ESTIMATES OF THE TOTAL ABSOLUTE ERROR OF A NUMERICAL SOLUTION OF VOLTERRA INTEGRAL EQUATIONS OF THE SECOND KIND

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Abstract. A total absolute error of approximate solution of Volterra integral equations of the second kind is considered. The error is not greater than the sum of errors of three kinds: inherent error, error because of inaccuracy of numerical algorithms and round-off error. The number of basic operations that are necessary for realization of these methods is estimated.

Key words. Volterra integral equation, error estimate, approximate-iterative method, accuracy, Chebyshev center

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1. Introduction. In this paper we deal with a Volterra integral equation of the second kind

$$x(t) = \int_0^t f(t,\tau,x(\tau))\,d\tau,\; t\in[0,T],\; x(0)=0, \tag{1}$$

where $f(t)$ is a function of some class F(to be considered below), and $x(t)$ – the desired solution.

The aim of this paper is to obtain estimates of the total absolute error of approximate solutionn of the equation (1) by some approximate-iterative methods. We also estimate numerical complexity of considered methods.

The inherent error, the error because of inaccuracy of numerical methods and round-off error are estimated (together or separately) for various problems in many papers, see e.g. [12, 2, 14, 5, 1] and the references therein.

We consider all the three kinds of of these errors. The total absolute error is not greater than their sum. From the estimates for total absolute error one can obtain requirements on the initial parameters in order to get the approximate solution with a pre-assigned accuracy. Further, we consider the problem of optimality of these methods from the viewpoint of [3]

2. The Inherent Error. Let $\mathcal{X}$ be a linear normed space of functions, that are defined on $[0,T]$, with a norm $\|*\|_t, 0<t\le T$, and satisfying the conditions

$$\|x_1x_2\|_t \le vrai \max_{0\le\tau\le t}|x_1(\tau)|\|x_2\|_t,$$

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(2) $$\left\|\int_0^t x(\tau)d\tau\right\|_t \le \int_0^t \|x\|_\tau d\tau,\ x_1, x_2 \in \mathcal{X}.$$

LEMMA 2.1. *For the cases $\mathcal{X} \equiv L_p(0,T)$, $1 \le p < \infty$ and $\mathcal{X} \equiv C([0,T])$ the conditions (2) hold.*

Proof. The first inequality in (2) is evident. The proof of second inequality in case $\mathcal{X} \equiv C$ is almost evident. In case $\mathcal{X} \equiv L_p$ the second inequality follows from Minkowski's general inequality [8]:

$$\left[\int \left|\int x(t,\tau)d\tau\right|^p dt\right]^{1/p} \le \int \left[\int |x(t,\tau)|^p \, dt\right]^{1/p} d\tau,\ 1 \le p < \infty,$$

where we assume $x(t,\tau) \equiv x(t-\tau)$, $x(t-\tau) = 0$ for $\tau > t$. □

We suppose that the equation (1) has an unique solution $x \in \mathcal{X}$ for all $f \in F$ and that $F \subset \mathcal{X}$. When f is running through F, the corresponding solutions of (1) are running through some set X. Without loss of information, just all the set X is the approximate solution of (1). But for convenience of further applications one usually takes some specific $\hat{x}$, as a solution corresponding to some $\hat{f} \in F$. In this case the desired absolute inherent error , is equal to

(3) $$\Delta_1 = \sup_{x \in X} \|x - \hat{x}\|_t,\ 0 < t \le T.$$

If we denote

$$\delta f = \int_0^t [f(t,\tau,\hat{x}(\tau)) - \hat{f}(t,\tau,\hat{x}(\tau))]d\tau,\ \delta x = x(t) - \hat{x}(t),$$

than we easily obtain

$$\delta x = \int_0^t [f(t,\tau,x(\tau)) - f(t,\tau,\hat{x}(\tau))]d\tau + \delta f,$$

and if f is Lipschitz-continuous

(4) $$\|f(*,x_1) - f(*,x_2)\| \le L\|x_1 - x_2\|,$$

then based on Lemma 2.1 and the Gronwall's inequality [9], we have

$$\|\delta x\|_t \le L \int_0^t \|\delta x\|_\tau d\tau + \|\delta f\|_t$$

and

(5) $$\Delta_1 \le L \int_0^t \|\delta f\|_\tau e^{L(t-\tau)} d\tau + \|\delta f\|_t \le \max_{0 \le \tau \le t} \|\delta f\|_\tau e^{Lt},\ 0 < t \le T.$$

Thus, following theorem is proved:

THEOREM 2.1. *If the function f satisfies (4) and properties (2) hold, then the estimate (5) holds for the inherent error (3).*

Remark. For the case of space $\mathcal{X} \equiv C$ this estimate is well-known [10].

Further, let instead of (4) the following condition

$$L'(x_1 - x_2) \le f(*, x_1) - f(*, x_2) \le L''(x_1 - x_2),\ -L \le L', L'' \le L \tag{6}$$

holds. Then we can obtain the following estimate

$$\Delta_1 \le \max\left(2, e^{\max(L',L'')t}\right) \max_{0\le\tau\le t} |\delta f|. \tag{7}$$

In particular, if $\max(L', L'') \le 0$ then

$$\Delta_1 \le 2 \max_{0\le\tau\le t} |\delta f| \tag{8}$$

inequality holds. Consequently, the next theorem is true:

THEOREM 2.2. *Let the condition (4) be substituted by (6) in the assumptions of the Theorem 2.1. Then for the inherent error (3) the estimates (7) and (8) hold.*

It should be emphasized that if (6) holds, than

$$L' \int_0^t x(\tau)d\tau + f^0(t) \le x(t) = \int_0^t [f(t, \tau, x(\tau)) - f(t, \tau, 0)]d\tau + f^0(t) \le$$

$$L'' \int_0^t x(\tau)d\tau + f^0(t),\ f^0(t) = \int_0^t f(t, \tau, 0)d\tau,$$

and based on these relations in some cases it is possible to estimate $|x(t)|$ from above and from below. These estimates with regard to (7) and (8) may give more accurate estimates for the inherent error.

Let a and b be such constants that

$$\hat{f}(t, \tau, x) = \sup_{f\in F} f(t, \tau, x) \ge ax + b,\ a > 0, \tag{9}$$

and

$$L_L \delta x \le f(*, x) - f(*, \hat{x}) \le L_R \delta x,\ \ \hat{x}(t) = \int_0^t \hat{f}(t, \tau, \hat{x}(\tau))]d\tau. \tag{10}$$

Then

$$\hat{x}(t) \ge a \int_0^t \hat{x}(\tau)d\tau + bt,\ \delta f = \int_0^t [f(t, \tau, \hat{x}(\tau) - \hat{f}(t, \tau, \hat{x}(\tau))]d\tau \le 0$$

and, similarly to previous arguments,

$$|\hat{x}(t)| > \frac{|b|}{a} \left|1 - e^{-at}(1 + at)\right| e^{at},$$

$$L_L \int_0^t \delta f e^{L_L(t-\tau)} d\tau + \delta f \leq \delta x \leq L_R \int_0^t \delta f e^{L_R(t-\tau)} d\tau + \delta f,$$

whence it follows that

$$|\delta x| \leq \max(|L_L|, |L_R|) \int_0^t |\delta f| e^{L_L(t-\tau)} d\tau + |\delta f|,$$

$$\|\delta x\|_t \leq \max\left(2, e^{L_L t}\right) \max_{0 \leq \tau \leq t} \|\delta f\|_\tau$$

and

$$\left\| \frac{\delta x}{\hat{x}} \right\|_t \leq \frac{a}{|b|} e^{-at} \frac{1}{|1 - e^{-at}(1+at)|} \max\left(2, e^{L_L t}\right) \max_{0 \leq \tau \leq t} \|\delta f\|_\tau, \tag{11}$$

$$0 < t \leq T.$$

Therefore, we have proved the next theorem:

THEOREM 2.3. *Let conditions (2), (10) and (9) hold. Then the estimate (11) is true for relative inherent error* $\left\| \frac{\delta x}{\hat{x}} \right\|$.

Finally, we will show how one can estimate the least feasible value

$$\Delta_1^* \equiv \rho^* = \inf_{\hat{x} \in \mathcal{X}} \sup_{x \in X} \|x - \hat{x}\|. \tag{12}$$

We assume that $f \in F$ have everywhere continuous partial derivative $\partial f / \partial x$ and that when f is running through F, then derivative $\partial f / \partial x$ runs F'. Let Φ be some set with the norm $\| * \|_\Phi$. We denote by φ^* the Chebyshev center and by ρ_φ^* the Chebyshev radius of Φ:

$$\rho_\varphi^* = \inf_{\varphi_1} \sup_{\varphi \in \Phi} \|\varphi_1 - \varphi\|_\Phi = \sup_{\varphi \in \Phi} \|\varphi^* - \varphi\|_\Phi,$$

where the lower bound is taking on the set of functions φ_1,such that $\|\varphi_1 - \varphi\|_\Phi$ has a meaning. Let f^*, $(\partial f / \partial x)^*$ and x^* are Chebyshev centers of the sets F, F' and X respectively.

THEOREM 2.4. *Let*

$$\|f^* - f_*\| = o\left(\sup_{f \in F} \|f - f^*\|\right), \; \partial f_* / \partial x = (\partial f / \partial x)^*,$$

and the set F be symmetric relative to its Chebyshev center f^*. Then the solution x_* of equation

$$x_*(t) = \int_0^t f^*(t, \tau, x_*(\tau)) d\tau \tag{13}$$

is asymptotically optimal with respect to accuracy, that is

$$\|x^* - x_*\| = o\left(\sup_{x \in X} \|x - x^*\|\right) = o(\Delta_1^*),$$

when $\Delta_1^* \to 0$.

Proof. We have

$$x - x_* = \int_0^t [f(t,\tau,x) - f^*(t,\tau,x_*)]d\tau = \int_0^t [f_*(t,\tau,x) - f_*(t,\tau,x_*)]d\tau +$$

$$\int_0^t [f^*(t,\tau,x) - f_*(t,\tau,x)]d\tau + \int_0^t [f(t,\tau,x) - f^*(t,\tau,x)]d\tau =$$

$$\int_0^t \frac{\partial f_*}{\partial x}(x_*)(x - x_*)d\tau + \int_0^t [f(t,\tau,x) - f^*(t,\tau,x)]d\tau + o(\|x - x_*\|),$$

then the proof is reduced to the linear case which is considered in [7]. □

As a result of Theorem 2.4 the following conclusion can be reached: any approximate solution of equation (13) with accuracy of the order higher than ρ_F^* will be asymptotically optimal. If f^* is given accurately enough, then the same conclusion is true.

3. The Error of Method. We solve (1) first on the segment $[0,\Delta T]$,$\Delta T \le T$. Let us consider two methods of approximate-iterative type [6]: first for the case when f is not smooth enough, and second for the case when f is an analytical function.

Let $f(t,\tau,x)$ satisfies condition (6) and has the i-th, $i = \overline{0,2}$, continuous partial derivatives with respect to all variables, that are not greater than the corresponding values $L_i = L_i(f)$ (in the domain of existence of the solution). Then we will take as an approximate solution of equation (1) the approximation to

$$x^N(t) = \sum_{\nu=0}^{N-1} (x^{\nu+1} - x^\nu);\ x^{\nu+1}(t) = \int_0^t f(t,\tau,x^\nu(\tau))d\tau,\ x^0 = 0, \tag{14}$$

where the integrals will be calculated by the trapezoidal rule [4].

We have on $[0,\Delta T]$

$$\|x^1 - x^0\|_t = \|f^0\|_t,\ f^0 = \int_0^t f(t,\tau,0)d\tau,$$

$$\|x^2 - x^1\|_t \le \max_{0\le\tau\le t} \|f^0\|_\tau Lt,\ \ldots,\ \|x^{\nu+1} - x^\nu\|_t \le \max_{0\le\tau\le t} \|f^0\|_\tau \frac{(Lt)^\nu}{\nu!},\ldots,$$

whence it follows that the error of the method may be estimated by

$$\Delta_{2,t}^1 = \|x - x^N\|_t \le \sum_{\nu=N}^{\infty} \|x^{\nu+1} - x^\nu\|_t \le$$

$$\max_{0\le\tau\le t} \|f^0\|_\tau \frac{(Lt)^N}{N!} \left(1 + \frac{Lt}{N+1} + \frac{(Lt)^2}{(N+1)(N+2)} + \cdots\right) \le$$

$$\max_{0\le\tau\le t}\|f^0\|_\tau\frac{1}{\sqrt{2\pi N}}\left(\frac{Lte}{N}\right)^N\left(1+\frac{Lt}{N+1}+\frac{(Lt)^2}{(N+1)(N+2)}+\cdots\right), \tag{15}$$

$$0\le t\le \Delta T.$$

Let us divide the segment $[0,\Delta T]$ into M parts with the points

$$t_j=j\Delta t,\ \Delta t=\Delta T/M,\ j=\overline{1,M}, \tag{16}$$

and let

$$\tilde x_j^{\nu+1}=\Delta t\left[\frac{f(t_j,0,0)+f(t_j,t_j,\tilde x_j^\nu)}{2}+\sum_{s=1}^{j-1}f(t_j,t_s,\tilde x_s^\nu)\right]=$$

$$\int_0^{t_j}{}'f(t_j,\tau,\tilde x^\nu(\tau))d\tau, \tag{17}$$

where by $'g(\tau)$ is denoted the polygon:

$$'g(\tau)=g(t_s)\frac{t_{s+1}-\tau}{\Delta t}+g(t_{s+1})\frac{\tau-t_s}{\Delta t},\ \tau\in[t_s,t_{s+1}],$$

$\tilde x^\nu(\tau)$ is a piecewise constant function, which is equal to $\tilde x_s^\nu$ for any $\tau\in[t_s,t_{s+1}]$.

It is well-known [4] that

$$\int_0^{t_j}|g-'g|d\tau\le\frac{L_2(g)}{12}(\Delta t)^2t_j.$$

Consequently,

$$|\tilde x_j^{\nu+1}-x^{\nu+1}(t_j)|\le\frac{L_2(f(*,x^\nu))}{12}(\Delta t)^2t_j+L\Delta t\left(\sum_{s=1}^{j-1}|\tilde x_s^\nu-x_s^\nu|+\frac{|\tilde x_j^\nu-x_j^\nu|}{2}\right)\le$$

$$\frac{L_2(f(*,x^\nu))}{12}(\Delta t)^2t_j+Lt_j\max_{1\le s\le j}|\tilde x_s^\nu-x_s^\nu|.$$

We obtain:

$$|(x^{\nu+1})'|\le L_0+L_1\Delta T$$

and

$$|(x^{\nu+1})''|\le 3L_1+L_0L_1+(L_1^2+L_2)\Delta T.$$

By combining the previous estimates we get the conclusion that following estimates

$$L_2(f(*,x^\nu))\le d=L_2+L_2L_0^2+3L_1^2+2L_0L_2+L_0L_1^2+$$

$$(2L_0L_1L_2 + L_1^3 + 3L_1L_2)\Delta T + L_2L_1^2(\Delta T)^2 \tag{18}$$

and

$$\Delta x^\nu \le \frac{dt_j}{12}(\Delta t)^2\frac{1-(Lt_j)^\nu}{1-Lt_j} < \frac{dt_j}{6}(\Delta t)^2,\ Lt_j \le L\Delta T \le 1/2.$$

hold. Considering (15) it is not difficult now to obtain the following estimate

$$\Delta_{2,t_j} = |x(t_j) - \tilde{x}_j^N| \le L_0t_j\frac{1}{\sqrt{2\pi N}}\left(\frac{e}{2N}\right)^N\left(1+\frac{1}{2(N+1)}+\cdots\right)+$$

$$\frac{dt_j}{6}(\Delta t)^2,\ Lt_j \le 1/2,\ 1 \le j \le M. \tag{19}$$

Thus, we can formulate the next proposition.

THEOREM 3.1. *The error of the method (14)–(17) can be estimated by (18)–(19), and the number of evaluations of function f is not more than $(M+1)(M+2)N/2$.*

We emphasize, that in the same way we can estimate the error and the number of basic operation, if instead of the trapezoidal rule we apply the Simpson rule [4] (when $f(t,\tau,x)$ has derivatives of 4-th order). It might be often more convenient to have the approximate solution in the form of corresponding spline [10] instead of the table $\tilde{x}_j^N$. In the case when order of smoothness f is lower one can use numerical integration in (14) based on fraction-rational approximation of f [10] or on the Monte-Carlo type methods [13].

Let $\tilde{x}(t)$ be an approximate solution of equation (1). We consider now the case $t \in [k\Delta T, (k+1)\Delta T]$, $k = \overline{1, R-1}$, $R\Delta T = T$. Further, let

$$x^k(t) = \int_0^{k\Delta T} f(t,\tau,\tilde{x}(\tau))d\tau + \int_{k\Delta T}^t f(t,\tau,x^k(\tau))d\tau,$$

$$x^{k,\nu}(t) = \int_0^{k\Delta T} f(t,\tau,\tilde{x}(\tau))d\tau + \int_{k\Delta T}^t f(t,\tau,x^{k,\nu-1}(\tau))d\tau,$$

$$x^{k,0}(t) = \int_0^{k\Delta T} f(t,\tau,\tilde{x}(\tau))d\tau,\ \nu = \overline{1,N}; \tag{20}$$

$$\tilde{x}_j^{k,\nu} = \tilde{x}_j^{k,0} + \int_{k\Delta T}^{t_j^k}{}' f(t_j^k,\tau,\tilde{x}^{k,\nu-1}(\tau))d\tau,$$

$$\tilde{x}_j^{k,0} = \int_0^{k\Delta T}{}' f(t_j^k,\tau,\tilde{x}(\tau))d\tau,\ t_j^k = k\Delta T + j\Delta t,$$

$$j = \overline{1,M},\ \Delta t = \Delta T/M,\ \nu = \overline{1,N}, \tag{21}$$

where $\tilde{x}^{k,\nu}(\tau)$ is a piecewise constant function, equals to $\tilde{x}_s^{k,\nu}$ on the interval (t_s^k, t_{s+1}^k) and

$$\tilde{x}(t) = \tilde{x}(t_j)(t-t_j)/\Delta t + \tilde{x}(t_{j+1})(t_{j+1}-t)/\Delta t,\ t \in [t_j, t_{j+1}].$$

Then, by analogy with (18),(19) from (20),(21), we can obtain

$$\Delta_{2,T}^1 = \max_{t\in[0,T]} |x(t) - \tilde{x}(t)| \le$$

$$\frac{3L_0T}{N!} + \frac{dT}{2}(\Delta t)^2 + \frac{1}{8}[3L_1 + L_0L_1 + (L_1^2 + L_2)\Delta T](\Delta t)^2,$$

(22) $$j = \overline{1, M},\ k = \overline{1, R-1};\ R\Delta T = T;\ \Delta t = \Delta T/M.$$

Therefore we have proved the next theorem:

THEOREM 3.2. *The estimate of error of the method (21) is given by inequalities (18),(22). The number of evaluations of the function f is not greater than the quantity $(M+1)(M+2)NR/2$.*

Let us consider now the second case when $f(z, v, w)$ is an analytical function of three complexe variables z, v, w in the domain

(23) $$\mathcal{D} = \mathcal{E}_r \times \mathcal{E}_r \times \mathcal{D}(H),$$

where $\mathcal{E}_r$ is a closed domain, bounded with the Zhukowski's ellipse

$$\partial\mathcal{E}_r = \left\{(x, y)|x = \frac{\Delta T}{2} + a_r \cos t,\ y = b_r \sin t,\ t \in [-\pi, \pi]\right\},$$

$$a_r = \Delta T(r + r^{-1})/4,\ b_r = \Delta T(r - r^{-1})/4,\ r > 1,$$

and the domain $\mathcal{D}(H)$ is a closed disk with a center in O and radius H.

Let, as in [6],

$$w(z) = \int_0^z f(z, s, w(s))ds,\ w^\nu(n; z) = \int_0^z L_n(f(*, w^{\nu-1}(n; *), \sigma))d\sigma =$$

(24) $$\sum_{i=0}^{n} f(z, x_i, w^{\nu-1}(n; x_i)) \int_0^z \pi_i(\sigma)d\sigma,$$

$$w^\nu(n; x_j) = w_j^\nu = \frac{\Delta T}{2}\sum_{i=0}^{n} a_{ij} f(x_j, x_i, w_i^{\nu-1});\ j = \overline{0, n};\ \nu = \overline{0, N};$$

$$a_{ij} = \frac{2}{\Delta T}\int_0^{x_j} \pi_i(\sigma)d\sigma =$$

$$\frac{\varepsilon_i}{n}\left[1-c_j+\frac{c_i}{2}(1-c_{2j})+\sum_{\nu=2}^{n}\varepsilon_\nu c_{i\nu}\left(\frac{c_{j(\nu-1)}}{\nu-1}-\frac{c_{j(\nu+1)}}{\nu+1}-\frac{2}{\nu^2-1}\right)\right],$$

$$\varepsilon_0=\varepsilon_n=0.5;\ \varepsilon_j=1,\ j=\overline{1,n-1};$$

$$c_k=\cos\frac{k\pi}{n};\ x_j=x_j(n)=\frac{\Delta T}{2}\left(1-\cos\frac{j\pi}{n}\right).$$

THEOREM 3.3. *Let in the equation*

$$w(z)=\int_0^z f(z,s,w(s))ds \tag{25}$$

the function $f(z,s,w)$ *be analytical in the domain* $\tilde{\mathcal{D}}=\mathcal{E}_{\tilde r}\times\mathcal{E}_{\tilde r}\times\mathcal{D}(\tilde H)$, *(see (23)),* $\Delta\tilde T>0$, $\tilde H>0$, $\tilde r>1$. *Further, assume that for some fixed* $r\in[1,\tilde r]$ *and some natural numbers* n *and* N *there are numbers* $H\le\tilde H$, $\Delta T\le\Delta\tilde T$ *such that the inequality*

$$H>(a_r+\Delta T/2)(\|f\|_{C(\tilde{\mathcal{D}})}+L\Delta_{2,\Delta T}),\ a_r=a_r(\Delta T)=\Delta T(r+r^{-1})/4$$

holds, where by $\Delta_{2,\Delta T}$ *is denoted the right-hand side of the inequality*

$$|w(z)-w^N(n;z)|\le$$

$$L_0(f)\left[(\|L_n^0\|+1)\frac{r(1+r^{-2n-2})}{(1-q)(\tilde r-r)}(\frac{r}{\tilde r})^n(a_r+\frac{\Delta T}{2}+L^{-1}e^q\frac{q^{N+1}}{(N+1)!}\right], \tag{26}$$

$$\|L_n^0\|\le\frac{2}{\pi}\ln n+1,\ q=L(a_r+\Delta T/2)$$

and $L_0(f)=\|f\|_{C(\tilde{\mathcal{D}})}$, L *is the Lipschitz-constant of* f *with respect to* w *in* $\mathcal{D}$. *Then the polynomial* $w^N(n;z)$, *constructed by algorithm (24) approximates the solution* $w(z)$ *of (25) in the domain* $\mathcal{E}_r$ *such, that the inequality (26) holds.*

The proof of this theorem is similar to Theorem 2 [6].

It is not difficult to obtain from previous theorem the next one:

THEOREM 3.4. *If the segment* $[0,T]$ *is divided into the subsegments*

$$[(k-1)\Delta T,k\Delta T],\ L\Delta T\le 1/2,\ k=\overline{1,R},$$

and for each of them conditions of Theorem 3.3 hold, then the error $\Delta^2_{2,T}$ *of piecewise polynomial approximation (24) is estimated by inequality*

$$\Delta^2_{2,T}\le 4e^{LT}\Delta^2_{2,\Delta T};\ L\Delta T\le 1/2. \tag{27}$$

In this case the total number of evaluations of the function f *is not greater than the quantity* $RN(n+1)^2$.

Let us ilustrate the origin of constant 4. If $L\Delta T=1/2$ then we have

$$\Delta^2_{2,T}\le\Delta^2_{2,\Delta T}\sum_{k=1}^{R-1}ke^{(R-k)L\Delta T}\le e^{LT}\Delta^2_{2,\Delta T}\int_0^\infty xe^{-x/2}dx=4e^{LT}\Delta^2_{2,\Delta T}.$$

4. Round-off Error. Let us denote by f_l the value of f that is evaluated numerically, where l is the number of binary digits (floating point). Often the following estimate is true (see [4]):

$$\Delta_3^f = |f - f_l| \le C_f(|f|+1)2^{-l} \le C_f(L_0+1)2^{-l}, \tag{28}$$

where the constant C_f is not very large. Let us look into what additional error can contribute into the solution the error (28).

THEOREM 4.1. *If conditions (6),(8) and (29) hold then the following*

$$\Delta_3^1 \le \left[2.12L_0[(M+1)R+2]^2 + 4C_f(L_0+1)(M+1)R\right]2^{-l}, \tag{29}$$

$$2^{-l}(M+1)R \le 0.1$$

estimate is true.

Proof. Let us apply the estimates from [15] to scheme (21). Then we obtain estimate (see also (17)) for maximal error per step by one iteration

$$\Delta_{3,\Delta T}^1 \le 1.06\cdot 2^{-l}L_0[(M+1)R+2]^2/2 + \Delta_3^f(M+1)R;\ 2^{-l}(M+1)R \le 0.1.$$

As iterative process (21) converges (with respect to ν) with the ratio $L\Delta T \le 1/2$ final error on one step will be not large than $2\Delta_{3,\Delta T}^1$. This error will appear on every next step as an inherent error. That is why we conclude based on (8) that the total accumulated error will be not larger than $4\Delta_{3,\Delta T}^1$. □

THEOREM 4.2. *The round-off error of the approximate solution (24) can be estimated by*

$$\Delta_{3,\Delta T}^2 \le \frac{1}{1-q}[(n+3)^2L_0\max|a_{ij}|\cdot 1.06\cdot 2^{-l-1} + (\Delta_3^{a_{ij}}L_0 +$$

$$\Delta_3^f\max|a_{ij}|)(n+1)],\ (n+2)2^{-l} < 0.1;$$

$$\Delta_3^{a_{ij}} \le \frac{(n+7)\ln n}{n}\cdot 1.06\cdot 2^{-l} + 2\Delta_3^{\cos}\ln n;\ \Delta_3^{\cos} \le 2^{-l+1};$$

$$|a_{ij}| \le 3\frac{\ln n}{n};\ q = L\left(a_r + \frac{\Delta T}{2}\right) < 1. \tag{30}$$

The proof is similar to the proof of Theorem 4.1.

From Theorem 4.2 and the proof of Theorem 4.1 we get the following

Corollary. Let the conditions of Theorem 4.1 and (4) hold. Then the main term of round-off error of piecewise polynomial approximation is estimated by

$$\Delta_{3,T}^2 \le e^{LT}\cdot 4\Delta_{3,\Delta T}^2 \approx L_0e^{LT}2^{-l+3}\frac{4n\ln n}{1-q}, \tag{31}$$

$$(n+2)R\cdot 2^{-l} \le 0.1;\ L\Delta T \le 1/2.$$

5. The Total Absolute Error. By combining the above given estimates (8),(22) and (29) we deduce the estimate of the total absolute error of approximate solution of the equation (1) by the method (21) in Chebyshev's metric

$$\Delta^1 \leq 2 \max_{0\leq\tau\leq T} |\delta f| + \frac{3TL_0}{N!} + \frac{dT}{2}(\Delta t)^2 + \frac{1}{8}[3L_1 + L_0 L_1 +$$

$$(L_1^2 + L_2)\Delta T](\Delta t)^2 + [2.12 L_0[(M+1)R+2]^2 +$$

$$4C_f(L_0+1)(M+1)R]2^{-l};\ 2^{-l}(M+1)R \leq 0.1. \tag{32}$$

Similarly, combining estimates (5), (26), (27), (30) and (31) gives the estimate of total absolute error of a solution of equation (1) by method (24) (considering only the main terms and using $q \leq 1/2$, $r/\tilde{r} \leq 1/2$)

$$\Delta^2 \leq e^{LT}[\max_{0\leq\tau\leq t} |\delta f| + 4[2^{-n} L_0 \pi \ln n + \sqrt{e} L_0 L^{-1}/(2^N N!)] +$$

$$2^{-l+6} L_0 n \ln n];\ 2^{-l}(n+2)R \leq 0.1. \tag{33}$$

One can see from (32) that in order for Δ^1 to have the beforehand given order of decrease ε it is sufficient that R/M^2, R/N^N, and $(MR)^2 2^{-l}$ be of order ε. The number of operations of the first method M^2NR will be logarithmically equivalent to number of initial data M^2R when $\varepsilon \to 0$ if, for instance, $N \sim \ln(1/\varepsilon)$, $m^2R \sim 1/\varepsilon$. In this case if $\ln R$ has an order of $\ln(1/\varepsilon)$, then l will have the same order too. Thus, both hypothesis from [3] for the first method are satisfyed. Further, from (33) it follows that in order to derive the same result for the second method, it is sufficient that $e^R \ln n/2^n$, $e^R/(2^N N!)$ and $e^R n \ln n/2^l$ have an order of ε. If instead of simple iteration we apply the method, which combines the simple iteration and the Newton's method, then in the corresponding estimate of the total absolute error we have $1/2^{2^N}$ instead of part $1/(2N)^N$. In this case the property of logarithmic equivalence will be satisfied for the second method too.

Finally we mention that some of the results were announced in [11].

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