Sensitivity Analysis and Parametric Programming, T. Gal and H. Greenberg, eds., Kluwer Academic Press, Boston, MA, to appear 1997.

[2] H. GREENBERG, *Mathematical Programming Glossary*, World Wide Web, http://www-math.cudenver.edu/ hgreenbe/glossary/glossary.html, 1996.

[3] ——, *Rim sensitivity analysis from an interior solution*, CCM No. 86, Center for Computational Mathematics, Mathematics Department, University of Colorado at Denver, Denver, CO, 1996.

[4] B. JANSEN, *Interior point techniques in optimization: Complexity, sensitivity, and algorithms*, ph.d. thesis, Faculty of Technical Mathematics and Informatics/Computer Science, Delft University of Technology, Delft, The Netherlands, 1995 (also to appear 1997, Kluwer Academic Press).

[5] O. GÜLER AND Y. YE, *Convergence behavior of interior-point algorithms*, Mathematical Programming, 60 (1993), pp. 215–228.

[6] C. ROOS, T. TERLAKY, AND J.-P. VIAL, *Theory and Algorithms for Linear Optimization: An Interior Point Approach*, John Wiley & Sons, New York, NY, to appear 1997.