CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364

Fax: (303) 556-8550
Phone: (303) 556-8442
http://www-math.cudenver.edu/

February 1997 UCD/CCM Report No. 99

A Computational Method for Fuzzy Optimization

Weldon A. Lodwick and K. David Jamison

February 1997 **UCD/CCM Report No. 99**

A Computational Method for Fuzzy Optimization

Weldon A. Lodwick, K. David Jamison
Department of Mathematics, Campus Box 170,
University of Colorado at Denver
P.O. Box 173364,Denver, CO 80217-3364, U.S.A.
email: wlodwick@math.cudenver.edu, kjamison@math.cudenver.edu

ABSTRACT: In this paper we develop a method to solve unconstrained and constrained fuzzy optimization problems for a class of fuzzy functions. The class of fuzzy functions we consider for the optimization problems is the set of real-valued functions where one or more parameters/coefficients are fuzzy numbers. The focus of this research is to explore some theoretical results that lead to a practical computational method to solve fuzzy optimization problems.

Keywords: Fuzzy/interval number, fuzzy/interval function, fuzzy/interval optimization, computational methods, nonlinear programming

1. Introduction: This paper considers a fuzzy number as uncertainty in a given numeric quantity by defining a possibility distribution for the quantity. For example, *the value is observed to be about 2* might be represented by the triangular fuzzy number (1.5,2,2.5) (see [4]) which defines a possibility distribution for the observation. The class of fuzzy functions considered here is the class of real-valued functions with at least one coefficient being a fuzzy number. In particular, the fuzzy functions we use are real-valued functions where the coefficients (parameters) are explicitly delineated and denoted as a vector $\vec{a}$, so that $f(\vec{x}) \equiv f(\vec{a}, \vec{x})$ and it is a subset of the components of $\vec{a}$ that contain the uncertainty of the underlying cause/effect that the function models. Fuzzy optimization problems having one or more fuzzy (soft) inequalities are transformed into fuzzy parameters as will be shown. When one or more of the parameters are "fuzzified," we denote the fuzzy function $f(\tilde{a}, \vec{x})$. We point out that a variety of researchers use several definitions for a fuzzy function (see, for example, [1], [3],[5],[12], and [17]). However, all these researchers agree that a real-valued function with fuzzy coefficients (our $f(\tilde{a}, \vec{x})$ and $G(\tilde{b}, \vec{x})$) constitutes a fuzzy function, albeit, not the most general that can be obtained.

A decision maker (DM) is faced with optimizing a function subject to constraints where it is not known precisely what objective function and what the constraint functions are except possibilistically. In our case this means that the DM does not know precisely what the coefficients $\vec{a} \in \tilde{a}$ are, just that they belong to their respective possibilistic/fuzzy sets. Each instance of $\vec{a} \in \tilde{a}$, in general, yields a crisp nonlinear programming (nlp) problem for which an optimal solution is assumed to exit. In this situation, not only does the DM need to optimize over all decision variables $\vec{x}$ (the traditional nlp problem), but even before this, the DM must deal with the uncertainties $\tilde{a}$. An algorithm is developed in the sequel to solve the fuzzy nlp.

Throughout this paper we will identify a fuzzy subset with the symbol $\sim$ over a letter. For example if X is a set, then $\tilde{x}$ will be used to denote a fuzzy subset of X and $\tilde{x}_\alpha$ will denote the α-level of possibility for $\tilde{x}$; that is, it is the crisp set

$$\tilde{x}_\alpha = \left\{ x \;\middle|\; \begin{matrix} \mu_{\tilde{x}}(x) \geq \alpha \text{ for } \alpha \in (0,1] \\ \mu_{\tilde{x}}(x) > 0 \quad \text{for } \alpha = 0 \end{matrix} \right\}$$

where $\mu_{\tilde{x}}$ denotes the membership function of $\tilde{x}$. The set of x for which $\mu_{\tilde{x}}(x)>0$ is called the *support of* $\tilde{x}$. We also use interval notation as follows. When the context requires it, an interval number is superscripted so that $a^I = [a^-, a^+]$ where $a^- \leq a^+$ are the left and right endpoints of the interval a^I . We will use the midpoint of an interval so define $mid\{a^I\} = (a^- + a^+)/2$. The rules associated with interval arithmetic can be found in several texts (see [4] and [9] for example).

The fuzzy constrained minimization problem we study here is:

$$opt\ z = f(\tilde{a}, \vec{x}) \tag{1}$$

such that

$$G(\tilde{b}, \vec{x}) = (g_1(\tilde{b}, \vec{x}), \ldots, g_m(\tilde{b}, \vec{x}))^T \leq \vec{0} \tag{2}$$

$$\vec{x} \in \Omega_n = [x_1^-, x_1^+] \times \ldots \times [x_n^-, x_n^+] \tag{3}$$

where $\tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_p)^T$, $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_q)^T$ are vectors of fuzzy numbers considered here as the parameters (coefficients) of the problem, f, G are continuous for each α-level in the parameters and variables and *opt* is either min or max.

Given fuzzy numbers $\tilde{a}_i$, and $\tilde{b}_i$, let the α-level ($0 \leq \alpha \leq 1$) be denoted

$$a_i^I(\alpha) = [a_i^-(\alpha), a_i^+(\alpha)] \tag{4}$$

$$b_i^I(\alpha) = [b_i^-(\alpha), b_i^+(\alpha)], \tag{5}$$

so that the $\alpha-$level of the coefficient vectors are the closed and bounded interval vectors

$$\vec{a}^I(\alpha) = (a_1^I(\alpha), \ldots, a_p^I(\alpha))^T,$$

$$\text{and } \vec{b}^I(\alpha) = (b_1^I(\alpha), \ldots, b_q^I(\alpha))^T.$$

We will concentrate our analysis on the α-level of the constraint set given by (2)and (3); that is,

$$\Omega_\alpha^k = \{\vec{x} \mid g_k(\vec{b}_\alpha, \vec{x}) \leq 0, \vec{x} \in \Omega_n \text{ for some } \vec{b}_\alpha \in \vec{b}_\alpha^I \} \tag{6}$$

$$\Omega^\alpha = \cap_{k=1}^m \Omega_\alpha^k \tag{7}$$

In [7], Ω^α is shown to be compact so that our optimization problem is, in principle, well-defined. Moreover, if g_k is linear, then Ω_α^k is convex (see [13]). Thus, a fuzzy linear programming problem with crisp objective function coefficients becomes a convex programming problem for each α-level.

Remark 1: The fuzzy inequality $g_k(\tilde{b},\vec{x}) \lesssim \vec{c}$ is interpreted according to [19] where the range of violation is defined by a trapezoidal fuzzy number and the $\lesssim \vec{c}$ becomes $\leq \tilde{c}$ as given by the trapezoidal number. For example, if $x \lesssim 3$ and this is given as the trapezoidal number whose α-level is $[3, 7-4\alpha]$, then the inequality is changed to $g(\tilde{b},x) = x - \tilde{b} \leq 0$ where the α-level of $\tilde{b}$ is $b(\alpha) = [3, 7-4\alpha]$. Therefore, all fuzzy inequalities are transformed in this way into crisp inequalities with fuzzy coefficients as given by (2).

Remark 2: Implicit in what follows is the fact that any fuzzy set is uniquely defined by specifying its α-level. Of course every fuzzy number, as we define a fuzzy number here, has a unique membership function which uniquely defines α-levels of the number.

The algorithms developed here transform the constrained optimization problem into an unconstrained one by folding the constraints into the objective function using penalties for violating the constraint (6) in the following way. Let

$$F(\tilde{a},\vec{x}) \equiv F(\tilde{a},\tilde{b},\tilde{c},\vec{x}) =$$

$$f(\tilde{a},\vec{x}) + \sum_{k=1}^{m} h_k(\tilde{a},\tilde{b},\tilde{c},\vec{x}) \max\{g_k(\tilde{b},\vec{x}),0\} \tag{8}$$

for $\vec{x} \in \Omega_n$ where h_k is the fuzzy penalty function for violating the k^{th} constraint and $\tilde{c}$ is an explicit fuzzy parameter of the penalty function h_k different from the fuzzy parameters that were originally given. We have renamed the right-most $\tilde{a}$ of (8) $\tilde{a} := (\tilde{a},\tilde{b},\tilde{c})^T$. Of course, an unconstrained fuzzy optimization problem has no penalty so that (8) is the general form for both constrained and unconstrained fuzzy optimization problems.

The second section discusses two approaches for obtaining a solution at each α -level which result in a closed interval of numbers for each decision variable $\vec{x}$. Each of these α-level solutions form fuzzy numbers and in the third section these will be analyzed with respect to obtaining an optimum over a set of fuzzy numbers. Since a membership function is uniquely defined by its α-level and each membership function uniquely defines a fuzzy set (number), the approaches taken here are well-defined. The fourth section contain the algorithms associated with the two approaches. The fifth section contains numerical examples and the last section has the conclusions.

2. Fuzzy Optimization: There is latitude in how to interpret what is meant by optimizing a fuzzy function subject to fuzzy constraint (see, for example, [1], [2], [3], [6], [8], [10], [11], [14], [15], and [18]). We point out that Luhandjula, [8], has a useful discussion of fuzzy optimization. We concentrate on a fuzzy optimization method which is defined next. It is clear that the transformation

(8) requires the checking of m! cases to cover all combinations for the maximum $\{g_k(\tilde{b},\vec{x}),0\}$. We are interested in computational methods so that we will specify a particular penalty function which will approximate the maximum. To this end, we approximate the penalty function by:

$$h_k(\tilde{a},\tilde{b},\tilde{c},\vec{x})\max\{g_k(\tilde{b},\vec{x}),0\} \equiv M_k(e^{g_k(\tilde{b},\vec{x})}-1) \tag{9}$$

if the optimization is a minimization and

$$h_k(\tilde{a},\tilde{b},\tilde{c},\vec{x})\max\{g_k(\tilde{b},\vec{x}),0\} \equiv M_k(1-e^{g_k(\tilde{b},\vec{x})}). \tag{10}$$

if the optimization is a maximization, where $M_k > 0$.

Remark 3: This form of the penalty function is useful for the fuzzy optimization problem not only because it approximates $h_k(\tilde{a},\tilde{b},\tilde{c},\vec{x})\max\{g_k(\tilde{b},\vec{x}),0\}$ and retains the spirit of fuzzy analysis by allowing the DM to violate constraints at a cost, but it makes the fuzzy optimization problem computationally more tractable..

We will use this form of the penalty function for our computational method below and to simplify the notation, will assume that we are minimizing so that we will use (9). Thus, for each possible decision $\vec{x}$, we wish to consider all possible outcomes. To consider all possible outcomes, we must allow a decision $\vec{x}$ the possibility of violating the constraint. For this reason, we work with the unconstrained version of the problem and use the penalized objective function $F(\tilde{a},\vec{x})$, $\vec{x}\in\Omega_n$ given by (8) using the penalty function (9) or (10). Therefore, each α-level is given by,

$$F_\alpha(\vec{a}_\alpha,\vec{b}_\alpha,\vec{x}) = f(\vec{a}_\alpha,\vec{x}) + \sum_{k=1}^{m} M_k(e^{g_k(\vec{b}_\alpha,\vec{x})}-1) \tag{11}$$
$$\vec{x}\in\Omega_n.$$

Next, we wish to compute the left and right endpoints of $F_\alpha(\vec{a}_\alpha,\vec{b}_\alpha,\vec{x})$ since we wish to know the membership function for the fuzzy function $\tilde{F}(\tilde{a},\vec{x})$. It will be this fuzzy function that will be defuzzified and optimized. For each $\vec{x}\in\Omega_n$, let

$$f_\alpha^-(\vec{x}) = \min\{f(\vec{a},\vec{x}),\vec{a}\in\vec{a}_\alpha^I\} \tag{12}$$
$$f_\alpha^+(\vec{x}) = \max\{f(\vec{a},\vec{x}),\vec{a}\in\vec{a}_\alpha^I\} \tag{13}$$

$$g_k^-(\vec{x}) = \min\{g_k(\vec{b},\vec{x}),\vec{b}\in\vec{b}_\alpha^I\} \tag{14}$$
$$g_k^+(\vec{x}) = \max\{g_k(\vec{b},\vec{x}),\vec{b}\in\vec{b}_\alpha^I\}. \tag{15}$$

This problem is well-defined by the compactness of the sets involved and the assumed continuity of the functions with respect to their parameters and variables so that in principle, these values are obtainable. Since $e^{g_k(\tilde{b},\vec{x})}$ is an increasing function, by the rules of interval arithmetic (see [9] or [4]), we obtain the right and left endpoint of $F_\alpha(\vec{a}_\alpha, \vec{b}_\alpha, \vec{x})$ as follows.

$$F_\alpha^-(\vec{x}) = f_\alpha^-(\vec{x}) + \sum_{k=1}^{m} M_k(e^{g_k^-(\vec{x})} - 1) \tag{16}$$

$$F_\alpha^+(\vec{x}) = f_\alpha^+(\vec{x}) + \sum_{k=1}^{m} M_k(e^{g_k^+(\vec{x})} - 1) \tag{17}$$

so that,

$$F_\alpha^I(\vec{x}) = [F_\alpha^-(\vec{x}), F_\alpha^+(\vec{x})]. \tag{18}$$

Remark 4: $F_\alpha^I(\vec{x})$ defines a fuzzy number since increasing α-levels are nested; that is, $F_{\alpha_2}^I(\vec{x}) \subseteq F_{\alpha_1}^I(\vec{x})$ for all $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus, the α-levels as defined yield a fuzzy number for each decision variable x.

We could define

$$z_\alpha = opt\ \{mid(F_\alpha^I(\vec{x})) \mid \vec{x} \in \Omega_n\}, \tag{19}$$

as some researchers have done (see for example [11]), but then z_α is the α-level optimum value of F(ã,$\vec{x}$). The decision variable $\vec{x}$ that is optimal at a given α-level, in general will not be related to an optimal decision for another α-level. That is, an optimal decision $\vec{x}_\alpha$ for a particular α may be a terrible decision for another α-level. By optimizing over the decisions at this point results in a severe loss of information about the shape of the distribution of the uncertainty. When the defuzzification process is the last step of the fuzzy optimization analysis, information about the uncertainty of the problem is retained as long as possible. For these reasons, we do not use (19) as an optimizing criterion.

Instead, we let the fuzzy number $\tilde{F}(\vec{x})$ be that fuzzy set whose membership function is obtained by defining its α-level to be (18) and we find an optimal decision based on this fuzzy relation. In this way, we keep all the information associated with the α-levels and decide based on the possibility function for each decision $\vec{x}$ rather than defuzzifying at each α-level as in (19).

3. The Optimum of a Set of Fuzzy Numbers - Defuzzification and Optimization: We note that there are a variety of ways to defuzzify fuzzy numbers. What is presented here is simply one way to handle this issue. The particular way that defuzzification is handled here allows for the development of our algorithm which is computationally tractable.. It is useful in that it weights the value of a fuzzy number according to its α-level and (essentially) the location of the midpoint.

For fuzzy numbers $\tilde{F}(\vec{x})$ obtained from (18), we compute the defuzzification for each $\vec{x} \in \Omega_n$ as follows. Let

$$F^*(\vec{x}) = \int_0^1 \alpha(F_\alpha^-(\vec{a}_\alpha, \vec{b}_\alpha, \vec{x}) + F_\alpha^+(\vec{a}_\alpha, \vec{b}_\alpha, \vec{x})d\alpha \tag{20}$$

The optimal decision are those $\vec{x}$ such that

$$F^* = \min\{F^*(\vec{x}) \mid \vec{x} \in \Omega_n\}. \tag{21}$$

4. Computational Method for Fuzzy Optimization Problems: The above is now used to develop an algorithm for finding an optimum of a fuzzy optimization problem.

Algorithm:

Step 1: Compute $f_\alpha^-(\vec{x})$ (12) and $f_\alpha^+(\vec{x})$ (13).
Step 2: Compute $g_k^-(\vec{x})$ (14) and $g_k^+(\vec{x})$ (15).
Step 3: Compute $F_\alpha^-(\vec{x})$ (16) and $F_\alpha^+(\vec{x})$ (17).
Step 4: Compute $F^*(\vec{x}) = \int_0^1 \alpha(F_\alpha^-(\vec{x}) + F_\alpha^+(\vec{x})d\alpha$ (20).
Step 5: Compute $F^* = \min\{F^*(\vec{x}) \mid \vec{x} \in \Omega_n\}$ (21).

5. Numerical Examples: Three small examples are given. The first example is a particularly simple one so that the components of the algorithm can be clearly illustrated. For all our examples, $M_k = 1$ for clarity.

Example 1: We do this example in detail and omit them for the succeeding ones.

$\min\ -5x_1 - 4x_2$
subject to:
$x_1 \lesssim 3 \Rightarrow x_1 - \tilde{b}_1 \leq 0 \Rightarrow g_1(\tilde{b}, \vec{x}) = x_1 - [2+\alpha, 5-2\alpha] \leq 0$
$x_2 \lesssim 5 \Rightarrow x_2 - \tilde{b}_2 \leq 0 \Rightarrow g_2(\tilde{b}, \vec{x}) = x_2 - [4+\alpha, 7-2\alpha] \leq 0$
$x_1, x_2 \geq 0$,and $0 \leq \alpha \leq 1$
Step 1:
$f_\alpha^-(\vec{x}) = f_\alpha^+(\vec{x}) = -5x_1 - 4x_2.$
Step 2:
$g_1^-(\vec{x}) = \min\{g_1(\vec{b}, \vec{x}) = x_1 - b_1, b_1 \in [2+\alpha, 5-2\alpha]\} = x_1 - 5 + 2\alpha$
$g_1^+(\vec{x}) = \max\{g_1(\vec{b}, \vec{x}) = x_1 - b_1, b_1 \in [2+\alpha, 5-2\alpha]\} = x_1 - 2 - \alpha$
$g_2^-(\vec{x}) = \min\{g_2(\vec{b}, \vec{x}) = x_2 - b_2, b_2 \in [4+\alpha, 7-2\alpha]\} = x_2 - 7 + 2\alpha$
$g_2^+(\vec{x}) = \max\{g_2(\vec{b}, \vec{x}) = x_2 - b_2, b_2 \in [4+\alpha, 7-2\alpha]\} = x_2 - 4 - \alpha$
Step 3:
$F_\alpha^-(\vec{x}) = -5x_1 - 4x_2 + e^{x_1-5+2\alpha} - 1 + e^{x_2-7+2\alpha} - 1$
$F_\alpha^+(\vec{x}) = -5x_1 - 4x_2 + e^{x_1-2-\alpha} - 1 + e^{x_2-4-\alpha} - 1$
Step 4:
$F^*(\vec{x}) = \int_0^1 \alpha\{-5x_1 - 4x_2 - 2 + e^{x_1-5+2\alpha} + e^{x_2-7+2\alpha} - 5x_1 - 4x_2 - 2 +$
$e^{x_1-2-\alpha} + e^{x_2-4-\alpha}\}d\alpha$
$= -5x_1 - 4x_2 - 2 + \frac{e^2+1}{4}(e^{x_1-5} + e^{x_2-7}) + (1 - \frac{2}{e})(e^{x_1-2} + e^{x_2-4})$

Step 5: Compute the optimum, $F^* = \min F^*(\vec{x})$, $\vec{x} \geq 0$ yields a solution of
$x_1 = 4.6073$
$x_2 = 5.2870$
So that one concludes that it pays to violate the constraints from the crisp counterpart by about 1.5 units in the first variable and about .3 units in the second variable. These solutions are reasonable.

Example 2:(Example 12-6 [19] page 229 has [16] simple example. We note that there is a typographical error in the example as presented in [19] which is corrected below.)

min $z = -2x_1 - x_2$
subject to:
$x_1 \lesssim 3 \Rightarrow g_1(\tilde{b}, \vec{x}) = x_1 - [3, 9 - 6\alpha] \leq 0$
$x_1 + x_2 \lesssim 4 \Rightarrow g_2(\tilde{b}, \vec{x}) = x_1 + x_2 - [4, 8 - 4\alpha] \leq 0$
$0.5x_1 + x_2 \lesssim 3 \Rightarrow g_3(\tilde{b}, \vec{x}) = 0.5x_1 + x_2 - [3, 5 - 2\alpha] \leq 0$
$x_1, x_2 \geq 0$,and $0 \leq \alpha \leq 1$
$F_\alpha^-(\vec{x}) = -2x_1 - x_2 + e^{x_1 - 9 + 6\alpha} - 1 + e^{x_1 + x_2 - 8 + 4\alpha} - 1 + e^{0.5x_1 + x_2 - 5 + 2\alpha} - 1$
$F_\alpha^+(\vec{x}) = -2x_1 - x_2 + e^{x_1 - 3} - 1 + e^{x_1 + x_2 - 4} - 1 + e^{0.5x_1 + x_2 - 3} - 1$
Performing the integration, we obtain:
$F^*(x) = -2x_1 - x_2 - 3 + \frac{1}{2}e^{x_1 - 3} + \frac{1}{2}e^{x_1 + x_2 - 4} + \frac{1}{2}e^{0.5x_1 + x_2 - 3} +$
$\frac{1}{36}(5e^6 + 1)e^{x_1 - 9} + \frac{1}{16}(3e^4 + 1)e^{x_1 + x_2 - 8} + \frac{1}{4}(e^2 + 1)e^{0.5x_1 + x_2 - 5}$
The maximum is:
$x_1 = 3.6040$
$x_2 = 0.3566$
$z = -7.5647$
The crisp version of this problem has solution:
$x_1 = 3$
$x_2 = 1$
$z = 7.$
The fuzzy optimum solution is reasonable.

Example 3: The last numerical example is one adapted from a simple nonlinear optimization problem.

min $z = x_1^2 + 2x_1x_2 + 2x_2^2 - 10x_1 - 12x_2$
subject to:
$x_1 + 3x_2 \lesssim 8 \Rightarrow g_1(\tilde{b}, x) = x_1 + 3x_2 - [8, 10 - 2\alpha] \leq 0$
$x_1^2 + 2x_2^2 + 2x_1 - 2x_2 \lesssim 3 \Rightarrow g_2(\tilde{b}, x) = x_1^2 + 2x_2^2 + 2x_1 - 2x_2 - [3, 6 - 3\alpha]$
≤ 0
$F_\alpha^- = x_1^2 + 2x_1x_2 + 2x_2^2 - 10x_1 - 12x_2 - 2 + e^{x_1 + 3x_2 - 10 + 2\alpha} + e^{x_1^2 + 2x_2^2 + 2x_1 - 2x_2 - 6 + 3\alpha}$
$F_\alpha^+ = x_1^2 + 2x_1x_2 + 2x_2^2 - 10x_1 - 12x_2 - 2 + e^{x_1 + 3x_2 - 8} + e^{x_1^2 + 2x_2^2 + 2x_1 - 2x_2 - 3}$
Performing the integration, the following is obtained:
$F^*(x) = x_1^2 + 2x_1x_2 + 2x_2^2 - 10x_1 - 12x_2 - 2 + \frac{1}{2}e^{x_1 + 3x_2 - 8} +$
$\frac{1}{2}e^{x_1^2 + 2x_2^2 + 2x_1 - 2x_2 - 3} + \frac{1}{4}(e^2 + 1)e^{x_1 + 3x_2 - 10} + \frac{1}{9}(2e^3 + 1)e^{x_1^2 + 2x_2^2 + 2x_1 - 2x_2 - 6} +$
$\frac{1}{9}(e^3 - 1)e^{x_1^2 + 2x_2^2 + 2x_1 - 2x_2 - 6}$

The optimal solution is:
$x_1 = 0.9380$
$x_2 = 1.3357$
Objective function values is $z = -18.45$. The crisp version of this problem has a solution of
$x_1 = 0.7920$
$x_2 = 1.3027$
with objective function values is $z = -17.48$. Thus our solution is reasonable.

6. Conclusions: The results of this study shows that practical computation of solution to fuzzy optimization problems are possible and tractable. To be sure, there are two places where transformations are required which are more intense than the crisp counterpart and that is in steps 1-4 of the algorithm. However, these steps are, in principle tractable and can be done. In the defuzzification (step 4 of the algorithm), if the integral cannot be analytically performed, then added complexity arises from a numerical integration scheme. However, for the penalty function we use and the defuzzification that is used here, an analytic solution to the defuzzification is always possible for trapezoidal fuzzy coefficients and inequalities. In general, the highest price to be paid computationally arises in obtaining the left and right endpoints of the constraints (14) and (15) which corresponds to step 1 and 2 of the algorithm.

References

[1] J.J. Buckley, Joint solution to fuzzy programming problems, *Fuzzy Sets and Systems 72* (1995), 214-220.

[2] Fuller, R. and Zimmermann, Hans-Jürgen, Fuzzy reasoning for solving fuzzy mathematical programming problems,*Fuzzy Sets and Systems 60* (1993), 121-133.

[3] Jamison, K. D. and Lodwick, W. A., Minimizing unconstrained fuzzy functions, *UCD/CCM Report No. 80* (1996).

[4] A. Kaufmann and Gupta,M.M., *Introduction to Fuzzy Arithmetic Theory and Applications* (Van Nostrand Reinhold, New York, 1991).

[5] Lee, B.S. and Cho, S.J., Fixed point theorem for contractive-type fuzzy mappings, *Fuzzy Sets and Systems* 61 (1994), 309-312.

[6] Lodwick, W.A., Analysis of structure in fuzzy linear programming, *Fuzzy Sets and Systems* 38 (1990), 15-26.

[7] Lodwick, W.A. and Jamison, K.D., Interval methods and fuzzy optimization, *UCD/CCM Report No. 95(1996).*

[8] Luhandjula, M.K., Fuzzy optimization: An appraisal, *Fuzzy Sets and Systems* 30 (1989), 257-282.

[9] Moore, R.E., *Methods and Applications of Interval Analysis* (SIAM, Philadelphia, 1979).

[10] Saade, J. J., Maximization of a function over a fuzzy domain, *Fuzzy Sets and Systems 62* (1994), 55-70.

[11] Sakawa, M. and Yano, H., An interactive fuzzy satisficing method for multiobjective nonlinear programming problems with fuzzy parameters, *Fuzzy Sets and Systems 30* (1989), 221-238.

[12] Sasaki, M., Fuzzy functions, *Fuzzy Sets and Systems* 55(1993), 295-301.

[13] Soyster, A. L., Convex programming with set-inclusive constraints and applications to inexact linear programming, *Operations Research* 21(5) (1973), 1154-1157.

[14] Tanaka, H., Okuda, T. and Asai, K., On fuzzy mathematical programming, *J. of Cybernet.* 3 (1974), 37-46.

[15] Verdegay, J.L., Fuzzy mathematical programming, in: M.M. Gupta and E. Sanchez, Eds., *Fuzzy Information and Decision Processes* (North-Holland, Amsterdam, 1982), 231-237.

[16] Werners, B., Interaktive Entscheidungsunterstützung durch ein flexibles mathematicshches Programmierungssytem, Minerva Publikation, München, 1984.

[17] Zhang, D. and Guo, C., Fuzzy integrals of set-valued mappings and fuzzy mappings, *Fuzzy Sets and Systems* 75 (1995), 103-109.

[18] Zimmermann, Hans-Jürgen, Fuzzy mathematical programming, *Comput. and Oper. Res.* 10 (1983), 291-298.

[19] Zimmerman, Hans-Jürgen, *Fuzzy Sets and Applications*, Kluwer Nijhoff, Dordrecht, 1985.

CENTER FOR COMPUTATIONAL MATHEMATICS REPORTS

University of Colorado at Denver
P.O. Box 173364, Campus Box 170
Denver, CO 80217-3364

Fax: (303) 556-8550
Phone: (303) 556-8442
http://www-math.cudenver.edu/

83. F. Brezzi, L.P. Franca, T.J.R. Hughes and A. Russo, "Stabilization Techniques and Subgrid Scales Capturing."
84. J. Mandel, R. Tezaur and C. Farhat, "A Scalable Substructuring Method by Lagrange Multipliers for Plate Bending Problems."
85. K. Kafadar, P.C. Prorok and P.J. Smith, "An Estimate of the Variance of Estimators for Lead Time and Screening Benefit in Randomized Cancer Screening Trials."
86. H.J. Greenberg, "Rim Sensitivity Analysis from an Interior Solution."
87. K. Kafadar, "Two-Dimensional Smoothing: Procedures and Applications to Engineering Data."
88. L.P. Franca, C. Farhat, A.P. Macedo and M. Lesoinne, "Residual-Free Bubbles for the Helmholtz Equation."
89. Z. Cai, J.E. Jones, S.F. McCormick and T.F. Russell, "Control-Volume Mixed Finite Element Methods."
90. D.C. Fisher, K. Fraughnaugh and S.M. Seager, "Domination of Graphs with Maximum Degree Three."
91. L.S. Bennethum, M.A. Murad and J.H. Cushman, "Clarifying Mixture Theory and the Macroscale Chemical Potential."
92. L.S. Bennethum and T. Giorgi, "Generalized Forchheimer Equation for Two-Phase Flow Based on Hybrid Mixture Theory."
93. L.P. Franca and A. Russo, "Approximation of the Stokes Problem by Residual-Free Macro Bubbles."
94. H.J. Greenberg, A.G. Holder, C. Roos and T. Terlaky, "On the Dimension of the Set of Rim Perturbations for Optimal Partition Invariance."
95. W.A. Lodwick and K.D. Jamison, "Interval Methods and Fuzzy Optimization."
96. S.C. Billups, "Improving the Robustness of Complementarity Solvers Using Proximal Perturbations."
97. M. Zuev, "New Algorithms for Modeling of Curves and Surfaces."
98. L.S. Bennethum, M.A. Murad and J.H. Cushman, "Modified Darcy's Law, Fick's Law, and Terzaghi's Effective Stress Principle for Swelling Clay Soils."