THE DOMINATION-COMPLIANCE GRAPH OF A TOURNAMENT

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Abstract. Vertices x and y are a dominant pair in a tournament T if for all vertices $z \neq x, y$ either x beats z or y beats z. Vertices x and y are a compliant pair in a tournament T if for all vertices $z \neq x, y$ either z beats x or z beats y. Let DC(T) be the graph on the same vertex set as T with edges between pairs of vertices that are either a dominant pair or a compliant pair in T. We show that the maximum possible number of edges in DC(T) is $2(n-1)$ and this bound is sharp. In addition we obtain results about the structure of DC(T) such as forbidden subgraphs and the clique number. Since DC(T) is the complement of the competition/resource graph of a tournament complementary results are obtained for this graph.

1. Introduction. In 1994 Fisher, Lundgren, Merz, and Reid introduced and studied the concept of domination as associated in the setting modeled by a tournament. A *digraph* D is a set $V(D)$ of vertices and a set $A(D)$ of ordered pairs of vertices called arcs. We denote an arc from vertex x to vertex y by $(x, y) \in A(D)$ and say that x dominates y. For all vertices $x \in D$, the *out-set* of x, denoted $O_D(x)$ or $O(x)$, is the set of all vertices that x dominates. Similarly, the *in-set*, denoted $I_D(x)$ or $I(x)$, is the set of all vertices that dominate x. Let $d^+_D(x)$ or $d^+(x) = |O(x)|$ be the out-degree of x in the digraph D. A tournament is a digraph without loops in which every distinct pair of vertices is joined by exactly one arc. A *transmitter* is a vertex in a tournament with out-degree $n-1$. An *n-tournament* is a tournament on n vertices. A *regular* tournament is one where $d^+(x)$ is constant for all vertices x. A *reducible* tournament is a tournament whose vertex set can be partitioned into two nonempty sets A and B in such a way that every vertex in A dominates every vertex in B. See Moon [9], Reid and Beineke [11], and Reid [10] for more on tournaments.

We shall say that vertices x and y are a dominating pair in a tournament T if every other vertex $z \neq x, y$ is dominated by either x or y, i.e. x and y are a dominating pair if $O(x) \cup O(y) \cup \{x, y\} = V(T)$. The *domination* graph of a tournament T, denoted dom(T) , is the graph on the same vertex set as T with edges between vertices which form a dominating pair (see figure 1).

In [5] Fisher et. al. investigated the following question, "*Which graphs are the domination graph of a tournament*". They discovered the graphs in the following theorem are the only possibilities.

THEOREM 1.1. *([5]) Let T be a tournament. Then dom(T) is either an odd cycle with or without isolated and/or pendant vertices, or a forest of caterpillars.*

From this last theorem one should observe that the maximum possible number of edges in the domination graph of an n-tournament is n and this bound is achieved

[1] This research was partially supported by Research Contract N00014-91-J-1145 of the Office of Naval Research.

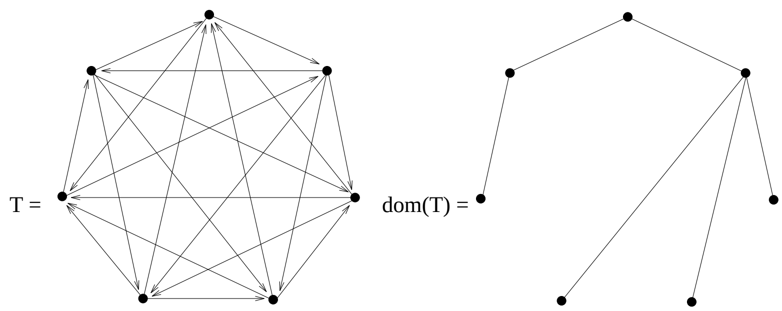


FIG. 1. *A tournament and its domination graph.*

when the domination graph is an odd cycle with or without pendant vertices.

The domination graph is closely related to the "compliance graph". Vertices x and y are said to be a compliant pair in a tournament T if every other vertex $z \neq x, y$ dominates either x or y, i.e. x and y are a compliant pair if $I(x) \cup I(y) \cup \{x, y\} = V(T)$. The compliance graph of a tournament, denoted com(T) , is the graph on the same vertex set as T with edges between vertices which form a compliant pair. Its easy to see that $com(\mathrm{T}) = dom(\widetilde{T})$, where $\widetilde{T}$ is the reversal of T. This relationship implies that every graph which is a domination graph is also a compliance graph.

Having defined the domination graph and the compliance graph we will now define the domination-compliance graph, the object of study for this paper. The *domination-compliance graph* of a tournament T, denoted DC(T) , is the graph on the same vertex set as T with edges between vertices which form either a dominating pair or a compliant pair. A quick observation one should make is that $DC(\mathrm{T}) = dom(\mathrm{T}) \cup com(\mathrm{T})$.

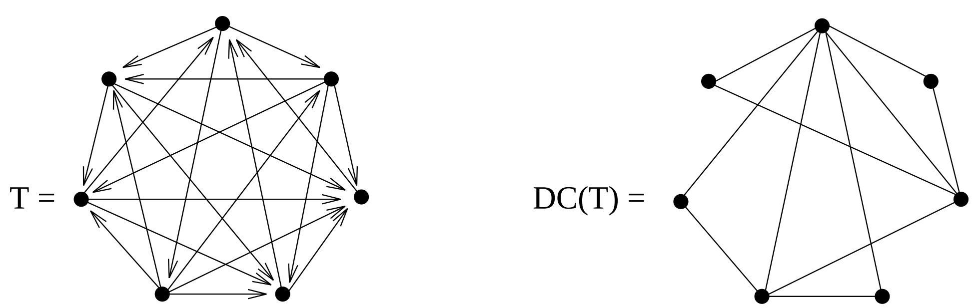


FIG. 2. *A 7-tournament and its domination-compliance graph.*

A graph which is closely related to the domination-compliance graph is the competition/resource graph. The *competition graph* of a tournament T, denoted C(T), is the graph on the same vertex set as T in which vertices x and y are adjacent if and only if there exist a vertex $z \neq x, y$ such that (x, z) and $(y, z) \in A(\mathrm{T})$. The *resource graph* of tournament T, denoted R(T), is the graph on the same vertex set as T in which vertices x and y are adjacent if and only if there exist a vertex $z \neq x, y$ such that (z, x) and $(z, y) \in A(\mathrm{T})$. It is not hard to see that $dom(\mathrm{T}) = (R(\mathrm{T})\ )^c$ and $com(\mathrm{T}) = (C(\mathrm{T})\ )^c$, which is true only when the digraph is a tournament. As a result of this relationship competition and resource graphs of tournaments have been characterized. See Fisher et. al. [5, 3, 4] for more about the competition and resource graph of a tournament.

The *competition/resource graph* of a tournament is the graph on the same vertex set as T in which vertices x and y are adjacent if and only if x and y are adjacent in both the competition graph and the resource graph. Note that $CR(\mathrm{T}) = C(\mathrm{T}) \cap R(\mathrm{T})$. Its important to point out that competition/resource graphs have been studied for arbitrary digraphs (see [7, 8, 12]) but not for tournaments.

This next result illustrates the relationship between the domination-compliance graph and the competition/resource graph.

LEMMA 1.2. *If T is a tournament then* $(CR(T)\ )^c = DC(T)$ *, where* ^{c} *is the complement operation.*

Proof. By DeMorgan's theorem,
$$\begin{aligned}(\mathrm{CR}(T)\ )^c &= (\mathrm{C}(T) \cap \mathrm{R}(T)\ )^c \\ &= (\mathrm{C}(T)\ )^c \cup (\mathrm{R}(T)\ )^c \\ &= \mathrm{com}(T) \cup \mathrm{dom}(T) \\ &= \mathrm{DC}(T)\ .\ \blacksquare\end{aligned}$$

From this lemma we can observe that results found for the domination-compliance graph of a tournament correspond to results for the competition/resource graph of a tournament. However since the domination-compliance graph generally has fewer edges than the competition/resource graph it is more convenient to state and prove results on the domination-compliance graph of a tournament. In addition, $dom(\mathrm{T}) \cap com(\mathrm{T})$, known as the *mixed pair graph* of a tournament, has been studied and characterized in [1] by Bowser, Cable and Lundgren. Therefore since the domination graph, compliance graph, competition graph, resource graph and the mixed pair graph of tournaments have all been studied, the only items remaining to be studied are the domination-compliance graph of tournaments and the competition/resource graphs of tournaments. In this paper we begin our examination of these last two items by first exhibiting an upper bound on the number of edges in the domination-compliance graph of a tournament. Second we give characterizations for the domination-compliance graph for two specific classes of tournaments.

2. Domination-Compliance Graph of Tournaments. *Which graphs can be the domination-compliance graph of a tournament?* We will start to partially answer this question by stating and proving an upper bound on the maximum possible number of edges on the domination-compliance graph. This in turn will lead to results concerning forbidden subgraphs.

We begin by defining a particular tournament on n vertices which will be very helpful in what follows.

DEFINITION 2.1. *Let* U_n *denote the tournament with vertices* $x_1, x_2, ..., x_n$ *in which* x_i *beats* x_j *if* $i - j$ *is either odd and negative or even and positive.*

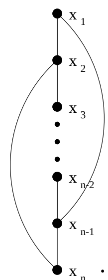
It should be observed that U_n is a regular tournament when n is odd and is a near regular tournament when n is even.

LEMMA 2.1 (FISHER ET. AL. [5]). *Let T be an n-tournament, for n odd. Then* $dom(T) = C_n$ *iff* $T \cong U_n$ *where* C_n *is the cycle on n vertices.*

Lemma 2.2. *If T is a regular tournament then every dominating pair is a compliant pair.*

Proof. Let T be a regular n-tournament, $n \geq 1$. Suppose $x, y \in V(\mathrm{T})$ such that $[x, y] \in dom(\mathrm{T})$. By way of contradiction, suppose $[x, y] \notin com(\mathrm{T})$. Then $\exists\, z \in V(\mathrm{T})$, $z \neq x, y$, such that x and y dominate z. WLOG suppose x dominates y. Then since $[x, y] \in dom(\mathrm{T})$ we have $|O(x) \cup O(y)| = n - 1$. But since $z \in O(x) \cap O(y)$ we get $|O(x) \cup O(y)| = \frac{n-1}{2} + \frac{n-1}{2} - |O(x) \cap O(y)| \leq \frac{n-1}{2} + \frac{n-1}{2} - 1 = n - 2$, a contradiction. ■

Definition 2.2. *Let P_n^* be the following graph,*



Lemma 2.3. *If $T \cong U_n$ then* $DC(T) \cong \begin{cases} P_n^* & \text{if } n \text{ is even} \\ C_n & \text{if } n \text{ is odd} \end{cases}$.

Proof. Let T be an n-tournament isomorphic to U_n, $n \geq 1$. First suppose that n is odd. Since U_n is a regular tournament, by lemmas 2.1 and 2.2 $dom(\mathrm{T}) \cong C_n$.

Now suppose that n is even. There is nothing to prove if $n = 2$, so assume that $n \geq 4$. Its obvious from the structure of U_n that $[x_i, x_{i+1}] \in DC(\mathrm{T})$, for $1 \leq i \leq n - 1$.

Suppose that $j \neq i + 1, i - 1$ and $i, j \in [2, 3, ...n - 1]$. We will show that x_i and x_j are not adjacent in $DC(\mathrm{T})$. If i and j are odd then x_i and x_j dominate x_1 and are dominated by x_2. If i and j are even then x_i and x_j dominate x_{n-1} and are dominated by x_n. If i and j are of opposite parity then since $|i - j| \geq 3$, x_i and x_j dominate x_{i+1} and are dominated by x_{i+2}.

Next suppose that $i = 1$, $j > 2$ and $j \neq n - 1$. We will show that x_1 and x_j are not adjacent in $DC(\mathrm{T})$. If j is even then x_1 and x_j dominate x_2 and are dominated by x_3. If j is odd then x_1 and x_j dominate x_n and are dominated by x_{n-1}. In a similar way we show that x_n is not adjacent to x_j for $j < n - 1$ and $j \neq 2$.

It remains to show that $[x_1, x_{n-1}]$ and $[x_2, x_n] \in DC(\mathrm{T})$. Since x_1 dominates x_j for j even and x_{n-1} dominates x_j for j odd it follows that $[x_1, x_{n-1}] \in dom(\mathrm{T}) \subseteq DC(\mathrm{T})$. Similarly since x_n is dominated by x_j for j odd and x_2 is dominated by x_j for j even it follows that $[x_2, x_n] \in com(\mathrm{T}) \subseteq DC(\mathrm{T})$. ■

Notice that the converse of this last lemma isn't true. For example consider the following 7-tournament and its corresponding domination-compliance graph.

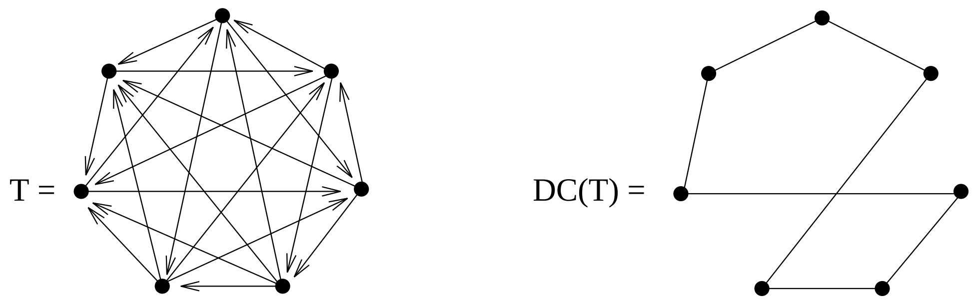


FIG. 3. *DC(T) is C_n but T is not isomorphic to U_n.*

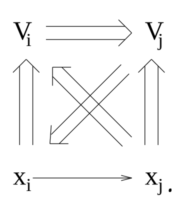
The next result is analogous to lemma 2.6 of [5].

LEMMA 2.4. *Let R be an induced subdigraph of a digraph D. Then the induced subgraph of DC(D) on the vertices of R is a subgraph of DC(R).*

Proof. Let $x, y \in R$ where $[x, y]$ is an edge in DC(D). Then either $O_D(x) \cup O_D(y) \cup \{x, y\} = V(D)$ or $I_D(x) \cup I_D(y) \cup \{x, y\} = V(D)$. Since $V(R) \subseteq V(D)$ we have that either $O_R(x) \cup O_R(y) \cup \{x, y\} = V(R)$ or $I_R(x) \cup I_R(y) \cup \{x, y\} = V(R)$. Thus $[x, y]$ is an edge in $DC(R)$. ■

As noted earlier Fisher et. al. in [5] showed that the possibilities for the domination graph of a tournament are an odd-spiked cycle with or without isolated and/or pendant vertices, or a forest of caterpillars. In a subsequent paper [4] they determined which connected graphs are the domination graph of a tournament. Jimenez and Lundgren in [6] determined "*Which tournaments yield a connected domination graph?*". The following result from that paper is useful in this setting.

LEMMA 2.5. *T is a tournament whose domination graph is an odd-spiked cycle if and only if T is a tournament whose vertex set V(T) can be partitioned into the following sets:*
$\{x_1, x_2, ..., x_k\}$, V_1, V_2, ..., V_k *with* $\{x_1, x_2, ..., x_k\} \cong U_k$*(k odd), the arcs within each* V_i *arbitrary and the other arcs prescribed by the following diagram*



We will now use this lemma to derive the following result.

LEMMA 2.6. *Let T be a tournament whose domination graph is an odd-spiked cycle without isolated vertices. Assume x_i and V_j are as described in Lemma 2.5. The following are the only possible edges in com(T) .*

i) Let $v \in V_i$. Then $[x_i, v] \in$ com(T) iff $d^+_{V_i}(v)$=0

ii) Let $v \in V_i$ and $w \in V_j$. Then $[v, w] \in$ com(T) iff $j \equiv i + 1$ mod(k) and $d^+_{V_j}(w) = 0$

iii) $[x_i, x_j] \in com(T)$ *iff* $j \equiv i+1 \ mod(k)$ *and* $V_i = \emptyset$

iv) Let $v \in V_i$. *Then* $[v, x_j] \in com(T)$ *iff* $j = i+2 \ mod(k)$ *and* $V_{i+1} = \emptyset$.

Proof. Let T be a tournament whose domination graph is an odd-spiked cycle without isolated vertices. T must have the structure which is prescribed in Lemma 2.5. First of all it is clear that no pair of vertices in V_i, $1 \le i \le k$ can be an adjacent pair in *com*(T) since they both dominate every vertex x_j which dominates x_i.

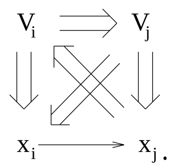
i) Let $v \in V_i$. Suppose $z \in O_{\mathrm{T}}(v) \cap O_{\mathrm{T}}(x_i)$ for some $z \in V(\mathrm{T})$. From the diagram in Lemma 2.5, $z \notin V_j$ for $j \ne i$ since if x_i dominates z, then z dominates v. In addition, $z \ne x_j$, for $j \ne i$ since if x_i dominates x_j, then x_j dominates v. So the only possibility is that $z \in V_i$. It follows that $[x_i, v] \in com(\mathrm{T})$ if and only if $d^+_{V_i} = 0$.

ii) Let $v \in V_i$ and $w \in V_j$. By Lemmas 2.1 and 2.2 we know that the dominant pairs in U_k are x_i, x_{i+1}, $1 \le i \le k-1$, and x_1, x_k. Since no other pairs of vertices in U_k are dominant pairs there exist a vertex x_l, $l \ne i, j$ in U_k such that x_i and x_j are dominated by x_l when $j \ne i+1 \ mod(k)$. Thus $[v, w] \notin com(\mathrm{T})$ when $j \ne i+1 \ mod(k)$ since v and w dominate x_l. Thus assume that $j = i+1 \ mod(k)$. Suppose $z \in O_{\mathrm{T}}(v) \cap O_{\mathrm{T}}(w)$ for some $z \in V(\mathrm{T})$. From the diagram in Lemma 2.5, $z \notin V_l$ for $l \ne j$ since if v dominates z then z dominates x_i which implies that x_{i+1} dominates z and which further implies z dominates w. In addition, $z \ne x_l$ for $l \ne i, j$ by a similar argument. Therefore the only possibility is that $z \in V_j$. It follows that $[v, w] \in com(\mathrm{T})$ if and only if $j = i+1 \ mod(k)$ and $d^+_{V_j}(w) = 0$.

iii) By Lemmas 2.1 and 2.2 we know that the compliant pairs in U_k are x_i, x_{i+1}, $1 \le i \le k-1$, and x_1, x_k. Since no other pairs of vertices in U_k are compliant pairs there exist a vertex x_m on U_k, $m \ne i, j$, such that x_i and x_j dominate x_m when $j \ne i+1 \ mod(k)$. Thus $[x_i, x_j] \notin com(\mathrm{T})$ when $j \ne i+1 \ mod(k)$ since x_j and x_i dominate x_m. Observe that $O_{\mathrm{T}-\{V_i\}}(x_i) \cap O_{\mathrm{T}-\{V_i\}}(x_{i+1}) = \emptyset$. To see this suppose that $O_{\mathrm{T}-\{V_i\}}(x_i) \cap O_{\mathrm{T}-\{V_i\}}(x_{i+1}) \ne \emptyset$. Then there exist a vertex $z \in V(\mathrm{T} - \{V_i\})$ such that (x_i, z) and $(x_{i+1}, z) \in A(\mathrm{T})$. Since *dom*(T) is assumed to be connected z must be adjacent to some z^* in $V(\mathrm{T} - \{V_i\})$. But then since $[z, z^*] \in dom(\mathrm{T})$ we must have that (z^*, x_i) and $(z^*, v) \in A(\mathrm{T})$, a contradiction. Therefore since $O_{\mathrm{T}-\{V_i\}}(x_i) \cap O_{\mathrm{T}-\{V_i\}}(x_{i+1}) = \emptyset$ and x_{i+1} dominates every vertex in V_i it follows that $[x_i, x_{i+1}] \in com(\mathrm{T})$ if and only if $V_i = \emptyset$.

iv) Let $v \in V_i$ and $j \ne i, i+2$. If x_j dominates x_i then both v and x_j dominate x_{j-2}. If x_i dominates x_j then both v and x_j dominate x_{j+1}. Thus $[v, x_j] \notin com(\mathrm{T})$ when $j \ne i, i+2$. Observe that $O_{\mathrm{T}-\{V_{i+1}\}}(v) \cap O_{\mathrm{T}-\{V_{i+1}\}}(x_{i+2}) = \emptyset$. To see this assume not. Then there must exist a vertex $z \in V(\mathrm{T} - \{V_{i+1}\})$ such that (v, z) and $(x_{i+2}, z) \in A(\mathrm{T})$. By parts i) and iii) we have that $O_{\mathrm{T}-\{V_{i+1}\}}(x_{i+1}) \cap O_{\mathrm{T}-\{V_{i+1}\}}(x_{i+2}) = \emptyset$ and $O_{\mathrm{T}-\{V_i\}}(x_i) \cap O_{\mathrm{T}-\{V_i\}}(v) = \emptyset$. Thus it follows that (z, x_i) and $(z, x_{i+1}) \in A(\mathrm{T})$, a contradiction. Therefore since $O_{\mathrm{T}-\{V_{i+1}\}}(v) \cap O_{\mathrm{T}-\{V_{i+1}\}}(x_{i+2}) = \emptyset$ and also since v and x_{i+2} dominate every vertex in V_{i+1} it follows that $[v, x_{i+2}] \in com(\mathrm{T})$ if and only if $V_{i+1} = \emptyset$. ■

Since *dom*(T) and *com*(T) are duals of each other in [6] we obtained a result analogous to Lemma 2.5 for the compliance graph of a tournament except the arcs are prescribed by the following diagram,



As a consequence of this we obtain the following result analogous to Lemma 2.6.

LEMMA 2.7. *Let T be a tournament whose compliance graph is an odd-spiked cycle without isolated vertices. Assume x_i and V_j are as described in preceeding discussion. The following are the only possible edges in dom(T) .*

i) Let $v \in V_i$. Then $[x_i, v] \in dom(T)$ iff v is a transmitter in T_{V_i}

ii) Let $v \in V_i$ and $w \in V_j$. Then $[v, w] \in dom(T)$ iff $j \equiv i+1 \ mod(k)$ and v is a transmitter in T_{V_i}

iii) $[x_i, x_j] \in dom(T)$ iff $j \equiv i+1 \ mod(k)$ and $V_{i+1} = \emptyset$.

iv) Let $v \in V_i$. Then $[v, x_j] \in dom(T)$ iff $j = i-2$ and $V_{i-1} = \emptyset$

THEOREM 2.8. *Let T be an n-tournament. The maximum possible number of edges in DC(T) is 2(n-1) and this bound is best possible.*

Proof. Suppose that the number of edges in DC(T) is $> \ 2(n-1)$. The only possible ways that this could happen is if we had the following configurations.

1. $|E(dom(T))| = n$, $|E(com(T))| = n$, and $|E(dom(T) \cap com(T))| = 0$.
2. $|E(dom(T))| = n$, $|E(com(T))| = n$, and $|E(dom(T) \cap com(T))| = 1$.
3. $|E(dom(T))| = n$, $|E(com(T))| = n-1$, and $|E(dom(T) \cap com(T))| = 0$.
4. $|E(dom(T))| = n-1$, $|E(com(T))| = n$, and $|E(dom(T) \cap com(T))| = 0$.

For each of the configurations above where $|E(dom(T))| = n$, dom(T) must be an odd cycle or an odd-spiked cycle. If dom(T) is an odd cycle we may conclude by Lemmas 2.1 and 2.2 that $|E(dom(T) \cap com(T))| = n$. Thus dom(T) must be an odd-spiked cycle. Similarly we may conclude that com(T) is an odd-spiked cycle if $|E(com(T))| = n$.

<u>case 1:</u> Since $|E(dom(T))| = n$ we may conclude by the above observation that dom(T) must be an odd- spiked cycle. Thus T must have the structure which is prescribed in Lemma 2.5. Since $|E(dom(T) \cap com(T))| = 0$, by Lemma 2.6 we have that for each V_i, $|V_i| \geq 3$ and T_{V_i} has no vertex of degree zero. This implies that com(T) has no edges. Thus we may conclude that this configuration doesn't exist.
By a similar argument we may conclude that configuration 3 does not exist.

<u>case 2:</u> Since $|E(dom(T))| = n$ we may conclude by the above observations that dom(T) is an odd-spiked cycle. Thus T must have the structure which is prescribed in Lemma 2.5. Since $|E(dom(T) \cap com(T))| = 1$ then one of the following edges of the form $[x_i, x_{i+1}]$ or $[v, x_i]$, where $v \in V_i$, must be in $E(dom(T) \cap com(T))$.

Suppose $[x_i, x_{i+1}] \in E(dom(T) \cap com(T))$, for some i. By Lemma 2.6 we may conclude that $V_i = \emptyset$. Since $|E(dom(T) \cap com(T))| = 1$, by Lemma 2.6 we have for each V_j, $j \neq i$, $|V_j| \geq 3$ and T_{V_j} has no vertex of degree zero. It follows that com(T) is a forest of caterpillars contradicting $|E(com(T))| = n$.

Next suppose that $[v, x_i] \in E(dom(T) \cap com(T))$, for some i. By Lemma 2.6 we may conclude that $d^+_{V_i}(v) = 0$. Since $|E(dom(T) \cap com(T))| = 1$, by Lemma 2.6 we have for each V_j, $j \neq i$, $|V_j| \geq 3$ and T_{V_j} has no vertex of degree zero. Once again it follows that com(T) is a forest of caterpillars contradicting $|E(com(T))| = n$.
Thus we may conclude that this configuration does not exist.

case 4: Since $|E(com(T))| = n$, by the above observations com(T) must be an odd-spiked cycle. Thus T must have the structure which is prescribed in the discussion preceding Lemma 2.7. Since $|E(dom(T) \cap com(T))| = 0$, by Lemma 2.7 we have that for each V_i, $|V_i| \geq 3$ and T_{V_i} has no transmitter. This implies that dom(T) is the trivial graph. Thus this configuration doesn't exist.

Therefore we may conclude that DC(T) has at most $2(n-1)$ edges.

To show that this bound is best possible let T $= U_{n-1} \Rightarrow \bullet$. For this tournament, $n-1$ edges are obtained from dom(T) and $n-1$ edges are obtained from com(T) (see figure 4). ■

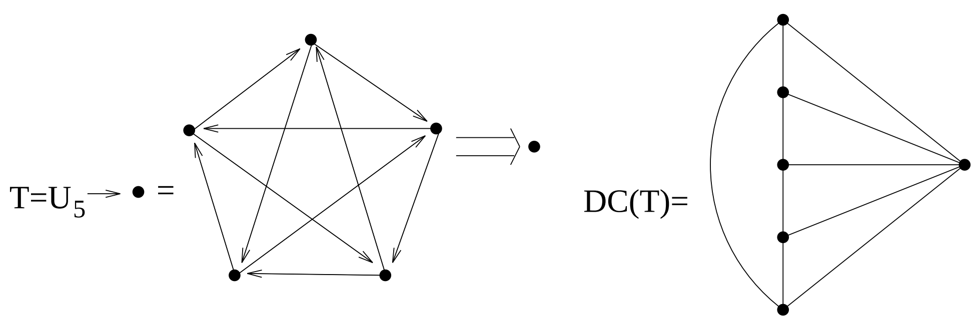


FIG. 4. *A 6-tournament whose domination-compliance graph has* $2(6-1) = 10$ *edges.*

3. Consequences of the upper bound. Theorem 2.1 gives a sharp upper bound on the maximum possible number of edges in the domination-compliance graph of a tournament. As a result it is straight forward to deduce results about forbidden subgraphs for the domination-compliance graph and graph parameters for the domination-compliance graph and the competition/resource graph. Below are some examples.

COROLLARY 3.1. *The minimum possible number of edges in the competition/resource graph of an n-tournament is* $\binom{n}{2} - 2(n-1)$.

Let G be a graph. A subset of vertices of G form a *clique* if there are edges between every pair of vertices in the subset. We define ω(G) (the *clique number* of G) to be the maximum size of a clique in G. A subset of vertices of G form an *independent set* if there are no edges between any two vertices in the subset. We define α(G) (the *independence number* of G) to be the maximum size of an independent set in G. Clearly, $\alpha(G) = \omega(\overline{G})$ where $\overline{G}$ is the complement of G.

COROLLARY 3.2. *For $n \geq 4$ let T be an n-tournament. The clique number of DC(T) is at most 4.*

Proof. Let T be an n-tournament, n≥5. Suppose that K_5 is a subgraph of DC(T) . Then by Lemma 2.4 there must exist a 5-tournament, T′, such that $DC(T') = K_5$. But this can't possibly happen since the maximum possible number of edges in $DC(T')$ is 8. Thus we may conclude that K_5 is not a subgraph of DC(T) , for any n-tournament T, where n≥4. ■

COROLLARY 3.3. *For n≥4 let T be an n-tournament. The independence number of the competition/resource graph of T is at most 4.*

4. Characterization for reducible and regular tournaments. Our intention of this investigation is to fully characterize the domination-compliance graphs of tournaments in order to achieve a result analogous to the results found in [5, 3, 4]. At this point we have not been able to achieve such a result. But if we restrict T to belong to one of the following two specific classes of tournaments, we can get such a result.

Recall that a tournament T is *reducible* if it is possible to partition its vertex set into two nonempty sets A and B in such a way that every vertex in A dominates every vertex in B. Its easy to see that reducible tournaments are precisely those which are not strong. A *regular* tournament is one in which every vertex has the same out-degree. Note that the only regular tournaments are those with an odd number of vertices. See Beineke and Reid [11] for more on reducible and regular tournaments. Let I_n be the graph with n isolated vertices.

LEMMA 4.1. *Let T be a reducible n-tournament with a partition of its vertex set into nonempty sets A and B such that A dominates B. If T has no transmitter or vertex of degree 0 then (1) dom(T) =* $dom(A) \cup I_{|B|}$ *and (2) com(T) =* $com(B) \cup I_{|A|}$.

Proof. Let T be a reducible n-tournament with a partition of its vertex set into nonempty sets A and B such that A dominates B. Assume T has no transmitter or vertex of degree 0.

(1). Clearly no pair of vertices in B can be a dominant pair in T, since A is nonempty. Since T doesn't have a transmitter, for every vertex $x \in V(\mathrm{T})$, $d^+(x) < n-1$. Thus no two vertices $x \in A$ and $y \in B$ can be a dominant pair. Therefore if $[x, y] \in dom(\mathrm{T})$, by the observations above, both x and y must belong to A. It follows that $[x, y] \in dom(A)$. Similarly, if $[x, y] \in dom(A)$ then $[x, y] \in dom(\mathrm{T})$ since A dominates B. Therefore $dom(\mathrm{T}) = dom(A) \cup I_{|B|}$.

(2). Clearly no pair of vertices in A can be a compliant pair in T, since B is nonempty. Since T does not have a vertex of degree 0, for every vertex $x \in V(\mathrm{T})$, $d^+(x) > 0$. Thus no two vertices $x \in A$ and $y \in B$ can be a compliant pair. Therefore if $[x, y] \in com(\mathrm{T})$, by the observations above, both x and y must belong to B. It follows that $[x, y] \in com(B)$. Similarly, if $[x, y] \in com(B)$ then $[x, y] \in com(\mathrm{T})$ since A dominates B. Therefore $com(\mathrm{T}) = com(B) \cup I_{|A|}$. ■

LEMMA 4.2. *If T is an n-tournament with a transmitter then dom(T) =* $K_{1,n-1}$.

Proof. Let T be an n-tournament with a transmitter, say x. Note that for any $y \in V(\mathrm{T}) - \{x\}$, $[x, y] \in dom(\mathrm{T})$ since x dominates all of $V(\mathrm{T}) - \{x\}$. Thus $K_{1,n-1} \subseteq dom(\mathrm{T})$. To show that $dom(\mathrm{T}) = K_{1,n-1}$, it suffices to show that no other pairs of vertices form a dominant pair. Let $y, z \in V(\mathrm{T}) - \{x\}$. Since (x, y) and $(x, z) \in A(\mathrm{T})$, $[y, z] \notin dom(\mathrm{T})$. ■

Lemma 4.3. *If T is an n-tournament with a vertex of degree 0 then com(T) $= K_{1,n-1}$.*

Proof. Let T be an n-tournament with a vertex of degree 0, say x. Note that for any $y \in V(\mathrm{T}) - \{x\}$, $[x, y] \in com(\mathrm{T})$ since x is dominated by every vertex in $V(\mathrm{T}) - \{x\}$. Thus $K_{1,n-1} \subseteq com(\mathrm{T})$. To show that $com(\mathrm{T}) = K_{1,n-1}$, it suffices to show that no other pairs of vertices form a compliant pair. Let $y, z \in V(\mathrm{T}) - \{x\}$. Since (y, x) and $(z, x) \in A(\mathrm{T})$, $[y, z] \notin com(\mathrm{T})$. ■

For the following theorem let an *odd-spiked cycle* denote a cycle with pendant vertices.

Theorem 4.4. *Let T be a reducible n-tournament. Then DC(T) is either,*

1) *two disjoint odd/odd-spiked cycles with or without isolated vertices*
2) *a forest of caterpillars with or without isolated vertices*
3) *an odd/odd-spiked cycle and a forest of caterpillars with or without isolated vertices*
4) $K_2 \vee I_{n-2}$
5) G

where G is the graph composed of a star with its center vertex adjacent to every vertex in one of the following: an odd cycle with or without isolated vertices and/or pendant vertices or a forest of caterpillars.

Proof. Let T a reducible n-tournament. Since T is reducible its vertex set can be partitioned into nonempty sets A and B such that A dominates B.

First assume that T has no transmitter or vertex of degree 0. Thus both $|A|$ and $|B| > 1$. By Lemma 4.1 and Theorem 1.1 $dom(\mathrm{T}) = dom(A) \cup I_{|B|}$, where $dom(A)$ is either an odd cycle with or without isolated and/or pendant vertices or a forest of caterpillars. Similarly, $com(\mathrm{T}) = com(B) \cup I_{|A|}$, where $com(B)$ is either an odd cycle with or without isolated and/or pendant vertices or a forest of caterpillars. In addition note that $E(dom(T)) \cap E(com(T)) = \emptyset$. Therefore since $DC(\mathrm{T}) = dom(\mathrm{T}) \cup com(\mathrm{T})$, it follows that $DC(\mathrm{T})$ is either two disjoint odd/odd-spiked cycles , a forest of caterpillars, or an odd/odd-spiked cycle and a forest of caterpillars all possibly occurring with isolated vertices.

Next assume T has both a transmitter, x, and a vertex of degree 0, y. By Lemmas 4.2 and 4.3 $dom(\mathrm{T}) = K_{1,n-1}$, $com(\mathrm{T}) = K_{1,n-1}$, and $E(dom(T)) \cap E(com(T)) = \{[x, y]\}$. Since $DC(\mathrm{T}) = dom(\mathrm{T}) \cup com(\mathrm{T})$ it follows that $DC(\mathrm{T}) = K_2 \vee I_{n-2}$.

Finally assume T has a transmitter, x, and no vertex of degree 0. Note that $A = \{x\}$ and $B = V(\mathrm{T}) - \{x\}$. By Lemmas 4.1 and 4.2 and Theorem 1.1 $dom(\mathrm{T}) = K_{1,n-1}$ and $com(\mathrm{T}) = com(V(\mathrm{T}) - \{x\})$ with 1 isolated vertex, where $com(V(\mathrm{T}) - \{x\})$ is either an odd cycle with or without isolated and/or pendant vertices or a forest of caterpillars. Notice that $E(dom(T)) \cap E(com(T)) = \emptyset$. Thus since $DC(\mathrm{T}) = dom(\mathrm{T}) \cup com(\mathrm{T})$, it follows that $DC(\mathrm{T})$ is the graph composed of a star with its center vertex adjacent to every vertex in one of the following: an odd cycle with or without isolated vertices and/or pendant vertices or a forest of caterpillars. A similar argument shows that $DC(\mathrm{T})$ is a graph G, where G is the same as above, when T has no transmitter and has a vertex of degree 0. ■

Now suppose that T is a regular n-tournament, n odd. Then by Lemma 2.2, $dom(\mathrm{T}) = com(\mathrm{T}) = DC(\mathrm{T})$. But then it follows by a recent theorem of Cho, Kim, and Lundgren [2] that $DC(\mathrm{T})$ is either an odd cycle or a forest of paths. So we have the following theorem.

THEOREM 4.5. *Let T be a regular n-tournament, n odd. The DC(T) is either an odd cycle or a forest of paths where some or all of the paths could be isolated vertices.*

As mentioned earlier reducible tournaments are precisely those tournaments which are not strongly connected. Having characterized the domination-compliance graphs for reducible tournaments it remains to characterize the domination-compliance graph of strongly connected tournaments. This problem will be dealt with in a subsequent paper.

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