CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364

Fax: (303) 556-8550
Phone: (303) 556-8442
http://www-math.cudenver.edu/

December 1996 UCD/CCM Report No. 95

Interval Methods and Fuzzy Optimization

Weldon A. Lodwick and K. David Jamison

December 1996 **UCD/CCM Report No. 95**

Interval Methods and Fuzzy Optimization

Weldon A. Lodwick, K. David Jamison
Department of Mathematics, Campus Box 170,
University of Colorado at Denver
P.O. Box 173364,Denver, CO 80217-3364, U.S.A.
email: wlodwick@math.cudenver.edu, kjamison@math.cudenver.edu

ABSTRACT: In this paper we develop methods to solve constrained fuzzy minimization problems for a class of fuzzy functions by using interval methods. The class of fuzzy functions we consider for the minimization problems is the set of real-valued functions where one or more parameters/coefficients are fuzzy numbers. The focus of this research is to explore some relationships between fuzzy set theory and interval analysis as it relates to optimization problems

Keywords: Fuzzy/interval number, fuzzy/interval function, fuzzy/interval optimization

Introduction: This paper considers a fuzzy number as uncertainty in a given numeric quantity by defining a possibility distribution for the quantity. For example, *the value is observed to be about 2* might be represented by the triangular fuzzy number (1.5,2,2.5) (see [4]) which defines a possibility distribution for the observation. The class of fuzzy functions considered here are those that are real-valued functions with at least one coefficient being a fuzzy number. In particular, the fuzzy functions we use are real-valued functions where the coefficients (parameters) are explicitly denoted as a vector $\vec{a}$, so that $f(\vec{x}) \equiv f(\vec{a}, \vec{x})$ and it is a subset of the components of $\vec{a}$ that contain the uncertainty of the underlying cause/effect that the function models. When one or more of the parameters are "fuzzified," we denote the fuzzy function $f(\tilde{a}, \vec{x})$.

A decision maker (DM) is faced with optimizing a function subject to constraints where it is not known precisely what objective function and what the constraint functions are except possibilistically. In our case this means that the DM does not know precisely what the coefficients $\vec{a} \in \tilde{a}$ are, just that they belong to their respective possibilistic/fuzzy sets. Each instance of $\vec{a} \in \tilde{a}$ yields a crisp nonlinear programming (nlp) problem for which an optimal solution exits in principle. In this situation, not only does the DM need to optimize over all decision variables $\vec{x}$ (the traditional nlp problem), but even before this, the DM must deal with the uncertainties $\tilde{a}$. The three approaches we present in the sequel to solve the fuzzy optimization problem are three ways that deal with the uncertainties $\tilde{a}$ and in turn, are chosen so that we can exploit interval methods.

Throughout this paper we will identify a fuzzy subset with the symbol $\sim$ over a letter. For example if X is a set, then $\tilde{\text{x}}$ will be used to denote a fuzzy subset of X and $\tilde{\text{x}}_\alpha$ will denote the α-level of possibility for $\tilde{\text{x}}$, i.e. it is the crisp set

$$\tilde{\text{x}}_\alpha = \left\{ \text{x} \;\middle|\; \begin{array}{ll} \mu_{\tilde{x}}(\text{x}) \geq \alpha & \text{for } \alpha \in (0,1] \\ \mu_{\tilde{x}}(\text{x}) > 0 & \text{for } \alpha = 0 \end{array} \right\}$$

where $\mu_{\tilde{x}}$ denotes the membership function of $\tilde{\text{x}}$. The set of x for which $\mu_{\tilde{x}}(\text{x})>0$ is called the *support of* $\tilde{x}$. We also use interval notation as follows. When the context requires it, an interval number is superscripted so that $a^I = [a^-, a^+]$ where $a^- \leq a^+$ are the left and right endpoints of the interval a^I . We will use the midpoint of an interval so define $mid\{a^I\} = (a^- + a^+)/2$. The rules associated with interval arithmetic can be found in several texts (see [4] and [8] for example).

The nonlinear fuzzy constrained minimization problem we study here is:

$$opt\ z = f(\tilde{a}, \vec{x}) \tag{1}$$

such that

$$G(\tilde{b}, \vec{x}) = (g_1(\tilde{b}, \vec{x}), \ldots, g_m(\tilde{b}, \vec{x}))^T \leq 0 \tag{2}$$

$$\vec{x} \in \Omega_n = [x_1^-, x_1^+] \times \ldots \times [x_n^-, x_n^+] \tag{3}$$

where $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_p)^T, \tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_q)^T$ are vectors of fuzzy numbers considered here as the parameters (coefficients) of the problem, f, G are continuous for each α-level in the parameter and variables and *opt* is either min or max.

Given fuzzy numbers $\tilde{a}_i$, and $\tilde{b}_i$, let the α-level ($0 \leq \alpha \leq 1$) be denoted

$$a_i^I(\alpha) = [a_i^-(\alpha), a_i^+(\alpha)] \tag{4}$$

$$b_i^I(\alpha) = [b_i^-(\alpha), b_i^+(\alpha)], \tag{5}$$

then the $\alpha-$level of the coefficient vectors are the closed and bounded interval vectors

$$\vec{a}^I(\alpha) = (a_1^I(\alpha), \ldots, a_p^I(\alpha))^T, \text{ and } \vec{b}^I(\alpha) = (b_1^I(\alpha), \ldots, b_q^I(\alpha))^T.$$

We will concentrate our analysis on the α-level of the constraint set given by (2)and (3); that is,

$$\Omega_\alpha^k = \{\vec{x} \mid g_k(\vec{b}_\alpha, \vec{x}) \leq 0, \vec{x} \in \Omega_n \text{ for some } \vec{b}_\alpha \in \vec{b}_\alpha^I \ \} \tag{6}$$

$$\Omega^{\alpha} = \cap_{k=1}^{m} \Omega_{\alpha}^{k} \tag{7}$$

In the sequel, we prove that Ω^{α} is compact so that our optimization problem is, in principle, well-defined.

We will have occasion to use the unconstrained version of the constrained optimization problem by folding the constraints into the objective function using penalties for violating the constraint (6) in the following way. Let

$$F(\tilde{a},\vec{x}) \equiv F(\tilde{a},\tilde{b},\tilde{c},\vec{x}) = f(\tilde{a},\vec{x}) + \sum_{k=1}^{m} h_k(\tilde{a},\tilde{b},\tilde{c},\vec{x}) \max\{g_k(\tilde{b},\vec{x}),0\} \tag{8}$$

for $\vec{x} \in \Omega_n$ and h_k is the fuzzy penalty function for violating the k^{th} constraint and $\tilde{c}$ is an explicit fuzzy parameter of the penalty function h_k different from the fuzzy parameters that were originally given. We have renamed the right-most $\tilde{a}$ of (8) $\tilde{a} := (\tilde{a},\tilde{b},\tilde{c})^T$.

The second section discusses how to obtain a solution at each α-level which will result in one fixed number for our first approach and a closed interval of numbers for each decision variable $\vec{x}$ for our second and third approaches. These α-level solutions form fuzzy numbers and these will be analyzed with respect to obtaining an optimum from a set of fuzzy numbers in the third section. Since a membership function is uniquely defined by its α-levels and each membership function uniquely defines a fuzzy set (number), the approach taken here is well-defined. The fourth section contains some theorems dealing with the properties of the constraint sets and the last section contains some conclusions and observations.

Remark 1: We interpret the fuzzy inequality constraint $G(\tilde{b},\vec{x}) \leq 0$ to be satisfied if **for some** $\vec{b}_{\alpha} \in \vec{b}_{\alpha}^{I}$ the inequality is satisfied for each $0 \leq \alpha \leq 1$ since all values of $\vec{b}_{\alpha}$ are at least α possible. If the constraint is interpreted as **for all** $\vec{b}_{\alpha} \in \vec{b}_{\alpha}^{I}$, the constraint is crisp.

Remark 2: If g_k is linear, then Ω_{α}^{k} is convex (see [12]). Thus, a fuzzy linear programming problem with crisp objective function coefficients becomes a convex programming problem.

Remark 3: We point out that a variety of researchers use several definitions for a fuzzy function (see, for example, [1], [3],[5],[11], and [15]). However, all these researchers agree that a real-valued function with fuzzy coefficients (our $f(\tilde{a},\vec{x})$ and $G(\tilde{b},\vec{x})$) constitutes a fuzzy function, albeit, not the most general that can be obtained.

Fuzzy Optimization: There is latitude in how to interpret what is meant by optimizing a fuzzy function subject to fuzzy constraint (see, for example, [1], [2], [3], [6], [7], [9], [10], [13], [14], and [16]). Since our focus is on the relationship between interval analysis and fuzzy set theory, we concentrate on three interpretations. One is based on [10], and two other approaches which we define below. We point out that Luhandjula, [7], has a useful discussion of fuzzy optimization approaches.

Approach 1: Sakawa and Yano, [10], interpret (1), (2) and (3) as follows. For a fixed α, find the optimal value over all parameters $\vec{a}_\alpha \in \vec{a}^I(\alpha)$ and $\vec{b}_\alpha \in \vec{b}^I(\alpha)$ and decision variables $\vec{x}$. That is, the parameters become decision variables along with $\vec{x}$ and are constrained by their respective intervals. In this case we have the following optimization problem:

$$z_\alpha = opt\ f(a_\alpha, \vec{x}) \tag{9}$$

such that

$$G(\vec{b}_\alpha, \vec{x}) = (g_1(\vec{b}_\alpha, \vec{x}), \ldots, g_m(\vec{b}_\alpha, \vec{x}))^T \leq 0 \tag{10}$$

$$\vec{x} \in \Omega_n, \vec{a}_\alpha \in \vec{a}^I(\alpha), \text{ and } \vec{b}_\alpha \in \vec{b}^I(\alpha). \tag{11}$$

It is clear that approach 1, under our initial assumptions, is simply a nlp which is well defined and in principle solvable using any number of techniques. What is also clear is that the solution from approach 1 is the most optimistic one possible and so is an upper bound for all fuzzy optimization solutions, though it assumes that the coefficients are under the DM's power to determine.

Regardless, each α-level yields a crisp number solution. Thus, the unique fuzzy number $\tilde{z}$ whose membership function is defined by the α-levels z_α and it is this fuzzy number $\tilde{z}$ that will be our fuzzy solution for approach 1. It remains to choose the optimal decision associated with this fuzzy number. This discussion is left to the next section.

Approach 2: The second approach to optimizing fuzzy functions is the following. For each possible decision $\vec{x}$, we wish to consider all possible outcomes. To consider all possible outcomes, we must allow a decision $\vec{x}$ the possibility of violating the constraint. For this reason, we work with the unconstrained version of the problem and use the penalized objective function $F(\tilde{a}, \vec{x})$, $\vec{x} \in \Omega_n$ given by (8). For each fixed $\vec{x} \in \Omega_n$, define

$$F_\alpha^-(\vec{x}) = \min\{F_\alpha(\vec{a}_\alpha, \vec{x}) \mid \vec{a}_\alpha \in \vec{a}^I(\alpha)\} \tag{12}$$

$$F_\alpha^+(\vec{x}) = \max\{F_\alpha(\vec{a}_\alpha, \vec{x}) \mid \vec{a}_\alpha \in \vec{a}^I(\alpha)\} \tag{13}$$

so that,

$$F^I_\alpha(\vec{x}) = [F^-_\alpha(\vec{x}), F^+_\alpha(\vec{x})]. \quad (14)$$

We could define

$$z_\alpha = opt\ \{mid(F^I_\alpha(\vec{x})) \mid \vec{x} \in \Omega_n\}, \quad (15)$$

but then z_α is the α-level optimum value of F(ã,$\vec{x}$). The decision variable $\vec{x}$ that is optimal at a given α-level, in general will not be related to an optimal decision for another α-level. That is, an optimal decision $\vec{x}_\alpha$ for a particular α may be a terrible decision for another α-level. By optimizing over the decisions at this point causes loss of information about the shape of the distribution of the uncertainty. The defuzzification process should be the last step of the process. For these reasons, we do not use (15) as an optimizing criterion.

Instead, we let fuzzy number $\tilde{F}(\vec{x})$ be that fuzzy set whose membership function is obtained by defining its α-level to be (14) and we find an optimal decision based on this fuzzy relation. In this way, we keep all the information associated with the α-levels and decide based on the possibility function for each decision $\vec{x}$ rather than defuzzifying at each α-level as in (15). How to obtain the optimal decision for these fuzzy numbers considering all α-levels is developed in the next section. Therefore, unlike approach 1, we have a fuzzy solution to our problem, which does NOT assume that the DM has the ability to control the choice of the parameters ã and $\tilde{b}$.

Approach 3: We call this the "Method of Minimum Regrets." This approach, like approach 2, considers the possibility of a constraints violation. Thus the penalized function, F(ã,$\vec{x}$), is used. Consider a DM who is faced with uncertainty in the parameters of function F. That is, the DM knows the objective function F possibilistically. If the DM knew what the parameter values were, an optimal decision could in principle be taken. That is, given fixed values $\vec{a}_\alpha \in \vec{a}^I(\alpha)$, $\vec{b}_\alpha \in \vec{b}^I(\alpha)$, and $\vec{c}_\alpha \in \vec{c}^I(\alpha)$, let

$$p(\vec{a}_\alpha) \equiv p(\vec{a}_\alpha, \vec{b}_\alpha, \vec{c}_\alpha) = opt\{F(\vec{a}, \vec{x}) \mid \vec{x} \in \Omega_n\}. \quad (16)$$

This is the optimal decision that, in theory, can be made given complete knowledge about the actual value of the parameters. Since the parameters are not know with certainty, the DM measures the affect of making a particular decision $\vec{x}$ by analyzing the absolute value of two differences, one between what is the best possible set of parameters that could occur for this particular decision $\vec{x}$ minus the optimal decision over all possible decisions given these best parameters and the other between what is the worse possible set of parameters that could occur for this particular decision $\vec{x}$ minus the optimal decision over all possible decisions given these worse parameters. These two values form an interval for each $\vec{x}$ which then defines a fuzzy number for each decision variable $\vec{x}$. From this set of fuzzy numbers we can, with methods discussed in the next section, obtain the decision which is an optimum.

Let's attempt to interpret what this might mean in the case that our original optimization problem were a linear programming problem. In this case the DM has a crisp linear program to solve but does not have enough information to fully characterize the linear program other than by using fuzzy coefficients. The objective of this third approach is to obtain a result from taking a decision that is as close to the optimal result as possible; i.e., to minimize the possible difference between the result the DM actually achieves (by making a decision) and the optimal result achievable given complete knowledge (of the parameters).

Formally, for each $\vec{x} \in \Omega_n$ let

$$e_\alpha^-(\vec{x}) = \min\{\mid F(\vec{a}_\alpha, \vec{x}) - p(\vec{a}_\alpha) \mid, \vec{a}_\alpha \in \vec{a}^I(\alpha)\}, \quad (17)$$
$$e_\alpha^+(\vec{x}) = \max\{\mid F(\vec{a}_\alpha, \vec{x}) - p(\vec{a}_\alpha) \mid, \vec{a}_\alpha \in \vec{a}^I(\alpha)\}, \quad (18)$$

so that for each $\vec{x}$ we have,

$$e_\alpha^I(\vec{x}) = [e_\alpha^-(\vec{x}), e_\alpha^+(\vec{x})]. \quad (19)$$

Thus, we have a fuzzy number $\tilde{e}(\vec{x})$ for each $\vec{x} \in \Omega_n$ uniquely defined by its membership function whose α-levels is given (19). We show how to obtain an optimal decision associated with this fuzzy relation, $\tilde{e}(\vec{x})$.

Remark 4:[3] show that all possible differences, $\mid F(\vec{a}_\alpha, \vec{x}) - p(\vec{a}_\alpha) \mid$, are contained in $e_\alpha^I(\vec{x})$ so that (19) is indeed a well-defined interval; i.e., it is connected for each $\vec{x}$. Thus, $e_\alpha^I(\vec{x})$ represents the range of all possible differences between a given decision $\vec{x}$ and the optimal outcome the DM could have achieved were knowledge sufficient to realize an a-priori crisp set of parameters.

As with approach 2, we have a fuzzy number (one for each decision) which must be minimized to obtain the solution to the fuzzy optimization problem and as in the previous approach, the ability of the DM to determine the value of ã and $\tilde{\text{b}}$ is not assumed. The discussion of how to pick the decision variable $\vec{x}$, which minimizes $\tilde{e}(\vec{x})$ over the set of fuzzy numbers is left to the next section. Before we develop this, we present an example.

Example: Consider a company that seeks to maximize its profits where the profit per commodity sold is know to lie in the interval [1,2] and there is a holding cost for unsold items known to be between 2 and 3; that is, in the interval [2,3]. Suppose the number of commodities sold can be no more than the parameter b where $b \in [10, 12]$. Thus, our fuzzy constrained optimization problem is:

$$\begin{aligned} \max f(\tilde{a}, \vec{x}) &= \tilde{a}x && (20)\\ \textit{subject to} &: \ 0 \leq x \leq \tilde{b} && (21)\\ \tilde{a} &= [1,2],\ \tilde{b} = [10,12],\ x \in [0,20] && (22) \end{aligned}$$

Note: Here, we have $x \in \Omega_1 = [0, 20]$ and in our constrained case, we do not allow the upper bound of the constraint function at each α-level to be violated;i.e., the penalty is infinite for violating the constraint. In essence, this means that $0 \leq x \leq 12$ for the constrained case and Ω_1 is redundant.

The unconstrained optimization problem resulting from folding into the objective function a penalty for violating the constraint is:

$$\begin{aligned} \max F(\tilde{a}, x) &= \tilde{a}x - \tilde{a}\max\{x - \tilde{b}, 0\} - \tilde{c}\max\{x - \tilde{b}, 0\} && (23)\\ \tilde{a} &= [1,2],\ \tilde{b} = [10,12],\ \tilde{c} = [2,3],\ x \in \Omega_1 = [0,20]. && (24) \end{aligned}$$

Unlike the constrained case, we allow the right-hand side constraint to be violated at a cost which is reflected in the penalty function

$$h(\tilde{a}, \tilde{b}, \tilde{c}, x) = -\tilde{a}\max\{x - \tilde{b}, 0\} - \tilde{c}\max\{x - \tilde{b}, 0\}.$$

The first penalty $\tilde{a}\max\{x - \tilde{b}, 0\}$ is simply the dollar for dollar offset to the objective ax for commodities that cannot be sold. The second penalty $\tilde{c}\max\{x - \tilde{b}, 0\}$is the cost of holding the unsold units. In this example, all α-levels of our fuzzy parameters are equal since the fuzzy numbers are intervals. This simplifies the identification of the optimal decision which is done below. We formally address the issue of optimizing over more general fuzzy numbers in the next section.

Approach 1(see Sakawa and Yano [10]): This approach works with the constrained problem (20), (21), and (22). It is clear that $f(\tilde{a}, x) \leq 24$ and that for $a = 2$, $b = 12$, and $x = 12$ (which is feasible), $f(2, 12) = 24$.

Therefore the optimal value is $x^* = 12$.

Approach 2: This approach works with the unconstrained problem (23), and (24). To solve the above, we break up the interval into three subintervals $0 \le x \le 10$, $10 \le x \le 12$ and $12 \le x \le 20$. Moreover, since the fuzzy numbers are intervals and the value of all the α-levels are the same, the resulting fuzzy function is an interval. In this case, with the absence of any other information, the optimum value is the decision x which optimizes the midpoint of the interval.

$(i) 0 \le x \le 10$: x is certainly feasible and we have an unconstrained problem.

$$F^I(\tilde{a}, \vec{x}) = [1, 2]x = [x, 2x] = [F^-(x), F^+(x)]$$

so that

$$\max\{mid(F^I(\tilde{a}, \vec{x}))\} \equiv \max\{(F^-(x) + F^+(x))/2\} = \max 3x/2, 0 \le x \le 10$$

which results in $F(\tilde{a}^*, \vec{x}^*) = 15$ where $x^* = 10$.

$(ii) 10 \le x \le 12$:

Case 1: $x \le b$

$$F^I(\tilde{a}, \vec{x}) = [1, 2]x = [x, 2x] = [F^-(x), F^+(x)]$$

Case 2: $x \ge b$

$$F(\tilde{a}, \vec{x}) = ax - a(x - b) - c(x - b) = ab + c(b - x)$$

and we have,

$$\begin{aligned} 40 - 3x &\le x \le ab + c(b - x) \le 2x \\ [x, 2x] &\subseteq [40 - 3x, 2x] \\ F^I(\tilde{a}, \vec{x}) &= [40 - 3x, 2x] = [F^-(x), F^+(x)] \end{aligned}$$

$$\max\{mid(F(\tilde{a}, \vec{x}))\} \equiv \max\{(F^-(x)+F^+(x))/2\} = \max\{20-x/2\}, 10 \le x \le 12$$

which results in $F(\tilde{a}^*, \vec{x}^*) = 15$ where $x^* = 10$.

(iii) $12 \le x \le 20$: Since $x \ge b$, (ii) Case 2 above applies and the minimum for these values of x will occur at x = 12 resulting in a objective function value of 14 which is inferior to (ii).

Therefore, overall, the optimum is $F(\tilde{a}^*, \vec{x}^*) = 15$ where $x^* = 10$.

Approach 3 : This approach also works with the unconstrained problem (23), and (24). Since $\Omega_1 = [0, 20]$ for this example, we have

$$\begin{aligned} p(\vec{a}_\alpha, \vec{b}_\alpha, \vec{c}_\alpha) &= \max\{ax - a\max\{x - b, 0\} - c\max\{x - b, 0\} \mid x \in [0, 20]\} \\ &= ab. \end{aligned}$$

Moreover, for $x \leq b$, $F(\tilde{a}, \vec{x}) = \tilde{a}x$ and for $x \geq b$, $F(\tilde{a}, \vec{x}) = \tilde{a}b + \tilde{c}(b - x)$.

(i) $0 \leq x \leq 10$

Since $x \leq b$ for all x we have:

$$\begin{aligned} e^+(x) &= e^+_\alpha(x) = \max\{\mid ax - ab \mid,\ a \in [1, 2],\ b \in [10, 12]\} \\ &= 24 - 2x, \end{aligned}$$

$$\begin{aligned} e^-(x) &= e^-_\alpha(x) = \min\{\mid ax - ab \mid,\ a \in [1, 2]\},\ b \in [10, 12] \\ &= 10 - x, \end{aligned}$$

and

$$\tilde{e}(x) = e^I(x) = [10 - x, 24 - 2x].$$

Therefore, the optimal solution is the solution that minimizes the regret function;i.e.,

$$\min\{mid(e^I(x)) = 17 - 3x/2 \mid 0 \leq x \leq 10\} = 2$$

which occurs for x = 10.

(ii) $10 \leq x \leq 12$

We will compute e^+ first.

Case 1: $x \leq b$

$$\begin{aligned} e^+_\alpha(x) &= \max\{\mid ax - ab \mid,\ a \in [1, 2],\ b \in [10, 12]\} \\ &= 24 - 2x \end{aligned}$$

Case 2: $x > b$

$$\begin{aligned} e^+_\alpha(x) &= \max\{\mid ab + c(b - x) - ab \mid,\ a \in [1, 2],\ c \in [2, 3],\ b \in [10, 12]\} \\ &= \max\{c(x - b),\ c \in [2, 3],\ b \in [10, 12]\} \\ &= 3x - 30 \end{aligned}$$

Therefore, for both cases,

$$e^{+}(x) = e_{\alpha}^{+}(x) = \max\{24 - 2x, 3x - 30\}$$

Next we compute e⁻.

Case 1: $x \le b$

$$\begin{aligned} e^{-}(x) &= e_{\alpha}^{-}(x) = \min\{\mid ax - ab \mid,\ a \in [1,2],\ b \in [10,12]\} \\ &= 0 \end{aligned}$$

Case 2: $x > b$

$$\begin{aligned} e^{-}(x) &= e_{\alpha}^{-}(x) = \min\{\mid ab + c(b-x) - ab \mid,\ a \in [1,2],\ c \in [2,3],\ b \in [10,12]\} \\ &= \min\{c(x-b),\ c \in [2,3],\ b \in [10,12]\} \\ &= 0. \end{aligned}$$

$$\tilde{e}(x) = e^{I}(x) = [0, \max\{24 - 2x, 3x - 30\}].$$

So that the value of the minimum of the regret function is:

$$\min\{mid(e^{I}(x))\} = \min\{\max\{24 - 2x, 3x - 30\}/2\} \mid 10 \le x \le 12\} = 1.2$$

which occurs for $x^* = 10.8$ for this example.

(*iii*) $12 \le x \le 20$: Since $x \ge b$ (ii) Case 2 applies and the minimum will be $\min\{3x - 30\}$ for $12 \le x \le 20$, since $3x - 30 \ge 24 - 2x$. This results in $x^* = 12$ with the minimum regret function having a value of 3 {the midpoint of [0, 6]}.

Therefore, overall, the optimal decision is $x^* = 10.8$, with the value of the minimum regret function value being 1.2 for this example. All three approaches yield different solutions. It is therefore important that in fuzzy optimization, the sense in which the fuzzy optimum is being taken, be explicitly understood.

The Optimum of a Set of Fuzzy Numbers: We distinguish two cases associated with the three approaches defined above. For the number $\tilde{z}$ that is obtained from (9), the optimal decision variable $\vec{x}$ associated with the fuzzy optimum $\tilde{z}$ is the $\vec{x}$ that produces the α-level which is at the center of gravity of $\tilde{z}$. That is, let

$$z^* = \frac{\int \mu_{\tilde{z}}(z) z dz}{\int \mu_{\tilde{z}}(z) dz}.$$

For this particular z*, there is an associated α such that z*= z_α. The decision variables $\vec{x}$ which produced this z_α is then the associated optimal decision.

For numbers $\tilde{F}(\vec{x})$ obtained from the second approach, (14), we compute the center of gravity for each $\vec{x} \in \Omega_n$; that is,

$$F^*(\vec{x}) = \frac{\int \mu_{\tilde{F}(\vec{x})}(y) y dy}{\int \mu_{\tilde{F}(\vec{x})}(y) dy}.$$

The optimal decision are those $\vec{x}$ such that

$$F^* = opt\{F^*(\vec{x}) \mid \vec{x} \in \Omega_n\}.$$

Likewise, for numbers $\tilde{e}(\vec{x})$ obtained from the third approach, (19), we compute the center of gravity for each $\vec{x} \in \Omega_n$; that is,

$$e^*(\vec{x}) = \frac{\int \mu_{\tilde{e}(\vec{x})}(y) y dy}{\int \mu_{\tilde{e}(\vec{x})}(y) dy}.$$

The optimal decision are those $\vec{x}$ such that

$$e^* = \min\{e^*(\vec{x}) \mid \vec{x} \in \Omega_n\}.$$

We remark that the above is simply one of several ways to defuzzify the resulting fuzzy objective functions defined by (14) and (19). Regardless, each of the above defuzzifications results in a real-valued function which is then optimized. This is the general tenor of most fuzzy optimization approaches. So the process of fuzzy optimization can be viewed as one which treats the uncertainties as possibility values which are translated into fuzzy relationships. These fuzzy relationships in turn are transformed into a crisp real-valued functions which are optimized in some sense. Except for the first approach, this process, as we have developed it, requires interval computations; that is, each α-level requires the computation of interval functions.

Properties of Fuzzy Constraint Sets: It is clear that the properties of constraint set (6) are key to obtaining a computational method to solve (1), (2) and (3). We now focus on the constraint set (7). Since $g_k(\vec{b}_\alpha, \vec{x})$ is continuous in $\vec{b}_\alpha \in \vec{b}^I_\alpha$ and $x \in \Omega_n$ and both $\vec{b}^I_\alpha$ and Ω_n are compact sets, $\exists$ $(\underline{b}^k(\alpha), \underline{x}^k)$ and $(\overline{b}^k(\alpha), \overline{x}^k)$ such that

$$\underline{y}^k(\alpha) = g_k(\underline{b}^k(\alpha), \underline{x}^k) = \min\{g_k(\vec{b}_\alpha, \vec{x}), \vec{b}_\alpha \in b^I_\alpha \text{ and } \vec{x} \in \Omega_n\},$$

$$\overline{y}^k(\alpha) = g_k(\vec{b}^k(\alpha), \overline{x}^k) = \max\{g_k(\vec{b}_\alpha, \vec{x}), \vec{b}_\alpha \in \vec{b}^I_\alpha \text{ and } \vec{x} \in \Omega_n\},$$

Moreover,

$$\forall \vec{y}^*(\alpha) \text{ such that } \underline{y}^k(\alpha) \leq \vec{y}^*(\alpha) \leq \overline{y}^k(\alpha),$$
$$\exists\ (\vec{b}^k(\alpha), \vec{x}^k),\ \vec{b}_\alpha \in \vec{b}^I_\alpha,\ \vec{x} \in \Omega_n$$

and

$$\vec{y}^k(\alpha) = g_k(\vec{b}^k(\alpha), \vec{x}^k).$$

Remark 6: If $\underline{y}^k(\alpha) > 0$, then fuzzy constrained nonlinear programming problem is certainly infeasible. If $\overline{y}^k(\alpha) < 0$, then the kth constraint is certainly redundant.

Let

$$\Omega_p^\alpha = \{\vec{x} \mid G(b_\alpha, \vec{x}) \leq 0 \text{ for } \textbf{all } \vec{b}_\alpha \in \vec{b}^I_\alpha,\ \vec{x} \in \Omega_n\} \quad (25)$$

$$\Omega_o^\alpha = \{x \mid G(b_\alpha, \vec{x}) \leq 0 \text{ for } \textbf{some } \vec{b}_\alpha \in \vec{b}^I_\alpha,\ \vec{x} \in \Omega_n\} \quad (26)$$

where (25) is called the *pessimistic constraint set* and (26) is called the *optimistic constraint set.*

Theorem 1 *For all* $0 \leq \alpha \leq \beta \leq 1$, $\Omega_o^0 \supseteq \Omega_o^\alpha \supseteq \Omega_o^\beta \supseteq \Omega_o^1$ *and* $\Omega_p^0 \subseteq \Omega_p^\alpha \subseteq \Omega_p^\beta \subseteq \Omega_p^1$.
Proof. *The proof follows directly from the definitions.*□

Note that this theorem implies that the fuzzy number $\tilde{z}$ defined by the α-levels z_α given by (9) is indeed a fuzzy number since z_α will be monotone as a function of α.

Theorem 2 Ω^k_α *is a compact set for all* α *and* k, $1 \leq k \leq m$.
Proof. Ω^k_α *is bounded for each* k *since* $\Omega^k_\alpha \subseteq \Omega_n$ *and* Ω_n *is bounded. To show that* Ω^k_α *is closed, assume that* $\{\vec{x}^j\}$ *is a sequence of points in* Ω^k_α *and let* $\vec{x}^* = \lim_{j\to\infty} \vec{x}^j \in \Omega_n$. *Now, since* $\vec{x}^j \in \Omega^k_\alpha \Rightarrow \exists\ \vec{b}^j_\alpha \in \vec{b}^I_\alpha$ *such that* $g_k(\vec{b}^j_\alpha, \vec{x}) \leq 0$. *Since* $\vec{b}^I_\alpha$ *is compact,* $\exists\ b^*_\alpha$ *such that* $b^*_\alpha = \lim_{j\to\infty} \vec{b}^j_\alpha \in \vec{b}^I_\alpha$. *By assumption* $g_k(\vec{b}^j_\alpha, \vec{x})$ *is continuous in both* $\vec{b}_\alpha$ *and* $\vec{x}$ *so that* $\lim_{j\to\infty} g_k(\vec{b}^j_\alpha, \vec{x}^j) = g_k(\lim_{j\to\infty} \vec{b}^j_\alpha, \lim_{j\to\infty} \vec{x}^j) = g_k(\vec{b}^*_\alpha, \vec{x}^*) \leq 0 \Rightarrow \vec{x}^* \in \Omega^k_\alpha$ *and the theorem is proved.*□

Conclusions: The results of the study show that, to compute the solution to the fuzzy optimization problem with the fuzzy functions involved containing fuzziness via the parameters of the functions, the transformation to an interval optimization problem is needed. In addition, it was shown how to obtain a crisp decision associated with a fuzzy optimal solution.

References

[1] J.J. Buckley, Joint solution to fuzzy programming problems, *Fuzzy Sets and Systems 72* (1995), 214-220.

[2] Fuller, R. and Zimmermann, Hans-Jürgen, Fuzzy reasoning for solving fuzzy mathematical programming problems,*Fuzzy Sets and Systems 60* (1993), 121-133.

[3] Jamison, K. D. and Lodwick, W. A., Minimizing unconstrained fuzzy functions, *UCD/CCM Report No. 80* (1996).

[4] A. Kaufmann and Gupta,M.M., *Introduction to Fuzzy Arithmetic Theory and Applications* (Van Nostrand Reinhold, New York, 1991).

[5] Lee, B.S. and Cho, S.J., Fixed point theorem for contractive-type fuzzy mappings, *Fuzzy Sets and Systems* 61 (1994), 309-312.

[6] W.A. Lodwick, Analysis of structure in fuzzy linear programming, *Fuzzy Sets and Systems* 38 (1990), 15-26.

[7] Luhandjula, M.K., Fuzzy optimization: An appraisal, *Fuzzy Sets and Systems* 30 (1989), 257-282.

[8] Moore, R.E., *Methods and Applications of Interval Analysis* (SIAM, Philadelphia, 1979).

[9] Saade, J. J., Maximization of a function over a fuzzy domain, *Fuzzy Sets and Systems 62* (1994), 55-70.

[10] Sakawa, M. and Yano, H., An interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters, *Fuzzy Sets and Systems 30* (1989), 221-238.

[11] Sasaki, M., Fuzzy functions, *Fuzzy Sets and Systems* 55(1993), 295-301.

[12] Soyster, A. L., Convex programming with set-inclusive constraints and applications to inexact linear programming, *Operations Research* 21(5) (1973), 1154-1157.

[13] Tanaka, H., Okuda, T. and Asai, K., On fuzzy mathematical programming, *J. of Cybernet.* 3 (1974), 37-46.

[14] Verdegay, J.L., Fuzzy mathematical programming, in: M.M. Gupta and E. Sanchez, Eds., *Fuzzy Information and Decision Processes* (North-Holland, Amsterdam, 1982), 231-237.

[15] Zhang, D. and Guo, C., Fuzzy integrals of set-valued mappings and fuzzy mappings, *Fuzzy Sets and Systems* 75 (1995), 103-109.

[16] Zimmermann, Hans-Jürgen, Fuzzy mathematical programming, *Comput. and Oper. Res.* 10 (1983), 291-298.

CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364

Fax: (303) 556-8550
Phone: (303) 556-8442
http://www-math.cudenver.edu/

79. S.E. Payne, T. Pentilla and G.F. Royle, "Building a Cyclic q-Clan."
80. K.D. Jamison and W.A. Lodwick, "Minimizing Unconstrained Fuzzy Functions."
81. F. Brezzi, L.P. Franca, T.J.R. Hughes and A. Russo, "$b = \int g$."
82. L.P. Franca, C. Farhat, M. Lesoinne and A. Russo, "Unusual Stabilized Finite Element Methods and Residual-Free-Bubbles."
83. F. Brezzi, L.P. Franca, T.J.R. Hughes and A. Russo, "Stabilization Techniques and Subgrid Scales Capturing."
84. J. Mandel, R. Tezaur and C. Farhat, "A Scalable Substructuring Method by Lagrange Multipliers for Plate Bending Problems."
85. K. Kafadar, P.C. Prorok and P.J. Smith, "An Estimate of the Variance of Estimators for Lead Time and Screening Benefit in Randomized Cancer Screening Trials."
86. H.J. Greenberg, "Rim Sensitivity Analysis from an Interior Solution."
87. K. Kafadar, "Two-Dimensional Smoothing: Procedures and Applications to Engineering Data."
88. L.P. Franca, C. Farhat, A.P. Macedo and M. Lesoinne, "Residual-Free Bubbles for the Helmholtz Equation."
89. Z. Cai, J.E. Jones, S.F. McCormick and T.F. Russell, "Control-Volume Mixed Finite Element Methods."
90. D.C. Fisher, K. Fraughnaugh and S.M. Seager, "Domination of Graphs with Maximum Degree Three."
91. L.S. Bennethum, M.A. Murad and J.H. Cushman, "Clarifying Mixture Theory and the Macroscale Chemical Potential."
92. L.S. Bennethum and T. Giorgi, "Generalized Forchheimer Equation for Two-Phase Flow Based on Hybrid Mixture Theory."
93. L.P. Franca and A. Russo, "Approximation of the Stokes Problem by Residual-Free Macro Bubbles."
94. H.J. Greenberg, A.G. Holder, C. Roos and T. Terlaky, "On the Dimension of the Set of Rim Perturbations for Optimal Partition Invariance."