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# A Two Level Finite Element Method and its Application to the Helmholtz Equation

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# A TWO-LEVEL FINITE ELEMENT METHOD AND ITS APPLICATION TO THE HELMHOLTZ EQUATION

by

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**Abstract**

A two-level finite element method is introduced and its application to the Helmholtz equation is considered. The method retains the desirable features of the Galerkin method enriched with residual-free-bubbles, while it is not limited to discretizations using elements with simple geometry. The method can be applied to other equations and to irregular shaped domains.

*Preprint*

## 1. INTRODUCTION

The Galerkin finite element method using piecewise polynomials enriched with *residual-free bubble functions* seems to provide a general framework for discretizations [1-6, 9]. Partitioning our domain into a mesh of elements, the residual-free bubble functions are defined to be as rich as possible within an element. In other words, these functions are assumed to satisfy strongly the partial differential equations in the interior of the element, up to the contribution of the piecewise polynomial functions. In addition, they are also assumed to satisfy a homogeneous Dirichlet condition on the element boundary. The residual-free-bubbles represent the unresolvable part of the solution, whereas the piecewise polynomials are the resolvable part for the given mesh.

This decomposition of the solution into a piecewise polynomial plus residual-free bubbles produces the exact solution of linear differential equations in the one-dimensional case. Furthermore, by inspecting the method after we eliminate the residual-free bubbles, various successful discretization schemes are unveiled, such as upwinding for advective-diffusive equations, mass lumping for a model of the parabolic heat transfer equation, selective reduced integration with adjustment of coefficients for the deflection of a Timoshenko beam, etc [2, 4-6]. In higher dimensions the computation of the residual-free bubbles becomes a major task, in that only in limited situations (such as rectangular elements) one can employ classical analytical tools to get the exact solution within each element.

In this note we introduce a two-level finite element method consisting of a mesh for discretization and a submesh. The Galerkin method with piecewise polynomials augmented with residual-free bubble functions is used in the mesh and the submesh is employed for approximating the computations of the residual-free bubble basis functions.

The submesh is defined in the interior of each element, where a Galerkin-least-squares numerical method is used to approximate the residual-free bubble functions. Once these are determined, the effect of the residual-free bubbles on the piecewise polynomial part of the solution can be calculated to find the solution of the Galerkin method in the original mesh. This method does not suffer from drawbacks of having to solve analytically partial differential equations in the element interior, and therefore it is suitable for any irregular mesh, used in practice in finite element computations. Furthermore, it provides a systematic framework to generate discretizations.

We organize the paper as follows: first, we revisit the Galerkin finite element method using piecewise polynomials enriched with residual-free bubble functions in Section 2. In Section 3 we introduce a two-level finite element method and discuss its application to the Helmholtz equation. In Section 4 we present preliminary computations of this method for approximating the Helmholtz equation in two dimensions. Therein we verify the improvement in the accuracy of the computations when we compare to the Galerkin-least-squares method for the original mesh. We draw conclusions in Section 5.

## 2. REVIEWING RESIDUAL-FREE BUBBLES

Let us first consider this idea in general terms. We wish to approximate the abstract boundary-value problem defined in the domain $\Omega \subset I\!R^2$ and given by:

$$\begin{aligned} Lu &= f \qquad \text{in}\,\Omega\,, \\ u &= 0 \qquad \text{on}\,\Gamma = \partial\Omega\,, \end{aligned} \tag{1}$$

where $L$ is a linear differential operator, $u$ is the unknown scalar function and $f$ is a given source function. The standard Galerkin method is formulated in a subspace $V_h \subset V$,

where $V$ is the space of functions for which a solution of the continuous problem is sought. The Galerkin method consists of finding $u_h \in V_h$ such that

$$a(u_h, v_h) = (Lu_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h \,. \tag{2}$$

Here $V_h$ is defined in a partition of the domain into triangles and/or quadrilaterals in the usual manner (i.e., no overlapping, their union reproduces the domain, no vertex on the edges of a neighboring element, etc). Each member $u_h$ and $v_h$ of the space of functions $V_h$ is spanned by continuous piecewise polynomials plus bubble functions to be defined below, i.e.,

$$u_h = u_1 + u_b \,. \tag{3}$$

The bubble part of this space is subject to zero Dirichlet boundary condition on each element $K$ boundary, i.e.,

$$u_b = 0 \qquad \text{on } \partial K \,. \tag{4}$$

This feature of the bubble functions allows us to employ the classical *static condensation* procedure: first we set $v_h = v_{b,K}$ on $K$ (zero elsewhere) in (2) to obtain

$$a(u_1 + u_b, v_{b,K})_K = (f, v_{b,K})_K \,. \tag{5}$$

where the subscript $K$ indicates that integration is restricted to the element $K$. This part of the process provides us with the bubble-part of the solution, $u_b$, at each element as a function of the piecewise linear polynomial part of the solution, $u_1$.

The second part of the static condensation procedure consists in setting $v_h = v_1$ in (2), which gives

$$a(u_1 + u_b, v_1) = (f, v_1) \,, \tag{6}$$

or

$$a(u_1, v_1) + a(u_b, v_1) = (f, v_1) \,. \tag{7}$$

If we refer to the standard Galerkin method for piecewise linears, equation (7) shows that eliminating the bubbles yields to a formulation that may be viewed as modifying the variational formulation in the left hand side by the addition of $a(u_b, v_1)$.

We have still not defined the space of bubbles we will use. These are the residual-free bubbles which are bubble functions that satisfy the differential equations strongly in each element $K$, i.e.,

$$Lu_b = -(Lu_1 - f) \qquad \text{in } K\,. \tag{8}$$

Note that this choice of bubbles implies that equation (5) is satisfied automatically, since it is the variational equation for (8).

In order to solve (8) for $u_b$ subject to (4), we solve instead:

$$L\varphi_{i,K} = -L\psi_{i,K} \qquad \text{in } K\,, \tag{9}$$

$$\varphi_{i,K} = 0 \qquad \text{on } \partial K\,, \tag{10}$$

where the $\psi_{i,K}$'s are the local basis functions for $u_1$ and

$$L\varphi_{f,K} = f \qquad \text{in } K\,, \tag{11}$$

$$\varphi_{f,K} = 0 \qquad \text{on } \partial K\,. \tag{12}$$

Thus, if

$$u_{1|K} = \sum_{i=1}^{n_{en}} c_{i,K}\psi_{i,K} \tag{13}$$

where $n_{en}$ is the number of nodes per element, then

$$u_{b|K} = \sum_{i=1}^{n_{en}} c_{i,K}\varphi_{i,K} + \varphi_{f,K}\,, \tag{14}$$

with the same coefficients $c_{i,K}$'s.

Putting all together we first solve the differential equations (9)-(12) for the $\varphi_i$'s and $\varphi_f$, and by equation (7) we solve the matrix problem (by selecting $v_1 = \psi_i$): Find the coefficients $c_j$'s such that:

$$\sum_j c_j \left[a(\psi_j, \psi_i) + a(\varphi_j, \psi_i)\right] = (f, \psi_i) - a(\varphi_f, \psi_i)\,. \tag{15}$$

We now turn to the application of this methodology to the solution of the Helmholtz equation. For this equation we have

$$L = \Delta + k^2 I\,, \tag{16}$$

where $k$ is the wave number, and $I$ is the identity operator and $f = 0$. Herein we consider boundary conditions of the type:

$$u = g \qquad \text{on}\,\Gamma_1\,, \tag{17}$$

$$\frac{\partial u}{\partial n} = h \qquad \text{on}\,\Gamma_2\,, \tag{18}$$

where $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Substituting into (7), and integrating by parts, leads to

$$-(\nabla u_1, \nabla v_1) + k^2(u_1, v_1) + \sum_K k^2(u_b, v_1)_K = -(h, v_1)_{\Gamma_2}\,. \tag{19}$$

From eq. (15), it follows that the above equation can be rewritten in terms of the basis functions as

$$\sum_j c_j \left\{-(\nabla\psi_j, \nabla\psi_i) + k^2(\psi_j, \psi_i) + k^2(\varphi_j, \psi_i)\right\} = -(h, \psi_i)_{\Gamma_2}\,. \tag{20}$$

In [3] we have considered the solution of equations (9)-(10) by a change of variables:

$$\varphi_j = -\psi_j + \lambda_j\,, \tag{21}$$

and now it follows from (9)-(10) that $\lambda_j$ solves

$$L\lambda_j = 0 \qquad \text{in } K\,, \tag{22}$$

$$\lambda_j = \psi_j \qquad \text{on } \partial K\,, \tag{23}$$

and from (20) the matrix formulation simplifies to solving

$$\sum_K \sum_{j=1}^{n_{en}} c_j^K \{-(\nabla\psi_j, \nabla\psi_i)_K + k^2(\lambda_j, \psi_i)_K\} = -(h, \psi_i)_{\Gamma_2}\,. \tag{24}$$

For square and rectangular elements (22)-(23) can be solved using Separation-of-Variables [3]. For example, we can solve (22)-(23) on a square of side $a$, placed in the first quadrant of the Cartesian coordinates $x$-$y$. Thus, for the shape function $\psi_j$ with value one at $x = y = a$ we solve

$$\Delta\lambda_j + k^2\lambda_j = 0 \qquad \text{in } K\,, \tag{25}$$

$$\lambda_j = 0 \qquad \text{on } x = 0 \text{ or } y = 0\,, \tag{26}$$

$$\lambda_j = y/a \qquad \text{on } x = a\,, \tag{27}$$

$$\lambda_j = x/a \qquad \text{on } y = a\,. \tag{28}$$

By separation-of-variables we obtain the following exact solution

$$\lambda_j = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{2}{m\pi \sinh(\sqrt{\frac{m^2\pi^2}{a^2} - k^2}\, a)} \left\{ \sin(\frac{m\pi x}{a}) \sinh(\sqrt{\frac{m^2\pi^2}{a^2} - k^2}\, y) + \sin(\frac{m\pi y}{a}) \sinh(\sqrt{\frac{m^2\pi^2}{a^2} - k^2}\, x) \right\}. \tag{29}$$

The formulas for the shape function $\psi_j$ with value one at the other nodes are similar. Therefore, we are now able to substitute back into (24) and solve the system of equations using this residual-free-bubble approach.

*Remark:*

Clearly the derivation of the residual-free-bubbles presented in [3] can be extended to, and is limited to, rectangular elements. For arbitrarily shaped elements we need to consider strategies of approximating the computation of the residual-free bubble basis functions. One possibility is discussed next.

## 3. A TWO-LEVEL FINITE ELEMENT METHOD

Summing up, in general, we need to solve the differential equations for the bubble shape functions $\varphi_j$'s and $\varphi_f$, (equations (9)-(12)) and then supply those functions to the matrix formulation (15).

It turns out that the first step of this procedure may be as complex as trying to solve the original PDE problem, except if the domain can be discretized by a union of square (or rectangular) elements. In this case, one may want to consider the methodology as described in the previous section. The PDE problem may still be posed in a domain shape other than a square - an L-shaped domain, for example - which causes considerable difficulties to analytical techniques whereas the element problems for the bubbles continue to be posed in square elements amenable to analytical techniques.

This state of affairs is not satisfactory when we consider the needs of the users who deal with domain shapes far more complex than simple geometries. Ideally we need discretization schemes that will work in irregular meshes, so that discretizations in complex geometry domains and the usual tools, such as adaptive strategies with local refinements, will all work as in the usual finite element method code.

We wish to consider the following strategy. At the pre-processing stage of a finite element code, consider approximating the differential equations for the bubble shape

functions $\varphi_j$'s and $\varphi_f$, (equations (9)-(12)) by another finite element method. In other words for each element, we consider a sub-mesh (a mesh defined for each element - see Figure 1) where we will solve the equations (9)-(12) by a suitable finite element method. This can be done using a non-standard method: the Galerkin-least squares method (GLS), for example. We will need to repeat this procedure for each element, and for each basis function.

Let us now describe the application of this strategy to the Helmholtz problem using the GLS method to solve each bubble problem. We have in this case a two-level finite element method. At the global level we are using the Galerkin method with piecewise linears for the original method, and at each element level we discretize again and solve the bubble problems with GLS. Thus corresponding to (9)-(10) we need to solve in each element $K$ of the original mesh:

$$\Delta\varphi_{i,K} + k^2\varphi_{i,K} = -k^2\psi_{i,K} \qquad \text{in } K\,, \tag{30}$$

$$\varphi_{i,K} = 0 \qquad \text{on } \partial K\,, \tag{31}$$

To construct the GLS approximation of this differential problem, let us denote by $K^*$ an arbitrary element in the submesh with diameter $h^*$ and by $\psi_l^*$ the basis function for a piecewise linear interpolation in the submesh, so that our unknown bubble basis function $\varphi_{i,K}$ can be approximated by

$$\varphi_{i,K}^{h^*} = \sum_l c_l^{(i)}\psi_l^* \tag{32}$$

Here $l$ runs over all unknown interior nodes in the submesh, say through $N^*$. The index $i$ refers to the specific bubble function we are trying to solve for (Recall that we need to solve for a number of bubbles equals to the number of basis functions used to span the piecewise polynomial defined in the global element $K$).

We then formulate the GLS method in matrix formulation as: for each $i$ (from 1 to $n_{en}$) find $c_l^{(i)}, l = 1, 2, \ldots, N^*$ such that:

$$\sum_l c_l^{(i)} \left[ -(\nabla\psi_l^*, \nabla\psi_m^*) + k^2(1-\tau k^2)(\psi_l^*, \psi_m^*) \right] = -k^2(1-\tau k^2)(\psi_{i,K}, \psi_m^*) \qquad (33)$$

for $m = 1, 2, \ldots, N^*$. We use the stability parameter $\tau$ given in [7, 8] as:

$$\tau k^2 = 1 - \frac{\alpha_{GLS}}{\alpha}, \qquad (34)$$

with

$$\alpha = \frac{(kh^*)^2}{12}, \qquad (35)$$

and

$$\alpha_{GLS} = \frac{1}{2} \; \frac{1-\cos kh^*}{2+\cos kh^*}. \qquad (36)$$

Once the constants $c_l^{(i)}$'s are found, we substitute them in (32) to get the approximate residual basis function $\varphi_i^{h^*}$. These approximations are then assembled to the global problem (20) in place of the exact residual function $\varphi_i$. The global problem may now be solved for the piecewise-linear part of the solution (i.e., we determine the constants $c_i$'s in equation (20)). We then recover the bubble part of the solution from (14) with $\varphi_i^{h^*}$ in lieu of $\varphi_i$.

*Remarks:*

1) Note that the submeshes do not need to 'match', since each bubble problem is independent of the problem posed in the next element.

2) Since the bubble problems can be solved independently, this part of the algorithm may profit from a parallel processor machine to carry out these computations efficiently.

3) Cost effectiveness of this algorithm needs to be addressed systematically. Roughly, if the bubble problems are solved in a very coarse mesh (orders of magnitude coarser than the original mesh), then the cost for solving the linear system for the global nodes, equation (20), should still be the dominating part of the total CPU time used with this two-level method. The cost of solving (33) for each node and for each element, should be kept as a small portion of the overall algorithm.

4) One may also want to consider solving each bubble problem with the Galerkin method enriched with residual-free bubbles for each element in the submesh. Then, this will yield to another layer of methods that will be needed to approximate the new residual-free bubbles. The process may be repeated various times creating a multi-level finite element method.

5) Although we borrowed the nomenclature from the multigrid/domain decomposition literature, it should be clear that this methodology is being created to address the stability and accuracy limitations of standard finite element methods (using piecewise polynomials). In the other literature, the main concern is to find a methodology to accelerate the solution of large system of equations, assuming the underlying discretization problems have been solved by some suitable method. Herein we insist on the standard Galerkin method enriched by these residual-free bubble functions that need to be approximated, and the different levels are created to 'resolve' this 'unresolvable part of the solution', using the nomenclature introduced in [9].

## 4. SOME NUMERICAL RESULTS

In this section we report two series of experiments with the two-level finite element introduced in the previous section. We compare our results with the GLS method

defined in the entire domain. The two-level results are obtained by subdividing each element into 25 subelements: each subelement is generated by a bilinear mapping from a uniform subdivision with $5 \times 5$ subelements in the reference square domain. In the following numerical results, we have not added the bubble part of the solution to the bilinear part in the two-level method, since these bubble parts are small (but crucial for accuracy improvement of the bilinear part of the solution) in the experiments reported herein.

### 4.1 Scattering in an $L$-shaped domain

We consider scattering in an $L$-shaped domain as defined in Figure 2. The non-dimensional wave number is $kL = 8$, where $L$ is the width of the $L$-shaped domain.

We perform a convergence study using non-uniform meshes, which are superimposed on each numerical result. We select non-uniform meshes to illustrate the ability of the current two-level finite element method to handle arbitrarily shaped quadrilateral elements, an impossibility using the residual-free bubble analytical solutions developed in [3] (see eq. (29)). In [3], the results for the same problem is reported employing square shaped elements.

In Figures 3 to 5 we plot the contour values of the solutions for the two-level method versus the GLS method. For the coarsest mesh (Figure 3) we can already detect the main features of the exact solution using the two-level finite element. GLS will be comparable in accuracy only for the finest mesh (Figure 5).

### 4.2 Scattering in a $C$-shaped domain

We now consider a similar problem, posed in a $C$-shaped domain (see Figure 6 for a problem statement). The non-dimensional wave number is $kL = 5$.

We compare the results for the two-level method versus the GLS method in two different meshes in Figures 7 to 8. Similar to the previous example, the coarsest mesh results (Figure 7) indicate the main features of the exact solution using the two-level finite element. GLS results will only be comparable in accuracy for the finest mesh (Figure 8).

## 5. CONCLUSIONS

If computed exactly, residual-free bubbles yield the exact solution of linear one dimensional problems. For one-dimensional problems we can *derive* several numerical tricks such as upwinding, mass lumping and selective reduced integration [4].

For multi-dimensional problems the method is approximate, since, in general, the exact solution may not be linear on element boundaries, as required by the decomposition of the solution (see equation (3)). However, preliminary computations indicate improvement in the accuracy of the results when we compute these functions using analytical techniques such as separation-of-variables [3]. Analytical techniques are too restrictive and are limited to simple element geometries.

In this paper we presented a two-level finite element method based on the Galerkin method for the original mesh, and a Galerkin-least-squares method to solve the partial differential equations governing the residual-free bubble functions. This approach is general, and can be used to approximate intricate geometries with elements of arbitrary shape (subject to the usual regularity constraints). This method can be used in conjunction with adaptive strategies due to its ability to work with elements of different sizes and shapes.

Considerable work is needed to handle convergence analysis and cost-effectiveness

issues for this methodology. Nevertheless we have provided with a systematic framework for discretization that may prove useful for singularly perturbed problems, in particular.

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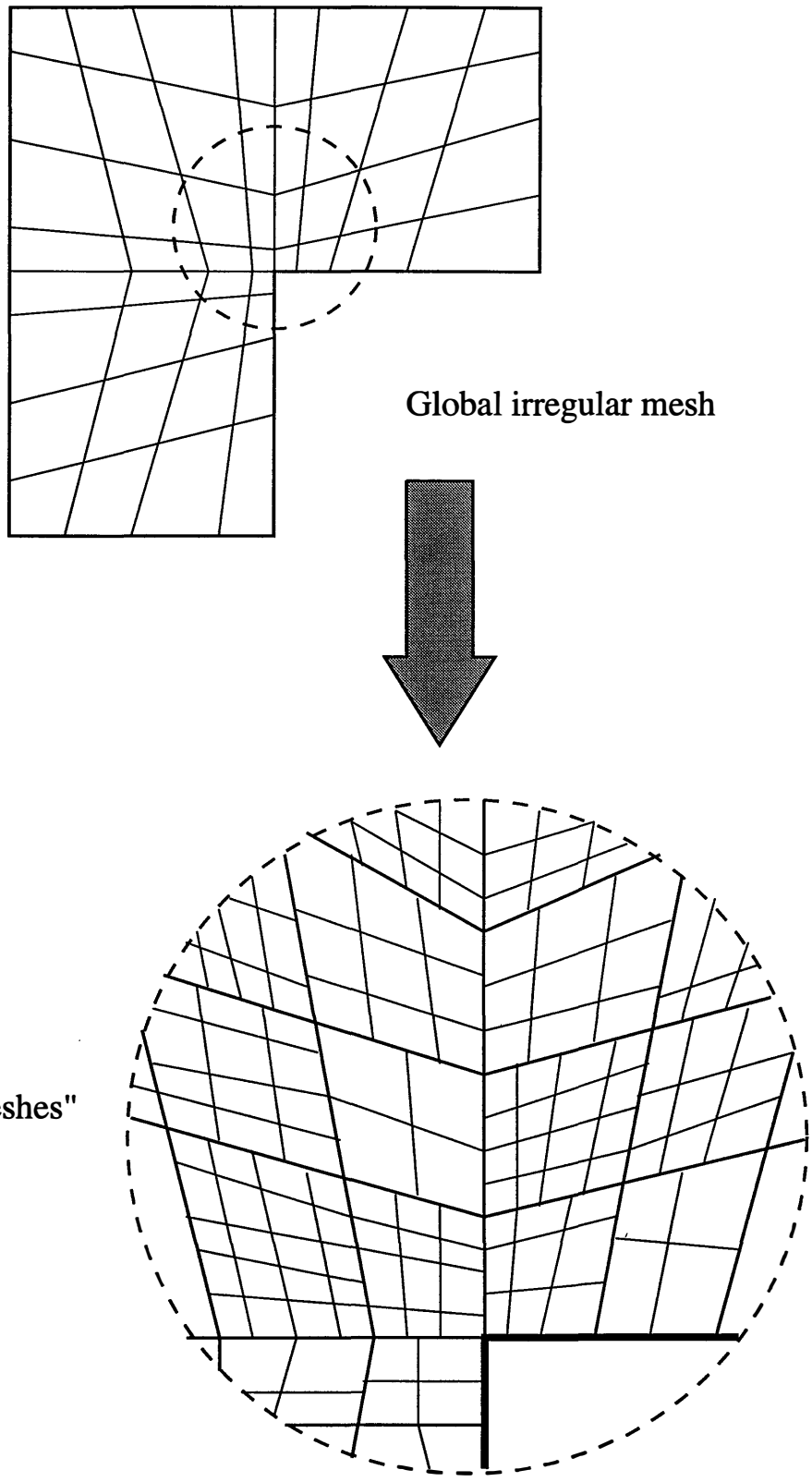


Figure 1: The global mesh for the Galerkin method and a submesh for approximating residual-free bubbles in a part of the domain.

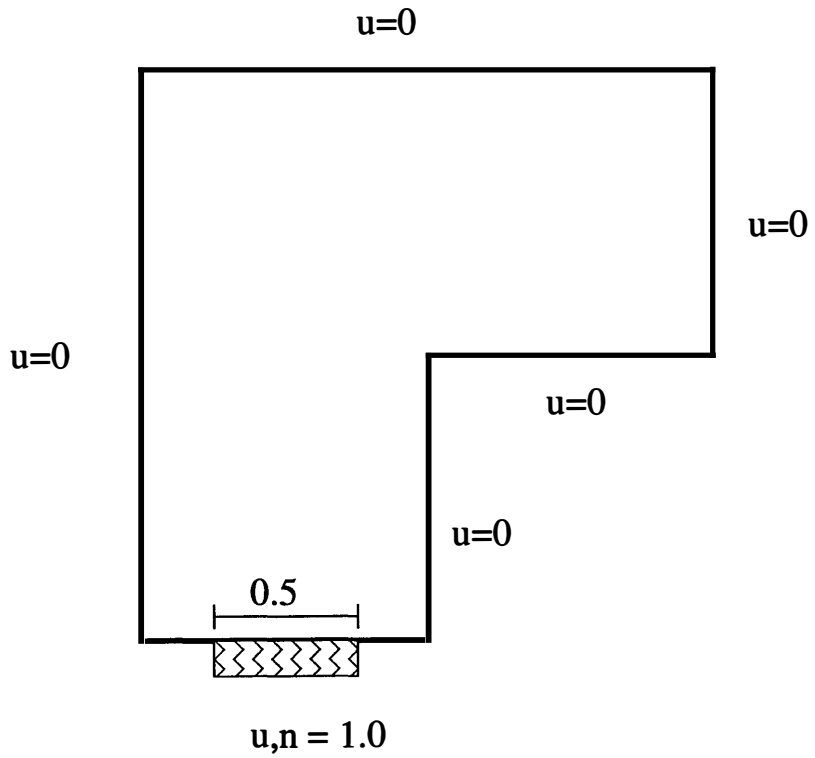


Figure 2: Scattering in an $L$-shaped domain composed by 3 unit side squares: boundary conditions.

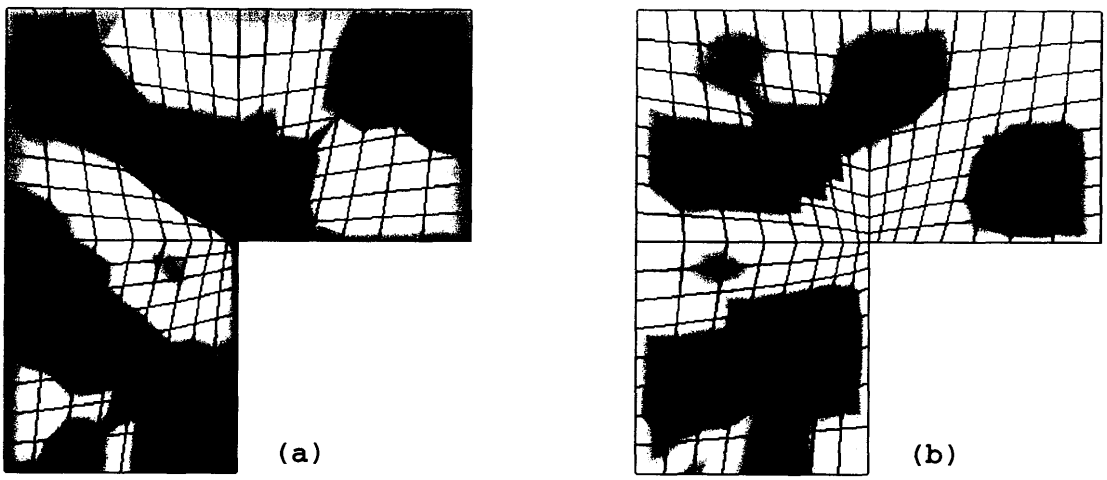


Figure 3: Scattering in an $L$-shaped domain with $3 \times (8 \times 8)$ quadrilaterals: a) The two-level finite element method; b) The GLS method with piecewise bilinears.

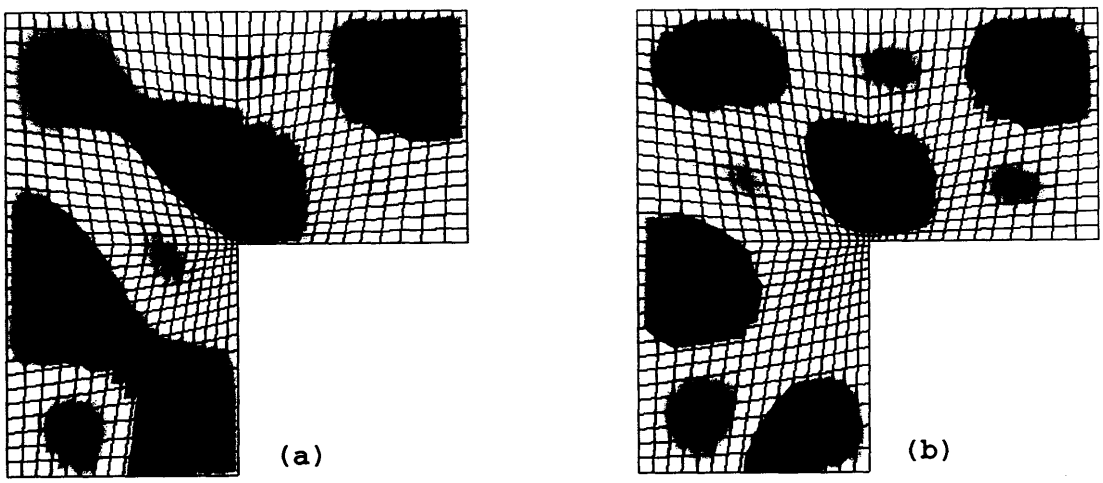


Figure 4: Scattering in an $L$-shaped domain with $3 \times (16 \times 16)$ quadrilaterals: a) The two-level finite element method; b) The GLS method with piecewise bilinears.

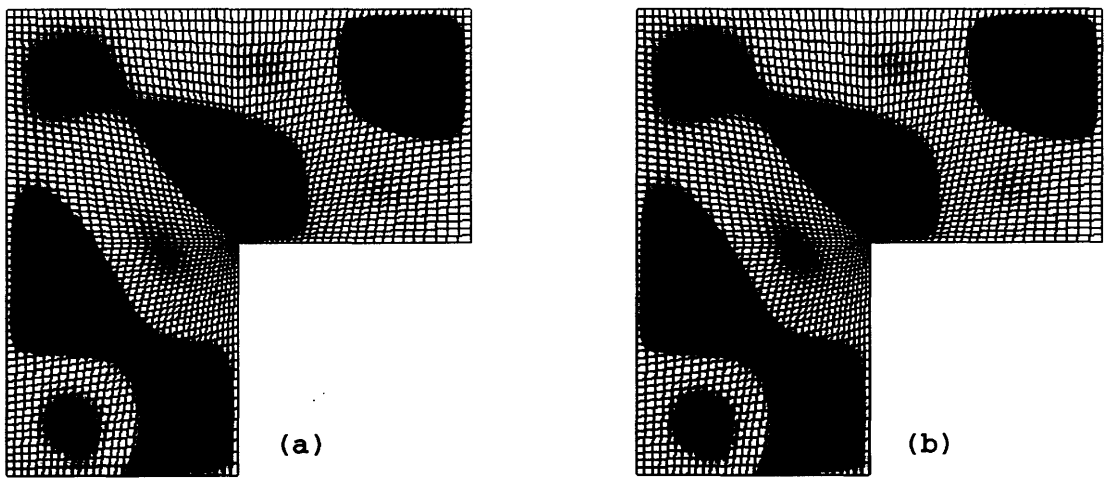


Figure 5: Scattering in an $L$-shaped domain with $3 \times (32 \times 32)$ quadrilaterals: a) The two-level finite element method; b) The GLS method with piecewise bilinears.

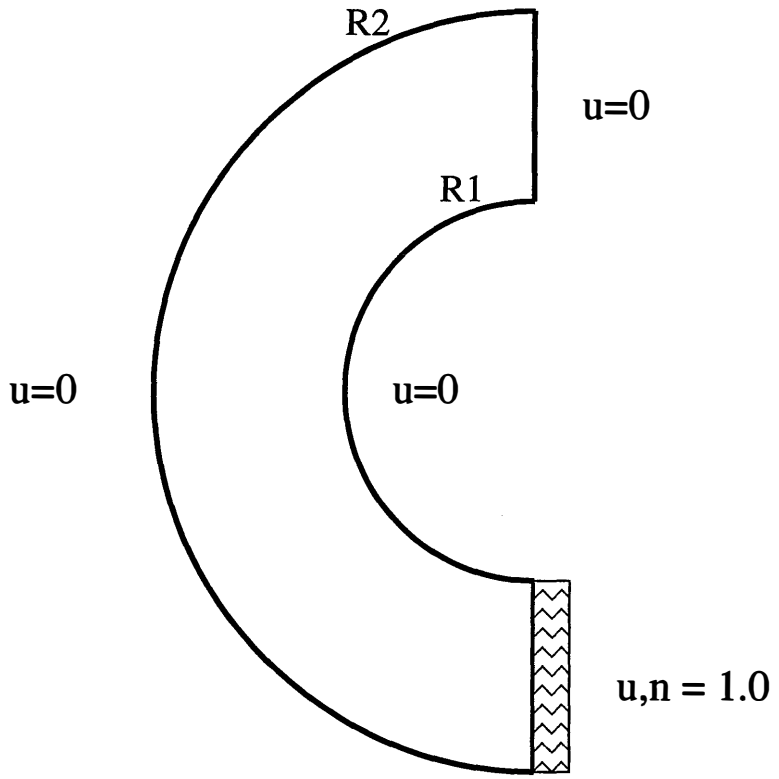


Figure 6: Scattering in a $C$-shaped domain based on concentric circles of radii $R1 = 1.0$ and $R2 = 2.0$: boundary conditions.

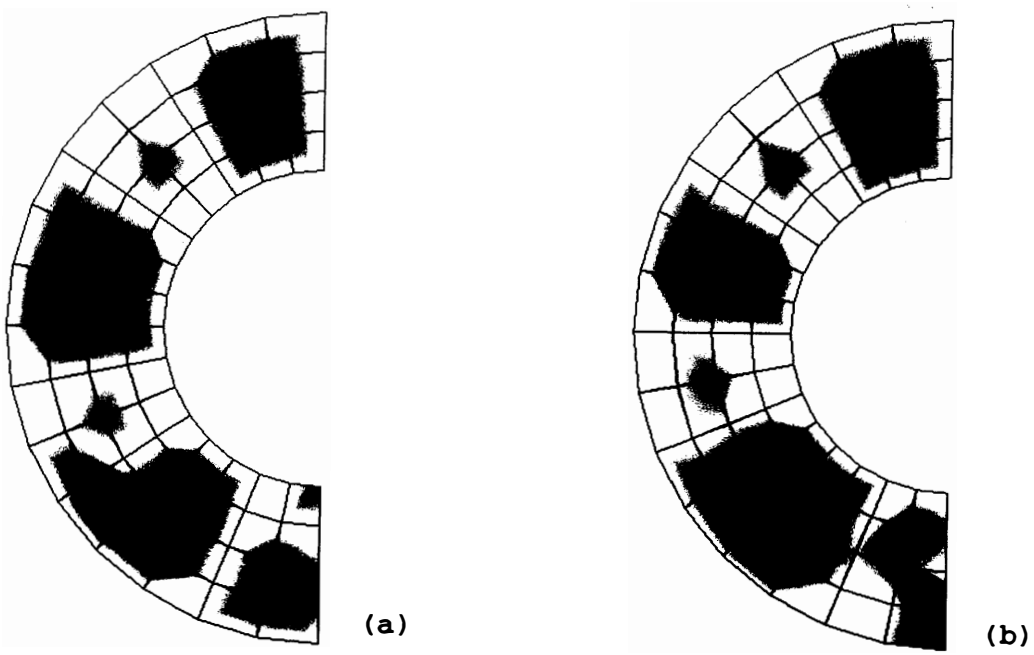


Figure 7: Scattering in a $C$-shaped domain with $4 \times 16$ quadrilaterals: a) The two-level finite element method; b) The GLS method with piecewise bilinears.

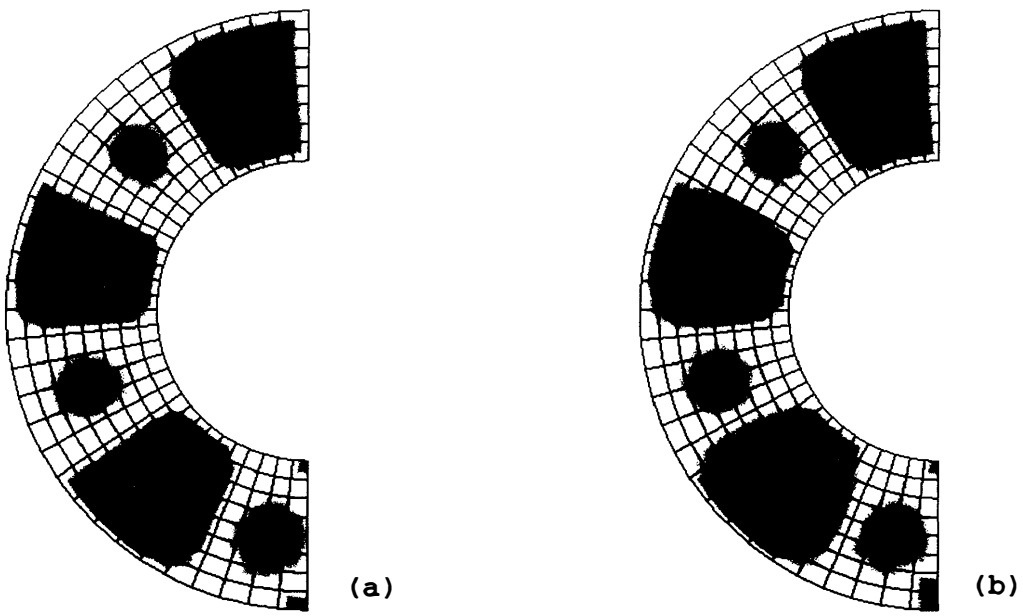


Figure 8: Scattering in a $C$-shaped domain with $8 \times 32$ quadrilaterals: a) The two-level finite element method; b) The GLS method with piecewise bilinears.

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