Matrix Sensitivity Analysis from an Interior Solution of a Linear Program

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Abstract

This paper considers the effect of changing matrix coefficients in a linear program after we have obtained an interior solution. Changes are restricted to where there remains an optimal solution to the perturbed problem (called "admissible"). Mills' minimax theorem provides one approach and has been used for similar sensitivity analysis from a basic optimum. Here we consider the effect on the optimal partition and how the analysis results relate to the classical approach that uses a basic solution.

Keywords: linear programming, sensitivity analysis, optimal partition, interior solution, computational economics.

Contents

1 Introduction

Consider the primal-dual linear programs:

$$\min\{cx : x \geq 0,\ Ax \geq b\} \qquad \max\{yb : y \geq 0,\ yA \leq c\},$$

where c is a row vector in $\mathbb{R}^n$, called *objective coefficients*; x is a column vector in $\mathbb{R}^n$, called *levels*; b is a column vector in $\mathbb{R}^m$, called *right-hand sides*; y is a row vector in $\mathbb{R}^m$ called *prices*; and A is an $m \times n$ matrix. Elements are denoted by subscripts, as c_j, b_i and A_{ij}; the j-th column of A is denoted A_j, and the i-th row by $A_{i\bullet}$.

In the study of how data changes affects an optimal solution, most attention has been on the *rim data:* (b, c), so-called because it appears on the rim of a schematic depicting an LP model. We suppose we have an optimal solution, and we are interested in the effect of perturbing A. This has been considered with respect to its effect on the optimal objective value, using Mills' minimax theorem [13], extended by Williams [18], which gives the directional derivative (when it exists). Subsequent results consider the range of a change for which a particular basis remains optimal. Here we develop analogous results for when the optimal partition, determined by having an interior solution, remains invariant.

The next section presents the most basic terms and concepts needed for what follows. More generally, the technical terms used throughout this paper are defined in the *Mathematical Programming Glossary* [8]. For a comprehensive background into the optimal partition, see [17, 19]. For particular applications to sensitivity analysis, see [6], and for underlying theory of rim sensitivity analysis from an interior solution, see [1, 7, 10, 11, 14]. As sensitivity analysis is extended to consider changing matrix coefficients, we shall be comparing our results with classical analysis from a basic solution. A key reference is [2] (also see some of the recent advances in [3]). The tolerance approach [15, 16] can also be extended with the use of generalized inverses, much like they use basis inverses.

2 Technical Background

Let $P(A)$ and $D(A)$ denote the primal and dual polyhedra, respectively. For $(x, y) \in P(A) \times D(A)$, we associate *surplus variables*, $s = Ax - b$ and *reduced costs*, $d = c - yA$. Let $P^*(A)$ and $D^*(A)$ denote the primal and dual optimality regions, respectively, which we suppose are not empty. The *support set* of a non-negative vector, v, is denoted: $\sigma(v) = \{k : v_k > 0\}$. Then, feasible primal and dual solutions, x^* and y^*, are optimal if, and only if, they satisfy *complementary* *slackness*: $\sigma(x^*) \cap \sigma(d^*) = \emptyset$ and $\sigma(s^*) \cap \sigma(y^*) = \emptyset$. As shown by Goldman and Tucker[4], there must exist a *strictly complementary* solution, whereby the support sets span the rows and columns: $\sigma(x^*) \cup \sigma(d^*) = \{1, \ldots, n\}$, and $\sigma(s^*) \cup \sigma(y^*) = \{1, \ldots, m\}$. This defines the (unique) *optimal partition*, obtained from any strictly complementary (i.e., interior) solution.

We are interested in the effect of changing the matrix to $A + \theta\,\delta A$ for $\theta > 0$, where δA is a fixed $m \times n$ matrix, called the *change direction*. One of our main concerns is how to establish a range of θ for which the optimal partition does not change. Analogous to the rim variation, we are also concerned with how the partition changes when it must change for a particular direction matrix. First, we recall a central minimax theorem that provides some information about the effect of changing A without regard to what kind of solution we have obtained (i.e., basic or interior). Then, in the next section, we review what is known about the effect on a basic optimum. This is in order to be self contained and establish some analogies we shall consider in the subsequent section that contains the new results.

Although we assume no rim variation, this analysis includes that case by suitable transformation. For example, augment primal variables x_0 and x_{n+1} to obtain the equivalent LP:

$$\min x_0 : x \geq 0, \begin{pmatrix} 1 & -c & 0 \\ 0 & A & -b \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ x_{n+1} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

The first constraint, $x_0 - cx \geq 0$, is the only one containing x_0, so the minimum value of x_0 equals the minimum of cx. The last two constraints, $x_{n+1} \geq 1$ and $-x_{n+1} \geq -1$ force $x_{n+1} = 1$, which then makes the remaining constraints equivalent to the original canonical system, $Ax \geq b$. Now the rim vectors, b and c, are included in the transformed matrix, so they can be perturbed as part of the matrix coefficient change.

We say ΔA is an *admissible* change if the LP defined by $A + \Delta A$ has an optimal solution — equivalently, if the primal and dual conditions are feasible:

$$\exists (x, y) \geq 0 \ni [A + \Delta A]x \geq b \text{ and } y[A + \Delta A] \leq c.$$

Let $\mathcal{A}$ denote the set of admissible changes. Unlike rim variation, this is not a convex set. To illustrate, consider the following

Example 1 $$\min\{-x : x \geq 0, \quad \begin{aligned} (-1 + \Delta A_{11})\, x &\geq -2 \\ (-2 + \Delta A_{21})\, x &\geq -2\}. \end{aligned}$$

When $\Delta A = 0$, the two constraints are simply upper bounds: $x \le 2$ and $x \le 1$. The set of admissible changes is when at least one of these constraints imposes an upper bound, which means at least one coefficient must be negative. Otherwise, if both coefficients are non-negative, the LP is unbounded by letting $x \to \infty$. Thus, the set of admissible changes is the following non-convex set:

$$\mathcal{A} = \left\{ \begin{bmatrix} \Delta A_{11} \\ \Delta A_{21} \end{bmatrix} : \Delta A_{11} < 1 \text{ or } \Delta A_{21} < 2 \right\}.$$

(Note that the changes $\left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right]$ are in $\mathcal{A}$, but their midpoint is not.)

We say δA is an *admissible direction* if there exists $\theta^* > 0$ such that $A + \theta\,\delta A$ is an admissible change for all $\theta \in [0, \theta^*)$. The set of admissible directions, $\mathcal{D}$, is composed of those δA for which the primal and dual feasibility conditions hold over an interval:

$$\mathcal{D} = \{\delta A \in R^{m \times n} : \exists \theta^* > 0 \ni \theta\,\delta A \in \mathcal{A}\ \forall \theta \in [0, \theta^*)\}.$$

Unlike the case of rim variation, we cannot say $\theta\,\delta A$ is an admissible change for $\theta \in (0, \theta^*)$ just because $\theta^*\,\delta A$ is admissible. However, the situation is more favorable than looking at the set of total changes, $\mathcal{A}$. Consider the previous example. For $i = 1, 2$, define

$$\theta^i = \begin{cases} 1/\delta A_{i1} & \text{if } \delta A_{i1} > 0 \\ \infty & \text{if } \delta A_{i1} \le 0 \end{cases}$$

and define $\theta^* = \sup\{\theta^1, \theta^2\}$. It follows that $\theta\,\delta A$ is an admissible change for all $\theta \in [0, \theta^*)$, and $\theta\,\delta A$ is not an admissible change for any $\theta > \theta^*$. That is, the set of feasible θ in $\mathbb{R}^+$ is a simple interval in this example.

On the other hand, consider the following

Example 2 The primal equations are as follows (let $c = 0$, so the objective does not matter):

$$\begin{pmatrix} 1+\theta & & \\ \theta & 1+\theta & \\ 1+\frac{1}{2}\theta & \theta & 1+\theta \end{pmatrix} x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(The equality system could be replaced by the equivalent canonical system, $[A + \theta\,\delta A]x \ge b$ and $[-A + \theta\,(-\delta A)]x \ge -b$.) For any $\theta \ge 0$, we have the following unique solution to the equations:

$$x_1(\theta) = \frac{1}{1+\theta}$$
$$x_2(\theta) = \frac{1}{(1+\theta)^2}$$
$$x_3(\theta) = \frac{\theta^2-\theta}{2(1+\theta)^3}$$

This is non-negative for $\theta = 0$ and $\theta \geq 1$, so the range of θ for which $\theta\,\delta A$ is an admissible change is not a simple interval. (The interval $(0,1)$ is missing because x_3 first becomes negative, then becomes positive as θ crosses 1.)

For $\delta A \in \mathcal{D}$, let $\theta^*(\delta A) \stackrel{\text{def}}{=} \sup\{\theta^*\colon\ \theta\,\delta A \in \mathcal{A} \text{ for } \textit{all } \theta \in [0,\theta^*)\}$, and let $\Theta(\delta A) \stackrel{\text{def}}{=} \{\theta\colon\ \theta\,\delta A \in \mathcal{A}\}$. We know from the examples that the range of θ for which $\theta\,\delta A$ is admissible may or may not be a simple interval — i.e., we can have $\Theta(\delta A) \neq [0,\theta^*(\delta A)]$ (or with the right end open).

The difference in the two examples is the structure of $\mathcal{A}$. Even though this is not convex in the first example, it is a union of polyhedra. In that case, we have the following.

Theorem 1 *Suppose* $\mathcal{A} = \bigcup_{k=1}^{k=K}\{\mathrm{P}_k\}$, *where each* P_k *is a polyhedron containing the origin. Then,* $\Theta(\delta A)$ *is a simple interval:* $[0,\theta^*(\delta A)) \subseteq \Theta(\delta A) \subseteq [0,\theta^*(\delta A)]$.

Proof: Let $\theta_k \stackrel{\text{def}}{=} \sup\{\theta\colon\ \theta\,\delta A \in \mathrm{P}_k\}$ for $k = 1,\ldots,K$. Since each P_k is convex, either $\theta_k = \infty$ or $[0,\theta_k\delta A) \subseteq \mathrm{P}_k$ and $\theta\delta A \notin \mathrm{P}_k$ for $\theta > \theta_k$ ($\theta_k\delta A$ may or may not be in P_k). There exists k for which $\theta^*(\delta A) = \theta_k$, so if $\theta^*(\delta A) = \infty$, $\Theta(\delta A) = [0,\infty)$. If each θ_k is finite, $\theta^*(\delta A) = \max\{\theta_k\}$. Suppose this maximum is for k^*. We have $\Theta(\delta A) = \{\theta \geq 0\colon\ \theta\,\delta A \in \mathrm{P}_{k^*}\} = [0,\theta_{k^*})$ or $[0,\theta_{k^*}]$ (depending on whether P_{k^*} contains the boundary point). □

Let $z(\delta A)$ denote the optimal objective value as a function of A, and let $Dz(A;\delta A)$ denote the directional derivative (when it exists):

$$Dz(A;\delta A) = \lim_{\theta\to 0_+} \frac{z(A+\theta\,\delta A) - z(A)}{\theta}$$

for $\delta A \in \mathcal{D}$. A key theorem, essentially due to Mills [13] and extended by Williams [18], gives the directional derivative of the optimal objective value when it exists. Let $\{\theta^k\} \to 0$, and define the set of optimal limit points:

$$P^\infty(A;\delta A) \times D^\infty(A;\delta A) \stackrel{\text{def}}{=} \{(x,y)\colon\ \exists\{(x^k,y^k)\} \in P^*(A+\theta^k\delta A) \times D^*(A+\theta^k\delta A) \ni \{(x^k,y^k)\} \to (x,y)\}.$$

In Mills' proof of his minimax theorem, he assumes normalization constraints, $\sum_j x_j = 1$ and $\sum_j y_j = 1$. This is natural in his game-theoretic context, where x and y are probabilities over a complete set of strategies, and it is often assumed in analysis of interior point methods (with scaling). The normalization is assumed to be unaffected by perturbation, which implies $P^\infty(A;\delta A) \times D^\infty(A;\delta A) \neq \emptyset$; otherwise, it is possible for there to be no limit points of any primal-dual optimal sequence. Here is an example (suggested by Stephen Billups):

Example 3 $\min x_2 + x_3$: $x \geq 0,\ -x_1 + \theta\, x_2 \geq 0,\ \theta\, x_1 - \theta\, x_3 \geq 0,\ x_3 \geq 1.$

For $\theta = 0$, the unique primal solution is $(0,0,1)$, and for $\theta > 0$, the unique primal solution is $(1, \frac{1}{\theta}, 1)$. The matrix change direction is thus admissible, but $P^\infty(A;\delta A) = \emptyset$.

Theorem 2 (Mills' Minimax) *Let δA be an admissible direction, and suppose $P^\infty(A;\delta A) \times D^\infty(A;\delta A) \neq \emptyset$. Then, the differential Lagrangian has a saddlepoint in the optimality region, and the saddlepoint value is the directional derivative of z:*

$$\begin{aligned} Dz(A;\delta A) &= \min_{x \in P^*(A)} \max_{y \in D^*(A)} \{-y(\delta A)x\} \\ &= \max_{y \in D^*(A)} \min_{x \in P^*(A)} \{-y(\delta A)x\}. \end{aligned}$$

This specializes to the one-sided derivatives when they exist:

$$\begin{aligned} \partial^+ z/\partial A_{ij} &= Dz(A; \mathrm{e}_i^T \otimes \mathrm{e}_j) &&= \min_x \max_y\{-y_i\, x_j\} &&= -y_i^{\min} x_j^{\max} \\ \partial^- z/\partial A_{ij} &= -Dz(A; -\mathrm{e}_i^T \otimes \mathrm{e}_j) &&= -\min_x \max_y\{\, y_i\, x_j\} &&= -y_i^{\max} x_j^{\min} \end{aligned}$$

where e_j is the j-th unit row vector (whose size is determined by context). When the associated price and quantity are unique ($y_i^{\min} = y_i^{\max} = y_i$ and $x_j^{\min} = x_j^{\max} = x_j$), this specializes to the complete derivative, $\partial z/\partial A_{ij} = -y_i x_j$.

The rest of this paper is concerned with the range of θ when the matrix change is $\theta\delta A$, where δA is a fixed direction of change. The special case of varying one coefficient, say A_{ij}, is with $\delta A = \mathrm{e}_i^T \otimes \mathrm{e}_j$. In the next section we review what is known about maintaining the optimality of a basis. Then, we derive similar results for maintaining the optimality of the partition.

A perspective to keep in mind when comparing classical methods that assume a basic solution is that those methods derive ranges for which the basis remains optimal. Here we obtain the (unique) optimal partition from an interior solution, and we are interested in two questions: *What is the range for* *which the optimal partition does not change?* and, if it must change immediately, *What is the range for which the new optimal partition remains optimal?* A companion paper [7] on rim analysis addressed both questions, but here we consider only the first question. The analysis highlights a fundamental difference in the sensitivity information sought, but the methodology we shall present has great similarity to classical analysis from a basic solution. (See [6] for practical situations where the optimal partition provides more information than a basis.)

3 Basic Ranges

Given an optimal basis, partition A in the usual way:

$$A = \begin{bmatrix} \mathbf{B} & \mathbf{N} \\ \mathbf{B}^* & \mathbf{N}^* \end{bmatrix} \begin{matrix} \leftarrow \text{Nonbasic } s \\ \leftarrow \text{Basic } s \end{matrix}$$
$$\begin{matrix} \uparrow & \uparrow \\ \text{Basic} & \text{Nonbasic} \\ x & x \end{matrix}$$

A basis $\mathbf{B}$ remains optimal if the following conditions are satisfied:

Primal

$$\begin{aligned} (\mathbf{B} + \theta\,\delta\mathbf{B})x_B &= b_N \\ (\mathbf{B}^* + \theta\,\delta\mathbf{B}^*)x_B &\geq b_B \\ x_B &\geq 0 \end{aligned}$$

Dual

$$\begin{aligned} y_N(\mathbf{B} + \theta\,\delta\mathbf{B}) &= c_B \\ y_N(\mathbf{N} + \theta\,\delta\mathbf{N}) &\leq c_N \\ y_N &\geq 0 \end{aligned}$$

Letting x^* and y^* denote the current optimal solution, $x^*_B = \mathbf{B}^{-1}b$ and $y^*_N = c_B\mathbf{B}^{-1}$, these conditions are equivalent to the following:

Primal

$$\begin{aligned} (\mathrm{I} + \theta\,\overline{\delta\mathbf{B}})\Delta x_B &= -\,\theta\,\overline{\delta\mathbf{B}}\,x^*_B \\ (\mathbf{B}^* + \theta\,\delta\mathbf{B}^*)\Delta x_B &\geq -s^*_B - \theta\,\delta\mathbf{B}^* x^*_B \\ \Delta x_B &\geq -x^*_B \end{aligned}$$

Dual

$$\begin{aligned} \Delta y_N(\mathrm{I} + \theta\,\widehat{\delta\mathbf{B}}) &= -\theta\, y^*_N\widehat{\delta\mathbf{B}} \\ \Delta y_N(\mathbf{N} + \theta\,\delta\mathbf{N}) &\leq d^*_N - \theta\, y^*_N\delta\mathbf{N} \\ \Delta y_N &\geq -y^*_N \end{aligned}$$

where $\overline{\delta\mathbf{B}} \stackrel{\text{def}}{=} \mathbf{B}^{-1}\delta\mathbf{B}$ and $\widehat{\delta\mathbf{B}} \stackrel{\text{def}}{=} \delta\mathbf{B}\,\mathbf{B}^{-1}$.

In general, when $\delta\mathbf{B} \neq 0$, the change vectors, Δx_B and Δy_N, are nonlinear functions of θ. Further, there is an additional condition that $\mathbf{B} + \theta\,\delta\mathbf{B}$ be nonsingular, making the range of θ not a simple interval. This latter difficulty is eliminated by imposing a restriction: $\theta < \theta^0$, where $\mathbf{B} + \theta\,\delta\mathbf{B}$ is nonsingular throughout $[0, \theta^0)$. Although the range is generally nonlinear for arbitrary change directions, this can still result in a *linear* inequality on θ.

One case that yields linear inequalities is when $\delta\mathbf{B} = 0$. Then, primal quantities and dual prices are unchanged: $\delta x = 0$ and $\delta y = 0$. The only conditions that limit the range of θ are the primal surplus changes: $\theta\,\delta\mathbf{B}^* x_B^* \geq -s_B^*$, and the dual reduced cost changes: $\theta\,\delta\mathbf{N} \leq d_N^*$.

Another approach, following Gal [2], is to suppose $\delta A = p \otimes q$, where p is a column vector in $\mathbb{R}^m$ and q is a row vector in $\mathbb{R}^n$. The case of changing row i is with $p = \mathrm{e}_i^T$ and q arbitrary (the change vector for the row). Similarly, the case of changing the j-th column is with p arbitrary and $q = \mathrm{e}_j$.

Partition $p = \begin{pmatrix} p_N \\ p_B \end{pmatrix}$ (conformal with b) and $q = (q_B\ q_N)$ (conformal with c), so we have the following:

$$\begin{aligned} \delta A &= \begin{pmatrix} p_N \\ p_B \end{pmatrix} \otimes (q_B\ q_N) \\ &= \begin{pmatrix} p_N \otimes q_B & p_N \otimes q_N \\ p_B \otimes q_B & p_B \otimes q_N \end{pmatrix} \begin{matrix} \leftarrow \text{tight rows } (s_N = 0) \\ \leftarrow \text{surplus rows } (y_B = 0) \end{matrix} \\ & \quad\ \begin{matrix} \uparrow & \uparrow \\ \text{basic} & \text{nonbasic} \\ \text{columns} & \text{columns} \\ (d_B = 0) & (x_N = 0) \end{matrix} \end{aligned}$$

We have $x_B(\theta) = [\mathbf{B} + \theta\, p_N \otimes q_B]^{-1} b_N$ and $y_N(\theta) = c_B [\mathbf{B} + \theta\, p_N \otimes q_B]^{-1}$, where θ is restricted so that $\det[\mathbf{B} + \theta\, p_N \otimes q_B] \neq 0$. Then, Gal applies Bodewig's formula:

$$[\mathbf{B} + \theta\, p_N \otimes q_B]^{-1} = \mathbf{B}^{-1} - \frac{\theta\, \mathbf{B}^{-1} p_N \otimes q_B \mathbf{B}^{-1}}{1 + \theta\, q_B \mathbf{B}^{-1} p_N}$$

to obtain the following nonlinear functions of θ:

$$\begin{aligned} x_B(\theta) &= \left[\mathrm{I} - \frac{\theta\, \overline{p}_N \otimes q_B}{1 + \theta\, \eta}\right] x_B^* \\ y_N(\theta) &= y_N^* \left[\mathrm{I} - \frac{\theta\, p_N \otimes \widehat{q}_B}{1 + \theta\, \eta}\right] \end{aligned} \qquad (1)$$

where $\overline{p}_N \stackrel{\text{def}}{=} \mathbf{B}^{-1}p_N, \widehat{q}_B \stackrel{\text{def}}{=} q_B\mathbf{B}^{-1}$ and $\eta \stackrel{\text{def}}{=} q_B\mathbf{B}^{-1}p_N$.

The nonsingularity condition is now evident: $1 + \theta\,\eta > 0$. Define $\theta^0 \stackrel{\text{def}}{=} -1/\eta$ if $\eta < 0$; otherwise, $\theta^0 \stackrel{\text{def}}{=} \infty$. The nonsingularity condition is the strict inequality, $\theta < \theta^0$.

Given this nonsingularity condition, the non-negativity of x_B and y_N become linear inequalities in θ:

$$x_B(\theta) \geq 0 \quad \Leftrightarrow \quad (1+\theta\,\eta)x^*_B \geq \theta\,(q_B x^*_B)\overline{p}_N$$
$$y_N(\theta) \geq 0 \quad \Leftrightarrow \quad (1+\theta\,\eta)y^*_N \geq \theta\,(y^*_N p_N)\widehat{q}_B$$

The remaining conditions are the non-negativities of the basic surplus variables (s_B) and the nonbasic reduced costs (d_N). These are equivalent to the following linear inequalities:

$$s_B(\theta) \geq 0 \quad \Leftrightarrow \quad s^*_B + \theta\Big(\eta s^*_B + (B^*\overline{p}_N - p_B)\,q_B x^*_B\Big) \quad \geq 0$$
$$d_N(\theta) \geq 0 \quad \Leftrightarrow \quad d^*_N + \theta\Big(\eta d^*_N + y^*_N p_N(\widehat{q}_B\mathbf{N} - q_N)\Big) \quad \geq 0.$$

(In the derivation, recall $\eta = q_B\mathbf{B}^{-1}p_N = q_B\overline{p}_N$, which eliminates the quadratic term.)

These linear inequalities determine the greatest value of θ for which this basis remains optimal. The result is either an interval of the form $[0, \theta^*]$ with θ^* a finite value determined by the non-negativity inequalities, or the result is an interval of the form $[0, \theta^*)$ with $\theta^* = \infty$ or $\theta^* = -1/\eta > 0$.

Consider the special case of changing one coefficient: $p = \mathrm{e}^T_i$ and $q = \mathrm{e}_j$. If it is in $\mathbf{N}$, the only effect of changing $\mathbf{N}_{ij}$ is to change its own reduced cost: $\Delta d_N = -\theta\, y^*_N(\mathrm{e}^T_i \otimes \mathrm{e}_j)$, so $\Delta d_k = 0$ for $k \neq j$ and $\Delta d_j = -\theta\, y^*_i$. The range of θ is thus determined simply: $\theta \leq d^*_j/y^*_i$ (this is defined to be ∞ if $y^*_i = 0$, even if $d^*_j = 0$). Thus, the range of this coefficient for which this basis remains optimal is $(-\infty, \mathbf{N}_{ij} + d^*_j/y^*_i]$.

In the above inequalities, we have $p_B = 0, q_B = 0$ and $\eta = 0$, so $x_B(\theta) = x^*_B$, $y_N(\theta) = y^*_N$ and $s_B(\theta) = s^*_B$. Further, $p_N = \mathrm{e}^T_i$ and $q_N = \mathrm{e}_j$, so $d_N(\theta) = d^*_N + \theta\,(y^*_N \mathrm{e}^T_i(-\mathrm{e}_j)) = d^*_N - \theta\, y^*_i \mathrm{e}_j$. This is what we just noted: $d_k(\theta) = d^*_k$ for $k \neq j$, and $d_j(\theta) = d^*_j - \theta\, y^*_i$.

It is useful to interpret this by activity analysis, where each column represents an activity's input (if $A_{ij} < 0$) or output (if $A_{ij} > 0$). (Details of this can be found in [9].) In the present case, this says that this activity (j) is currently not economical, relative to the basic prices. It remains uneconomical if this coefficient is decreased because that means the unit output is less or that the input requirement is more. On the other hand, if the coefficient is increased, it eventually reaches a value when it is equally economical to enter it into the basis, namely when its reduced cost becomes zero (at $\theta = d_j^*/y_i^*$).

If the coefficient is $\mathbf{N}_{ij}^*$, nothing is affected, so the range of this coefficient is $(-\infty, \infty)$. In terms of the derived inequalities, we have $p_N = 0$, $p_B = e_i^T$, $q_B = 0$ and $q_N = e_j$, so all solution variables remain constant. In words, this says it does not matter the rate that the j-th activity consumes or produces the i-th entity since it has no value (i.e., $y_i^* = 0$).

If the coefficient is $\mathbf{B}_{ij}^*$, the only effect is to change the level of surplus: $\Delta s_B = \theta(e_i^T \otimes e_j)x_B^* = \theta\, e_i^T x_j^*$. The range of θ is thus determined simply: $\theta \geq -s_i^*/x_j^*$ (this is defined to be $-\infty$ if $x_j^* = 0$, even if $s_i^* = 0$). The complete range is thus $[-s_i^*/x_j^*, \infty)$. In words, this says this coefficient can increase indefinitely without changing the optimality of the basis because it simply increases the level of surplus. If the coefficient decreases, however, it means that the level of surplus also decreases and is eventually driven to zero. At that point ($\theta = -s_i^*/x_j^*$), the i-th row becomes tight and the basis is on the threshold of becoming infeasible. Figure 1 shows the ranges of these three cases.

Figure 1: Range of A_{ij} not in the Basis Kernel for the Basis to Remain Optimal

Submatrix	$\Delta A_{ij}^{\min}$		$\Delta A_{ij}^{\max}$	
$\mathbf{N}^*$	$-\infty$		∞	
$\mathbf{B}^*$	$-s_i/x_j$	if $x_j > 0$	∞	
	$-\infty$	if $x_j = 0$		
$\mathbf{N}$	$-\infty$		d_j/y_i	if $y_i > 0$
			∞	if $y_i = 0$

Now consider changing $\mathbf{B}_{ij}$. There are four non-negativity conditions plus the nonsingularity condition, which are given in figure 2. For example, the inequalities imposed by the k-th basic variable are:

$$x_k/(\beta_{ki}x_j - \beta_{ji}x_k)^- \leq \Delta B_{ij} \leq x_k/(\beta_{ki}x_j - \beta_{ji}x_k)^+.$$

At most one of these inequalities restricts the change: an upper bound is imposed if $\beta_{ki}x_j - \beta_{ji}x_k > 0$; a lower bound is imposed if $\beta_{ki}x_j - \beta_{ji}x_k < 0$.

Figure 2: Range of $\mathbf{B}_{ij}$ for the Basis (B) to Remain Optimal

Condition	$\Delta\mathbf{B}_{ij}^{\min}$	$\Delta\mathbf{B}_{ij}^{\max}$
$\mathbf{B}$ nonsingular†	$-1/\beta_{ji}^{+}$	$-1/\beta_{ji}^{-}$
$x_B(\theta) \ge 0$	$x_k/(\beta_{ki}x_j - \beta_{ji}x_k)^-$	$x_k/(\beta_{ki}x_j - \beta_{ji}x_k)^+$
$s_B(\theta) \ge 0$	$s_k/(\beta^*_{ki}x_j - \beta_{ji}s_k)^-$	$s_k/(\beta^*_{ki}x_j - \beta_{ji}s_k)^+$
$y_N(\theta) \ge 0$	$y_k/(y_i\beta_{jk} - y_k\beta_{ji})^-$	$y_k/(y_i\beta_{jk} - y_k\beta_{ji})^+$
$d_N(\theta) \ge 0$	$-d_k/(y_i\beta_{j\bullet}\mathbf{N}_k + \beta_{ji}d_k)^+$	$-d_k/(y_i\beta_{j\bullet}\mathbf{N}_k + \beta_{ji}d_k)^-$

†The nonsingularity condition is a strict bound.

Notation: $\beta = \mathbf{B}^{-1}$; $\beta^* = \mathbf{B}^*\mathbf{B}^{-1}$; $v^+ = \max\{0, v\}$; $v^- = \min\{0, v\}$.

These bounds are derived as follows (given θ satisfies the nonsingularity condition, $1 + \theta\,\beta_{ji} > 0$):

$$\begin{aligned} x_B(\theta) \ge 0 \quad &\Leftrightarrow [\mathbf{B} + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j]^{-1} b_N \quad && \ge 0 \\ &\Leftrightarrow [\mathrm{I} + \theta\,\beta_{\bullet i} \otimes \mathrm{e}_j]^{-1}\mathbf{B}^{-1} b_N && \ge 0, \end{aligned}$$

where $\beta_{\bullet i}$ is the i-th column of $\mathbf{B}^{-1}$. Letting E denote the elementary column matrix, $\mathrm{I} + \theta\,\beta_{\bullet i} \otimes \mathrm{e}_j$, we have

$$x_B(\theta) \ge 0 \quad \Leftrightarrow \quad \mathrm{E}^{-1} x_B \ge 0.$$

The inverse of E is another elementary column matrix:

$$\mathrm{E}^{-1} = \begin{bmatrix} \mathrm{I} & \dfrac{-\theta\,\beta_{1i}}{1 + \theta\,\beta_{ji}} & 0 \\ & \vdots & \\ 0 & \dfrac{1}{1 + \theta\,\beta_{ji}} & 0 \\ & \vdots & \\ 0 & \dfrac{-\theta\,\beta_{mi}}{1 + \theta\,\beta_{ji}} & \mathrm{I} \end{bmatrix} \begin{matrix} \\ \\ \leftarrow \text{row } j \\ \\ \\ \end{matrix}$$

$$\uparrow$$
$$\text{column } j$$

This implies $E^{-1}x_B$ is the vector with entries $x_k - x_j\theta\,\beta_{ki}/(1+\theta\,\beta_{ji})$ for $k \neq j$, and the j-th entry is $x_j/(1+\theta\,\beta_{ji})$. Since we assume the nonsingularity condition, $1+\theta\,\beta_{ji} > 0$, it follows that $x_j(\theta) \geq 0$ with no further restriction on θ. For $k \neq j$, the k-th basic variable imposes the inequality:

$$x_k - \frac{x_j\theta\,\beta_{ki}}{1+\theta\,\beta_{ji}} \geq 0$$
$$\Leftrightarrow$$
$$x_k + \theta(x_k\beta_{ji} - x_j\beta_{ki}) \geq 0 \quad (\text{given } 1+\theta\,\beta_{ji} > 0).$$

This does not limit θ if $x_j\beta_{ki} = x_k\beta_{ji}$. Otherwise, this is either an upper bound or a lower bound, depending upon the sign of the difference, as given in figure 2. (We can allow $k = j$ since that holds identically.)

Following the theory for rim variation[5], define the *range of compatibility* of an optimal basis, $\mathbf{B}$:

$$\rho(\mathbf{B};\delta A) \stackrel{\text{def}}{=} \sup\{\theta\colon\ \mathbf{B} \text{ is an optimal basis throughout } [0,\theta]\}.$$

Then, let the *basic spectrum* be $\rho^*(\delta A) \stackrel{\text{def}}{=} \sup\{\rho(\mathbf{B};\delta A)\colon\ \mathbf{B} \text{ is an optimal basis}\}$. Computation of $\rho(\mathbf{B};\delta A)$ is equivalent to standard parametric programming when $\delta A = p \otimes q$ since the resulting inequalities are linear in θ. Other directions of change lead to rational functions of θ.

Let $\mathcal{B}$ denote the set of *compatible* change directions:

$$\mathcal{B}_k \stackrel{\text{def}}{=} \{\delta A\colon\ \rho(\mathbf{B}^k;\delta A) > 0\}$$
$$\mathcal{B} \stackrel{\text{def}}{=} \bigcup_k \mathcal{B}_k = \{\delta A\colon\ \rho^*(\delta A) > 0\}.$$

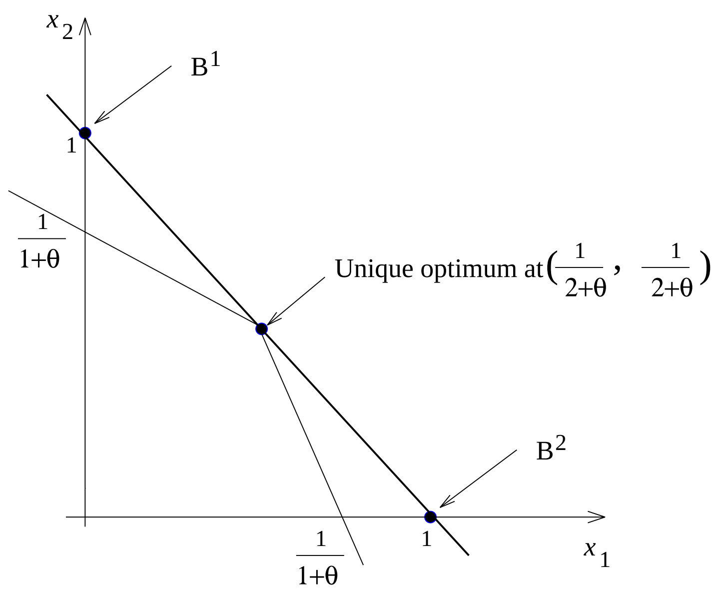
One of the fundamental theorems of compatibility for rim variation is that the sets of change directions span the set of admissible directions. The analogous property here is that $\mathcal{B} = \mathcal{D}$, but this is not true, as the following example shows.

Example 4 $\min x_1 + x_2\colon\ x \geq 0,\ (1+\theta)x_1 + x_2 \geq 1,\ x_1 + (1+\theta)x_2 \geq 1.$

At $\theta = 0$, the optimal extreme points are $(0,1)$ and $(1,0)$ from bases $\mathbf{B}^1 = [A_2\ \ -e_1]$ and $\mathbf{B}^2 = [A_1\ \ -e_2]$, respectively. As shown in figure 3, this direction of change creates an extreme point at the midpoint of the current optimality region $(\frac{1}{2+\theta}, \frac{1}{2+\theta})$ for $\theta > 0$. Further, under perturbation, this new extreme point is the unique optimum, with basis $[A_1 + \theta\,\delta A_1\ A_2 + \theta\,\delta A_2]$, so none of the currently optimal bases remain optimal. We thus have $\rho^*(\delta A) = 0$, but $\delta A \in \mathcal{D}$.

Consider the set of all bases. If one is not feasible in the primal or the dual of the current LP, it remains infeasible in a small enough neighborhood. So, for

Figure 3: Example with $\mathcal{B} \neq \mathcal{D}$



θ sufficiently small (but positive), the optimal bases cannot include those that have an infeasibility. If none of the currently optimal bases is optimal for θ near zero, there are only two possibilities: the LP has no solution (i.e., $\delta A \notin \mathcal{D}$), or a submatrix becomes a basis for positive θ that is currently not a basis (as in Example 4). If the latter does not happen for a particular change direction, that direction is in the span of the currently optimal bases (i.e., in $\mathcal{B}$). The pathology can happen, however, even if the change direction is restricted to be one column. This is shown with the following

Example 5 $\max x_1 : x \geq 0,\ x_1 - \theta\, x_2 = 1,\ \theta\, x_2 + x_3 = 1.$

At $\theta = 0$, the LP has the uniquely optimal basis: $[A_1\ A_3]$ with value 1. For any $\theta > 0$, the uniquely optimal basis is $[A_1\ A_2]$ with value 2 (note the discontinuity in the maximum value as a function of θ).

We leave open how $\mathcal{B}$ might relate to $\mathcal{D}$; the classical theory is what is reported by Gal, which is how to compute $\rho(\mathbf{B}; p \otimes q)$ for any optimal basis, $\mathbf{B}$. (Recall that this leads to linear inequalities, which is why standard methods of parametric programming apply. Also note, however, that other conditions lead to linear inequalities. One such condition is $\delta\mathbf{B} = 0$, and there are others, such as the special structure of when b and c are put into the matrix — we know that leads to a system of linear inequalities on θ.) In the next section we use an analogous approach to determine when the optimal partition does not change.

4 Partition Invariance

In this section we are concerned with matrix coefficient changes for which the optimal partition does not change. To begin, let (x^0, y^0) denote an optimal interior solution, thereby determining the optimal partition by their support sets.

Partition A according to the optimal partition:

$$A = \begin{bmatrix} B & N \\ B^* & N^* \end{bmatrix} \begin{matrix} \leftarrow \sigma(y^0) = \text{rows active in some optimal solution} \\ \leftarrow \sigma(s^0) = \text{rows inactive in all optimal solutions} \end{matrix}$$

$$\begin{matrix} \uparrow & \uparrow \\ \sigma(x^0) & \sigma(d^0) = \text{columns inactive in all optimal solutions} \\ = \text{columns active in some optimal solution.} & \end{matrix}$$

The submatrices are not to be confused with the partition induced by a basic solution. Unless the solution is unique, $B \neq \mathbf{B}$, and similarly for the other submatrices of A whose definition is induced by the optimal partition.

We refer to $\sigma(y^0) \times \sigma(x^0)$ as the *active set* of rows and columns, which define the *active submatrix*, B. The rows are active because they never have surplus in any optimal solution (i.e., $s_i = 0$ in every optimal solution for all $i \in \sigma(y^0)$), and for each row we have an optimal solution where its price is positive. The columns are active because they never have a positive reduced cost (i.e., $d_j = 0$ in every optimal solution for all $j \in \sigma(x^0)$), and for each column we have an optimal solution where its level is positive. The complementary rows and columns are called *inactive*. Those rows never have a positive price, and each inactive row has a positive surplus in at least one optimal solution. The inactive columns never have a positive level, and each inactive column has a positive reduced cost in at least one optimal solution.

Partition the rim data vectors conformally: $b = \begin{pmatrix} b_N \\ b_B \end{pmatrix}$ and $c = (c_B \ c_N)$. Also, $x = \begin{pmatrix} x_B \\ x_N \end{pmatrix}$, $s = \begin{pmatrix} s_N \\ s_B \end{pmatrix}$, $y = (y_N \ y_B)$, and $d = (d_B \ d_N)$. So, the original linear programs are equivalent to the following primal-dual pair:

Primal	Dual
$\min c_B x_B + c_N x_N$:	$\max y_N b_N + y_B b_B$:
$B x_B + N x_N - s_N = b_N$	$y_N B + y_B B^* + d_B = c_B$
$B^* x_B + N^* x_N - s_B = b_B$	$y_N N + y_B N^* + d_N = c_N$
$x,\ s \geq 0$	$y,\ d \geq 0.$

Maintaining the partition conditions, $x_N = 0, s_N = 0, y_B = 0$ and $d_B = 0$, the partition does not change for $\theta > 0$ if there exists $u = x_B + \Delta x_B > 0$ and $v = y_N + \Delta y_N > 0$ to satisfy the following:

$$(B + \theta\,\delta B)u = b_N \qquad v(B + \theta\,\delta B) = c_B \tag{2}$$

$$(B^* + \theta\,\delta B^*)u > b_B \qquad v(N + \theta\,\delta N) < c_N. \tag{3}$$

We can see immediately, if $\delta B = 0$, the partition remains optimal for θ satisfying

$$\theta\,\delta B^* x_B^0 > -s_B^0 \qquad \text{and} \qquad \theta\, y_N^0 \delta N < d_N^0. \tag{4}$$

This is only sufficient, using $\Delta x = 0$ and $\Delta y = 0$; it is possible there exists $(u, v) \neq (x_B^0, y_N^0)$ such that the invariance conditions are satisfied with a greater value of θ.

Following [7], let $\tau(\delta A)$ denote the range for which the partition remains optimal for $\delta A \in \mathcal{D}$. When changing one coefficient, the entries in figure 1 give values of τ, thus making the range of invariance for changes outside the active submatrix easy to determine. If the coefficient is ${N^*}_{ij}$, there is no restriction so $\tau(\delta A) = \infty$. If the coefficient is ${B^*}_{ij}$, only the primal inequality applies: $\theta\, x_j^0 > -s_i^0$. If the coefficient is N_{ij}, only the dual inequality applies: $\theta\, y_i^0 < d_j^0$. These agree with what is shown in figure 1, except the solution values are interior, not necessarily basic.

Identifying the active submatrix is the key, and when it remains unchanged, so does the optimal partition. Although this follows immediately, this fact is important enough to state as the following.

Theorem 3 *If the active submatrix remains unchanged ($\delta B = 0$), the optimal partition is invariant on $[0, \theta^*)$ for some $\theta^* > 0$. Further, the range of each coefficient outside the active submatrix, $\tau(\mathrm{e}_i^T \otimes \mathrm{e}_j)$, is at least as great as the values given in figure 1. Within that range, $(x^0,\ y^0)$ remains an interior optimum (with $s(\theta) = s^0 - \theta\,\mathrm{e}_i\, x_j^0$ if B_{ij}^* changes, and $d(\theta) = d^0 - \theta\,\mathrm{e}_j\, y_i^0$ if N_{ij} changes).*

The range of θ obtained from figure 1 is for the particular interior solution obtained. Since this applies to every interior solution, the value of τ is the greatest among all interior solutions:

$$\tau = \sup\{\frac{s_i}{x_j}\} \quad \text{if only } B^*_{ij} \text{ changes (and } \delta B^*_{ij} = 1);$$

$$\tau = \sup\{\frac{d_j}{y_i}\} \quad \text{if only } N_{ij} \text{ changes (and } \delta N_{ij} = 1).$$

The economic inferences are stronger than those of a basic optimum. For the coefficient in N^*, we can say there is no effect in *every* optimal solution, not just for one basis. In fact, the coefficient need not be in $\mathbf{N}^*$ associated with a basis, even if it is in N^* associated with the optimal partition. In the other two cases, to say that the optimal partition does not change gives a qualitative result: *inactive activities continue to be inactive for the range specified in every optimal solution to the perturbed LP.*

Now consider the perturbation of one coefficient in the active submatrix, say B_{ij}. The invariance conditions reduce to the following:

$$\begin{array}{rclcrcl} (B + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j)\, u & = & b_N & \quad & v\,(B + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j) & = & c_B \\ B^* u & > & b_B & & v\, N & < & c_N \\ u & > & 0 & & v & > & 0. \end{array}$$

Suppose B is $m' \times n'$ and let M^+ denote an arbitrary generalized inverse of any matrix M. We know we have a solution for $\theta = 0$, so $BB^+ b_N = b_N$ and $c_B B^+ B = c_B$ for every generalized inverse of B. For the partition to remain optimal, this needs to be true for $\theta > 0$ (but sufficiently small). Even further, we want to consider a solution of the form $u(\theta) = x^0_B + f(\theta)$ and $v(\theta) = y^0_N + g(\theta)$, where f and g are continuous in a neighborhood of $\theta = 0$, and $f(0) = g(0) = 0$. Then, not only does this satisfy the equations, but also it satisfies the strict inequalities for θ sufficiently small.

Substituting $x^0_B + f(\theta)$ into the primal equation and $y^0_N + g(\theta)$ into the dual, these functions exist if, and only if,

$$(B + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j)(B + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j)^+\, \mathrm{e}_i x^0_j = \mathrm{e}_i x^0_j$$

and

$$y^0_i \mathrm{e}_j (B + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j)^+ (B + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j) = y^0_i \mathrm{e}_j$$

for every generalized inverse, $(B + \theta\, \mathrm{e}_i^T \otimes \mathrm{e}_j)^+$, in which case we have

$$f(\theta) = -\theta\, x_j (B + \theta\, \mathrm{e}_i \otimes \mathrm{e}_j)^+ \mathrm{e}_i + [\mathrm{I} - (B + \theta\, \mathrm{e}_i \otimes \mathrm{e}_j)^+ (B + \theta\, \mathrm{e}_i \otimes \mathrm{e}_j)]\, \mu(\theta)$$

for $\mu(\theta) \in \mathbb{R}^{n'} \ni \mu$ is continuous in a neighborhood of $\theta = 0$ and $\mu(0) = 0$; and

$$g(\theta) = -\theta\, y_i^0 \mathrm{e}_j (B + \theta\, \mathrm{e}_i \otimes \mathrm{e}_j)^+ + \nu(\theta)\, [\mathrm{I} - (B + \theta\, \mathrm{e}_i \otimes \mathrm{e}_j)(B + \theta\, \mathrm{e}_i \otimes \mathrm{e}_j)^+]$$

for $\nu(\theta) \in \mathbb{R}^{m'} \ni \nu$ is continuous in a neighborhood of $\theta = 0$ and $\nu(0) = 0$.

We begin with the special case where the active submatrix can be permuted to the following form:

$$B \;=\; \begin{bmatrix} \overline{B} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

where $\overline{B}$ is nonsingular. Then,

$$B^+ = \begin{bmatrix} \overline{B}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Conformally partition $b_N = \begin{pmatrix} \overline{b}_N \\ b_0 \end{pmatrix}$ and $c_B = \begin{pmatrix} \overline{c}_B & c_0 \end{pmatrix}$. Since x_B is a solution to $Bu = b_N$, and y_N is a solution to $vB = c_B$, we must have $BB^+ b_N = b_N$ and $c_B B B^+ = c_B$. Substituting the above, we have

$$\begin{bmatrix} \overline{B} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{B}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \overline{b}_N \\ b_0 \end{pmatrix} = \begin{pmatrix} \overline{b}_N \\ b_0 \end{pmatrix}$$

and

$$(\overline{c}_B\ c_0) \begin{bmatrix} \overline{B} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{B}^{-1} & 0 \\ 0 & 0 \end{bmatrix} = (\overline{c}_B\ c_0).$$

These imply $b_0 = 0$ and $c_0 = 0$, so the active portion of the rim vector is $b_N = \begin{pmatrix} \overline{b}_N \\ 0 \end{pmatrix}$ and $c_B = \begin{pmatrix} \overline{c}_B & 0 \end{pmatrix}$.

Further, assume the changes (δB) are in the nonsingular matrix, $\overline{B}$, and the zeroes remain unchanged. Then, for θ sufficiently small,

$$(B + \theta\, p \otimes q)^+ \;=\; \begin{bmatrix} (\overline{B} + \theta\, \overline{p} \otimes \overline{q})^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

We now use Bodewig's formula to obtain the following:

$$f(\theta) = \begin{pmatrix} -\dfrac{\theta\, \overline{B}^{-1} \overline{p} \otimes \overline{q}\, \overline{x}_B^0}{1 + \theta\, \overline{q}\overline{B}^{-1} \overline{p}} \\ U(\theta) \end{pmatrix}$$

and

$$g(\theta) = \left(-\frac{\theta\, \overline{y}_N^0 \overline{p} \otimes \overline{q}\overline{B}^{-1}}{1 + \theta\, \overline{q}\overline{B}^{-1}\overline{p}} \quad V(\theta) \right),$$

where we have partitioned $x_B^0 = \begin{pmatrix} \overline{x}_B^0 \\ U(0) \end{pmatrix}$ and $y_N^0 = \begin{pmatrix} \overline{y}_N^0 & V(0) \end{pmatrix}$. The vectors U and V correspond to the row and column zeroes in B, respectively. Since the current solution is strictly complementary, $U(0) > 0$ and $V(0) > 0$, and we can continue to choose $U(\theta) > 0$ and $V(\theta) > 0$, so it is only the other portion that limits θ.

Theorem 4 *If B satisfies (5) and δB satisfies (6), there exists $\theta^* > 0$ such that the optimal partition is invariant in $[0, \theta^*)$. Moreover, θ^* is determined by the following inequalities:*

$$\begin{array}{rlcl} \text{Nonsingularity:} & 1 + \theta\,\eta > 0, \text{ where } \eta \stackrel{\text{def}}{=} \overline{q}\overline{B}^{-1}\overline{p} & & \\ x_B > 0: & \theta\, \overline{B}^{-1}\overline{p}\,(\overline{q}\,\overline{x}_B^0) & < & (1+\theta\,\eta)\,\overline{x}_B^0 \\ s_B > 0: & \theta\,(\overline{B}^* + \theta\,\delta\overline{B}^*)\overline{B}^{-1}\overline{p}\,(\overline{q}\,\overline{x}_B^0) & > & -(1+\theta\,\eta)\,s_B^0 \\ y_N > 0: & \theta\,(\overline{y}_N^0\overline{p})\,\overline{q}\overline{B}^{-1} & < & (1+\theta\,\eta)\,\overline{y}_N^0 \\ d_N > 0: & \theta\,(\overline{y}_N^0\overline{p})\,\overline{q}\overline{B}^{-1}(\overline{N} + \theta\,\delta\overline{N}) & > & -(1+\theta\,\eta) d_N^0 \end{array}$$

In particular, if $\delta\overline{B}_{ji} = 1$ and all other coefficients remain unchanged (so $\overline{q} = \mathrm{e}_j$ and $\overline{p} = \mathrm{e}_i^T$), the optimal partition is invariant for the following:

$$\begin{array}{rcccl} & & \theta & < & -\dfrac{1}{\beta_{ji}^-} \\ \dfrac{x_k^0}{(\beta_{ik}x_j^0 - \beta_{ji}x_k^0)^-} & < & \theta & < & \dfrac{x_k^0}{(\beta_{ik}x_j^0 - \beta_{ji}x_k^0)^+} \\ \dfrac{y_k^0}{(\beta_{kj}y_i^0 - \beta_{ji}y_k^0)^-} & < & \theta & < & \dfrac{y_k^0}{(\beta_{kj}y_i^0 - \beta_{ji}y_k^0)^+} \end{array}$$

where $\beta \stackrel{\text{def}}{=} \overline{B}^{-1}$.

These are linear inequalities if $\delta\overline{B}^* = 0$ and $\delta\overline{N} = 0$. Otherwise, the inequalities for $s_B > 0$ and $d_N > 0$ are quadratic, in which case θ^* is limited by the least positive root in order to have an interval of invariance.

Now consider the case where $\delta A = p \otimes q$, so we have the following block partition:

$$\begin{pmatrix} \delta B & \delta N \\ \delta B^* & \delta N^* \end{pmatrix} = \begin{pmatrix} p_N \otimes q_B & p_N \otimes q_N \\ p_B \otimes q_B & p_B \otimes q_N \end{pmatrix}.$$

Having considered the simple cases with either $p_N = 0$ or $q_B = 0$, we now consider these both nonzero.

The equations in (2) define the following primal-dual functions:

$$u(\theta) = (B + \theta\,\delta B)^+ b_N + [\mathrm{I} - (B + \theta\,\delta B)^+(B + \theta\,\delta B)]\,\mu(\theta) \tag{7}$$

$$v(\theta) = c_B(B + \theta\,\delta B)^+ + \nu(\theta)[\mathrm{I} - (B + \theta\,\delta B)(B + \theta\,\delta B)^+] \tag{8}$$

for $\mu(\theta) \in \mathbb{R}^{n'}$ and $\nu(\theta) \in \mathbb{R}^{m'}$ (given B is $m' \times n'$). In order that these equations have a solution, it is necessary that $(B + \theta\,\delta B)(B + \theta\,\delta B)^+ b_N = b_N$ and $c_B(B + \theta\,\delta B)^+(B + \theta\,\delta B) = c_B$ (we know these hold for $\theta = 0$).

In what follows let $p = p_N$ and $q = q_B$ for notational convenience. That is, let the change vectors define δB since the other changes appear only in strict inequalities that restrict θ to preserve the optimal partition.

Lemma 1 *Suppose B^+ is a generalized inverse of B such that*

$$BB^+p \otimes q + p \otimes qB^+B = p \otimes q + BB^+p \otimes qB^+B. \tag{9}$$

Then, Bodewig's formula generalizes as follows:

$$(B + \theta\, p \otimes q)^+ = B^+ - \frac{\theta\, B^+p \otimes qB^+}{1 + \theta\, qB^+p}. \tag{10}$$

Proof: Going through the (tedious) algebra, we obtain

$$(B + \theta\, p \otimes q)(B + \theta\, p \otimes q)^+(B + \theta\, p \otimes q) = B + \theta\, p \otimes q \quad + \\ \frac{\theta}{1 + \theta\, qB^+p}\left(BB^+p \otimes q + p \otimes qB^+B - p \otimes q - BB^+p \otimes qB^+B\right),$$

from which the result follows. □

Two special cases are as follows:

- $BB^+ = I$, such as when $\mathrm{rank}(B) = m'$ and $B^+ = B^T(BB^T)^{-1}$.
- $B^+B = I$, such as when $\mathrm{rank}(B) = n'$ and $B^+ = (B^TB)^{-1}B^T$.

We use these in the following.

Theorem 5 *Suppose* (9) *holds and* $1+\theta\,\eta>0$, *where* $\eta \stackrel{\text{def}}{=} qB^+p$. *Then, there exists* $\theta^*>0$ *such that the optimal partition remains invariant for* $A+\theta\,p\otimes q$, *for all* $\theta\in[0,\theta^*)$, *if, and only if,*

$$(\mathrm{I}-BB^+)p\otimes qB^+b_N=0 \text{ and } c_BB^+p\otimes q(\mathrm{I}-B^+B)=0,$$

in which case an interior solution is given by:

$$\begin{aligned} x_B(\theta) &= \quad (\mathrm{I}-\tfrac{\theta}{1+\theta\eta}B^+p\otimes q)\;x_B^0 \\ y_N(\theta) &= y_N^0\;(\mathrm{I}-\tfrac{\theta}{1+\theta\eta}B^+p\otimes q). \end{aligned}$$

Proof: In general, the equation $(B+\theta\,p\otimes q)u=b_N$ has a solution if, and only if, $(B+\theta\,p\otimes q)(B+\theta\,p\otimes q)^+b_N=b_N$ for any generalized inverse. Applying (9), this becomes $(B+\theta\,p\otimes q)(B^+-\frac{\theta}{1+\theta\,\eta}B^+p\otimes qB^+)b_N=b_N$. Since $BB^+b_N=b_N$ (because x_B^0 is a solution for $\theta=0$), upon multiplying by $(1+\theta\,\eta)/\theta$, this equation is equivalent to $(-BB^+p\otimes qB^++p\otimes qB^+(1+\theta\,\eta)-\theta\,p\otimes qB^+p\otimes qB^+)b_N=0$. Recognizing that the last term reduces to $\theta\eta p\otimes qB^+$, we obtain the stated condition. Further, a solution is of the form $u(\theta)=(B+\theta\,p\otimes q)^+b_N+(\mathrm{I}-(B+\theta\,p\otimes q)^+(B+\theta\,p\otimes q))\mu(\theta)$, and we take the particular solution with $\mu(\theta)=u^0$ (independent of θ) to obtain the stated value of $x_B(\theta)$. Since this is continuous in θ with $x_B(0)=x_B^0>0$, we can restrict θ small enough that the primal strict inequalities hold. A similar argument applies to obtain the dual equation and the interior solution, $y(\theta)$. □

This leads immediately to the following two corollaries because we can restrict θ to be small enough that the rank of the active submatrix does not change.

Corollary 5.1 *Suppose* $\mathrm{rank}(B)=m'$. *Then,* $\exists\theta^*>0$ *such that the optimal partition remains invariant throughout* $[0,\theta^*)$ *if, and only if,*

$$c_BB^T[BB^T]^{-1}p\otimes q(\mathrm{I}-B^T[BB^T]^{-1}B)=0.$$

Corollary 5.2 *Suppose* $\mathrm{rank}(B)=n'$. *Then,* $\exists\theta^*>0$ *such that the optimal partition remains invariant throughout* $[0,\theta^*)$ *if, and only if,*

$$(\mathrm{I}-B[B^TB]^{-1}B^T)p\otimes q[B^TB]^{-1}B^Tb_N=0.$$

We can combine our results to obtain the following.

Theorem 6 *Suppose*

$$A + \delta A \;=\; \begin{bmatrix} \begin{bmatrix} \overline{B} & 0 \\ 0 & 0 \end{bmatrix} & N \\ B^* & N^* \end{bmatrix} + \begin{bmatrix} \begin{bmatrix} \delta\overline{B} & 0 \\ 0 & 0 \end{bmatrix} & \delta N \\ \delta B^* & \delta N^* \end{bmatrix}.$$

Then, $\tau(\delta A) > 0$ if at least one of the following conditions hold:

1. $\delta\overline{B} = 0$;
2. $\operatorname{rank}(\overline{B}) = m'$ *and* $c_B B^T[BB^T]^{-1}p \otimes q(\mathrm{I} - B^T[BB^T]^{-1}B) = 0$;
3. $\operatorname{rank}(\overline{B}) = n'$ *and* $(\mathrm{I} - B[B^TB]^{-1}B^T)p \otimes q[B^TB]^{-1}B^Tb_N = 0$.

Proof: If $\delta\overline{B} = 0$, the result is contained in theorem 3. Otherwise, apply the appropriate corollary of theorem 5. □

Corollary 6.1 *If the primal-dual solution is unique, the optimal partition is invariant for any matrix change with sufficiently small norm.*

Proof: The uniqueness implies the solution is basic, so both rank conditions in theorem 6 are true. Thus, if ΔA is any matrix change, $\tau(\Delta A) > 0$, so scale ΔA by $s < \tau(\Delta A)$ to make the norm "sufficiently small". □

5 An Example

We consider an example in some depth to illustrate the theory. We are given three fuels from which to generate electricity: coal, oil and uranium. Define six activities, as follows:

PCL... purchase coal	GCL... generate electricity from coal
POL... purchase oil	GOL... generate electricity from oil
PUR... purchase uranium	GUR... generate electricity from uranium

Figure 4 shows the LP. The objective is to minimize cost, shown as the first row, while meeting the required electricity demand, shown as the last row. Rows BCL, BOL and BUR balance the associated fuels: what is purchased must be at least as great as what is used for generation. Row LNU limits nuclear generation (from uranium): $-x_{\text{GUR}} \geq -10$ ($\Leftrightarrow x_{\text{GUR}} \leq 10$). Finally, row DEL is the demand for electricity.

Figure 4: Electricity Generation Example

	— Purchase —			— Generate —					
	PCL	POL	PUR	GCL	GOL	GUR			
COST	18	15	20	.8	.6	.4	=	min	
BCL	1			−1			≥	0	balance coal
BOL		1			−1		≥	0	balance oil
BUR			1			−1	≥	0	balance uranium
LNU						−1	≥	−10	limit nuclear
DEL				.33	.3	.4	≥	10	demand electricity

The net cost in dollars per unit of electricity is the sum of purchase and generation costs divided by the yield: $(c_{Pf} + c_{Gf})/A_{DEL,Gf}$ for fuel f. These are as follows:

Coal	Oil	Uranium
$18.8/.33 = 56.97$	$15.6/.3 = 52$	$20.4/.4 = 51$

Generation from uranium is the least costly at \$51 per unit of electricity generated, so its level is as much as possible, which is 10 units. This generates 4 units of electricity, leaving 6 units of electricity demand to be generated by the next cheapest fuel, which is oil. Since the oil yield is .3, the levels of POL and GOL are 20, and this sets the marginal price of electricity at \$52. No electricity is generated from coal: the levels of PCL and GCL are zero in every optimal solution.

There are two optimal bases, whose tableaux are shown in figure 5. The two bases differ in which coal activities are present, to accommodate different events (see [5], as this is a part of *Compatibility Theory*). The first tableau has the purchase activity in the basis, which is compatible with increasing the right-hand side to require a coal stockpile. In that case the level of PCL simply increases to equal the requirement. The second tableau has the generation activity in the basis, which is compatible with decreasing the right-hand side, making free coal available. In that case the level of GCL increases to equal the free coal, displacing oil.

Figure 5: Optimal Tableaux for Electricity Generation Example

		GCL	BCL	BOL	BUR	DEL	LNU
PCL	0	−1	−1				
POL	20	1.1		−1		−3.333	−1.333
PUR	10				−1		1
GOL	20	1.1				−3.333	−1.333
GUR	10				1		1
−COST	516	1.64	18	15	20	52	.4

(a) Tableau 1 has Coal Purchase (PCL) in Basis $\mathbf{B}^1$, Compatible with Increasing b_{BCL} (Requiring Coal Stockpile)

		PCL	BCL	BOL	BUR	DEL	LNU
GCL	0	−1	1				
POL	20	1.1	−1.1	−1		−3.333	−1.333
PUR	10				−1		1
GOL	20	1.1	−1.1			−3.333	−1.333
GUR	10				1		1
−COST	516	1.64	16.36	15	20	52	.4

(b) Tableau 2 has Generation from Coal (GCL) in Basis $\mathbf{B}^2$, Compatible with Decreasing b_{BCL} (Making Free Coal Available)

Figure 6 shows the active submatrix, where the optimal partition has $\sigma(x) = \{\text{POL, PUR, GOL, GUR}\}$ (activities to generate electricity from oil and uranium).

Figure 6: Optimal Partition for the Electricity Generation Example

$$\sigma(x): \begin{matrix} \text{POL} & \text{PUR} & \text{GOL} & \text{GUR} \end{matrix} \qquad B = \begin{bmatrix} & & & \\ 1 & & -1 & \\ & 1 & & -1 \\ & & & -1 \\ & & .3 & .4 \end{bmatrix} \begin{matrix} \text{BCL} \\ \text{BOL} \\ \text{BUR} \\ \text{LNU} \\ \text{DEL} \end{matrix} \qquad \sigma(d): \begin{matrix} \text{PCL} & \text{GCL} \end{matrix} \qquad N = \begin{bmatrix} 1 & -1 \\ & \\ & \\ & \\ & .33 \end{bmatrix}$$

Suppose we want to vary the yield factors in the demand row. Let $\kappa = \Delta A_{\text{DEL},\text{GCL}}$, $\lambda = \Delta A_{\text{DEL},\text{GOL}}$ and $\mu = \Delta A_{\text{DEL},\text{GUR}}$ denote the changes in coal, oil and uranium yields, respectively, so

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa & \lambda & \mu \end{bmatrix}.$$

Figure 7 gives the primal-dual conditions for basis $\mathbf{B}^1$ to remain optimal. First, there is the *nonsingularity condition*: $\mathbf{B}$ is nonsingular $\Leftrightarrow .3 + \lambda \neq 0$. We strengthen this to $.3 + \lambda > 0$. Then, the primal conditions are simply $x_B = [\mathbf{B} + \Delta\mathbf{B}]^{-1} b_N \geq 0$ (s_B is vacuous). The dual conditions are $y_N = c_B[\mathbf{B} + \Delta\mathbf{B}]^{-1} \geq 0$ and $d_N = c_N - y_N(\mathbf{N} + \Delta\mathbf{N}) \geq 0$. The net result is a system of linear inequalities on the coefficient change values, shown in figure 8(a). By similar derivation, the primal-dual conditions yield the system in figure 8(b) for basis $\mathbf{B}^2$ to remain optimal.

Figure 7: Maintaining Optimality of $\mathbf{B}^1$

$$\begin{array}{lllllllll} x_{\mathrm{PCL}} & & & & & & & = & 0 \\ & x_{\mathrm{POL}} & & - & x_{\mathrm{GOL}} & & & = & 0 \\ & & x_{\mathrm{PUR}} & & & - & x_{\mathrm{GUR}} & = & 0 \\ & & & & & - & x_{\mathrm{GUR}} & = & -10 \\ & & & & (.3+\lambda)x_{\mathrm{GOL}} & + & (.4+\mu)x_{\mathrm{GUR}} & = & 10 \end{array}$$

$x \geq 0 \Longleftrightarrow \lambda > -.3$ and $\mu \leq .6$.

(a) Primal Conditions

$$\begin{array}{llllllll} y_{\mathrm{BCL}} & & & & & & = & 18 \\ & y_{\mathrm{BOL}} & & & & & = & 15 \\ & & y_{\mathrm{BUR}} & & & & = & 20 \\ & -y_{\mathrm{BOL}} & & & + & (.3+\lambda)y_{\mathrm{DEL}} & = & .6 \\ & & -y_{\mathrm{BUR}} & - \; y_{\mathrm{LNU}} & + & (.4+\mu)y_{\mathrm{DEL}} & = & .4 \end{array}$$

$y \geq 0 \Longleftrightarrow \lambda > -.3$ and $20.4\lambda - 15.6\mu \leq .12$.

Reduced cost for GCL :

$$-y_{\mathrm{BCL}} + (.33+\kappa)y_{\mathrm{DEL}} \leq .8$$

$\Longleftrightarrow 15.6\kappa - 18.8\lambda \leq .492$.

(b) Dual Conditions

Letting $P^k \stackrel{\text{def}}{=} \{(\kappa, \lambda, \mu) : \mathbf{B}^k \text{is optimal}\}$, we have $P^2 \subset P^1$ in this example. The reason is that the second basis contains the conditions for which it is feasible and economical to generate the electricity by using the most uranium possible (row LNU remains tight, with $x_{\text{GUR}} = 10$), then using oil for the remaining units. Changing the yield factors changes these proportions, but the primal conditions ensure that a total of 10 units of electricity is generated from these two fuels without using coal. The dual conditions ensure it remains economical to do so. The added dual condition to keep $\mathbf{B}^2$ optimal arises because the coal-fired generation activity is in the basis, and if its yield becomes low enough, it is not economical to activate it, even with free coal. The way this shows up mathematically is that the coal price (y_{BCL}) is driven to zero as κ is decreased, which is the same as saying that the reduced cost of the purchase activity, d_{PCL}, is driven to zero and would become negative with further decreases in the coal yield factor, rendering this basis as non-optimal.

Figure 8: Yield Changes that Preserve Optimality of Bases

$$\begin{array}{rrrcrl}
 & \lambda & & > & -.3 & \leftarrow \text{nonsingularity} \\
 & & \mu & \le & .6 & \leftarrow x_{\text{GOL}} \ge 0 \\
 & 20\lambda & -\ 15.6\mu & \le & .24 & \leftarrow y_{\text{LNU}} \ge 0 \\
15.6\kappa & -\ 18.8\lambda & & \le & .492 & \leftarrow d_{\text{GCL}} \ge 0
\end{array}$$

(a) $\mathbf{B}^1$ Remains Optimal

$$\begin{array}{rrrcrl}
 & \lambda & & > & -.3 & \leftarrow \text{nonsingularity} \\
 & & \mu & \le & .6 & \leftarrow x_{\text{GOL}} \ge 0 \\
 & 20\lambda & -\ 15.6\mu & \le & .24 & \leftarrow y_{\text{LNU}} \ge 0 \\
-15.6\kappa & +\ 18.8\lambda & & \le & 4.898 & \leftarrow y_{\text{BCL}} \ge 0 \\
15.6\kappa & -\ 18.8\lambda & & \le & .492 & \leftarrow d_{\text{PCL}} \ge 0
\end{array}$$

(b) $\mathbf{B}^2$ Remains Optimal

The conditions in figure 7 are also sufficient for the optimal partition to remain unchanged (except strict inequalities replace weak ones). In fact, the primal solution values are unique: $x_{\text{PCL}} = x_{\text{GCL}} = 0$, $x_{\text{PUR}} = x_{\text{GUR}} = 10$, $x_{\text{POL}} = x_{\text{GOL}} = \frac{6-10\mu}{.3+\lambda}$. The dual solution values are partly unique: $y_{\text{BOL}} = 15$, $y_{\text{BUR}} = 20$, $y_{\text{DEL}} = \frac{15.6}{.3+\lambda}$, $y_{\text{LNU}} = \frac{.4+\mu}{.3+\lambda}15.6 - 20.4$. The value of y_{BCL} can be any value in the interval (determined by the two optimal bases): $[\frac{.33+\kappa}{.3+\lambda}15.6, 18]$. An interior solution is any value in the interior of this interval, which yields the same optimal partition.

Choose a particular interior solution, say the midpoint: $y_{\mathrm{BCL}} = 9 + \frac{.33+\kappa}{.3+\lambda} 7.8$. This changes as a nonlinear function of the oil yield parameter, λ, and it is independent of the uranium yield parameter, μ. Another way to see this is that each interior solution, notably the analytic center, is a particular convex combination of the basic optima, and the basic optimal solutions change with those rational functions of the yield parameters.

Figure 9 gives the range of each yield parameter with the other two held fixed at their current values. Note that the range for the partition to remain optimal equals the first basic range. This occurs because the events for the change are the same:

- Coal yield can decrease indefinitely because it is not used and the decrease simply makes it even more uneconomical. It can increase only until it reaches .3615 (approximately), at which point it becomes equally economical as oil. For the basis to remain optimal, we have the weak inequality, $\kappa \leq .0315$, which allows another basis to become optimal (with no oil used). For the partition to remain optimal, we have the strong inequality, $\kappa < .0315$, which disallows the coal to be an alternative optimal solution.

- Oil yield can decrease until it becomes equally economical to use coal. At that threshold ($\lambda = -.0261$), both bases remain optimal but new bases become optimal that use coal instead of oil. The partition remains invariant as long as this threshold is not actually reached. Similarly, the upper limit is where oil becomes equally economical as uranium, and the same reasoning applies to give the same ranges (except the partition range is always a strict inequality).

- Uranium yield can decrease to become equally economical with oil, and the same reasoning applies to preserve the optimality of both bases and of the partition. The upper limit is where uranium can generate all electricity demanded (10), so no oil is used. Again, the same events apply to all three thresholds.

Now suppose we fix the relative rates of change and consider the one-parameter change, $\theta \, \mathrm{e}_5^T \otimes (0\ 0\ 0\ \kappa\ \lambda\ \mu)$. We use equations (1) to determine the spectrum of the two bases. From the tableau in figure 5(a), the inverse of

Figure 9: Individual Yield Ranges that Preserve Optimality

Fuel	Current value	Range for $\mathbf{B}^1$ to remain optimal min	max	Range for $\mathbf{B}^2$ to remain optimal min	max	Range for partition to remain optimal min	max
Coal	.33	$-\infty$	.3615	.0260	.3615	$-\infty$	.3615
Oil	.3	.2839	.3120	.2839	.3120	.2839	.3120
Uranium	.4	.3846	1.0000	.3846	1.0000	.3846	1.0000

$\mathbf{B}^1$ is given by:

$$\beta = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \overset{\begin{array}{ccccc} \text{BCL} & \text{BOL} & \text{BUR} & \text{LNU} & \text{DEL} \end{array}}{\begin{bmatrix} 1 & & & & \\ & 1 & & 1.333 & 3.333 \\ & & 1 & -1 & \\ & & & 1.333 & 3.333 \\ & & & -1 & \end{bmatrix}} \begin{array}{l} \text{PCL} \\ \text{POL} \\ \text{PUR} \\ \text{GOL} \\ \text{GUR} \end{array}$$

Further, $p_N = p = e_5^T$, $q_B = (0, 0, 0, \lambda, \mu)$, and $q_N = (\kappa)$. From this we obtain $\eta = 3.333\lambda$. The nonsingularity bound is thus $\theta\,(3.333\lambda) > -1$, which restricts θ only if the oil yield coefficient decreases (i.e., $\lambda < 0$). The primal solution values are

$$\begin{aligned} x_B(\theta) &= \left[I - \tfrac{\theta}{1+\theta(3.333\lambda)} \begin{pmatrix} 0 \\ 3.333 \\ 0 \\ 3.333 \\ 0 \end{pmatrix} \otimes (0\ 0\ 0\ \lambda\ \mu) \right] \begin{pmatrix} 0 \\ 20 \\ 10 \\ 20 \\ 10 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 20 \\ 10 \\ 20 \\ 10 \end{pmatrix} - \tfrac{\theta}{1+\theta(3.333\lambda)} \begin{pmatrix} 0 \\ 3.333 \\ 0 \\ 3.333 \\ 0 \end{pmatrix} (20\lambda + 10\mu) \end{aligned}$$

The only effect on primal feasibility is the second equation, which is the same as the fourth:

$$20 - \frac{\theta}{1 + \theta(3.333\lambda)} 3.333(20\lambda + 10\mu) \geq 0.$$

This imposes the bound:

$$\theta(33.33\mu) \leq 20$$

Note this depends only on μ, not on λ, because changes in the oil yield can be compensated by the level of generation, whereas we must maintain $x_{\text{GUR}} = 10$ to have no surplus capacity (i.e., to maintain s_{LNU} as nonbasic). The only way λ could affect the primal feasibility of this basis is for it to increase so much that too much electricity is generated, but that is circumvented by the nonsingularity condition. Also, κ does not appear in the primal conditions because $\mathbf{B}^*$ and s_B are vacuous.

Now consider the dual conditions:

$$y_N(\theta) = (18, 15, 20, .4, 52) - \frac{\theta}{1 + \theta\, 3.333\lambda} 52(0,\ 0,\ 0,\ 1.333\lambda - \mu,\ 3.333\lambda).$$

The first three prices (for purchasing coal, oil and uranium) are not changed. The last inequality imposes no restriction on θ because the price of electricity decreases only if the yield from oil increases, and the price cannot reach zero due to the nonsingularity condition. The only inequality that imposes a restriction is from the price of nuclear capacity:

$$y_{\text{LNU}}(\theta) \geq 0 \Leftrightarrow \theta\,(67.98\lambda - 52\mu) \leq .4.$$

The value of κ does not affect the dual prices of the rows, but it does appear in the reduced costs:

$$\begin{aligned} d_N(\theta) \geq 0 \quad &\Leftrightarrow \quad 1.64 + \theta\left(3.333\lambda(1.64) + 52(\lambda - \kappa)\right) \geq 0 \\ &\Leftrightarrow \quad \theta\,(52\kappa - 57.47\lambda) \leq 1.64. \end{aligned}$$

In summary, $\mathbf{B}^1$ remains an optimal basis for $\theta \in [0, \theta^*]$, where

$$\theta^* = \max\left\{\frac{20}{(33.33\mu)^+}, \frac{.4}{(67.98\lambda - 52\mu)^+}, \frac{1.64}{(52\kappa - 57.47\lambda)^+}\right\}.$$

A lower bound can be formed by considering θ coefficients negative. Moreover, a similar derivation applies to determine the range of $\mathbf{B}^2$, where only the dual price of coal (y_{BCL}) and the reduced costs of activities PCL and GCL differ.

Now consider the optimal partition. Since row BCL is zero and the remaining 4×4 submatrix is nonsingular, we could apply theorem 4 with

$$\overline{B}^{-1} = \begin{bmatrix} 1 & 0 & 1.333 & 3.333 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1.333 & 3.333 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

Then, $\overline{p} = \mathrm{e}_4$ and $\overline{q} = (0\ 0\ \lambda\ \mu)$, so $\eta = 3.333\lambda$. The inequalities are as follows (where s_B and $\overline{B}^*$ are vacuous):

Nonsingularity: $\theta\,(3.333\lambda) > -1$

$$x_B > 0: \quad \theta \begin{pmatrix} 3.333 \\ 0 \\ 3.333 \\ 0 \end{pmatrix} (20\lambda + 10\mu) < (1 + \theta\, 3.333\lambda) \begin{pmatrix} 20 \\ 10 \\ 20 \\ 10 \end{pmatrix}$$

$$y_N > 0: \quad \theta\, 52(0,\ 0,\ 1.333\lambda - \mu,\ 3.333\lambda) < (1 + \theta\, 3.333\lambda)(15,\ 20,\ .4,\ 52)$$

$$d_N > 0: \quad \theta\, 52(0,\ 3.333\lambda(.33 + \theta\,\kappa)) > -(1 + \theta\, 3.333\lambda)(d^0_{\mathrm{PCL}}\ ,\ d^0_{\mathrm{GCL}}\ )$$

We have left the alternative reduced costs of the inactive coal activities unspecified for now. They can be any value generated by an interior coal price: $y^0_{\mathrm{BCL}} \in (17.16, 18)$.

Removing redundancies, the primal inequalities reduce to the single inequality:

$$\theta\, 3.333(20\lambda + 10\mu) < (1 + \theta\, 3.333\lambda)20.$$

This is equivalent to $\theta\,(33.33\mu) < 20$, as in the bound for preserving the optimality of basis $\mathbf{B}^1$.

The first two dual inequalities do not restrict θ, and the last is implied by the nonsingularity condition. The only restrictive inequality is from the nuclear capacity price:

$$y_{\mathrm{LNU}}\,(\theta) > 0 \Leftrightarrow \theta\,(67.98\lambda - 52\mu) < .4.$$

Unlike the derivation of the range of optimality of $\mathbf{B}^1$, we obtain a quadratic inequality:

$$d_N(\theta) > 0 \quad \Leftrightarrow \quad \theta^2\, 52\lambda\kappa + \theta\,(52 + 3.333 d^0_{\mathrm{GCL}}\ )\lambda + d^0_{\mathrm{GCL}}\ > 0,$$

where d^0_{GCL} is any interior solution in $(0, 1.64)$. If the interior point reached happens to have d^0_{GCL} near 0, the inequality reduces to $\theta < \dfrac{1}{\kappa^+}$.

In summary, the range for which the optimal partition remains invariant is at least as great as the range of the first basis, $\rho(\mathbf{B}^1; \mathrm{e}_5 \otimes (0,\ 0,\ 0,\ \kappa,\ \lambda,\ \mu))$. (This is particular to this example, and need not be the case in general.)

Since we have $\operatorname{rank}(B) = n' = 4$, we could also apply corollary 5.1, and come to the same conclusion: no matter what relative rates are chosen, the optimal partition remains invariant over an interval, $[0, \theta^*)$, where $\theta^* > \rho^*(0,\ 0,\ 0,\ \kappa,\ \lambda,\ \mu)$. This invariance means no coal is used in *every* optimal solution to each perturbed problem.

6 Concluding Comments

The derivation of the range of a matrix change for which the optimal partition does not change is analogous to that of the optimality of a basis. The actual values, however, can differ. When the partition does not change, the inference is stronger: variables that are zero remain zero in *every* optimal solution, not just the one at hand. One expects this range to be less than that of a basis, but this need not be so. One reason comes from the results for rim variation where it is possible to vary the right-hand sides in a certain way so that the partition does not change over a larger interval than any of the optimal bases. This fact underscores the difference between the two kinds of solutions and their responses under perturbation.

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