THE FINITE RAY ELEMENT METHOD FOR THE HELMHOLTZ EQUATION OF SCATTERING: FIRST NUMERICAL EXPERIMENTS*

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1. Introduction. A new numerical method for the Helmholtz equation of scattering is presented. The method uses as basis function plane waves in a set of directions at every mesh node, multiplied by standard linear finite element basis functions. The motivation is that potentially one can use only a few basis functions with directions close to the dominant wave propagation directions, and with element size larger than the wavelength. This property was confirmed in our experiments and we have obtained high accuracy in a model problem using only few such directions. We have introduced an efficient criterion to identify dominant directions in a coarse approximation to the solution obtained by only few uniformly distributed directions present in the Finite Ray Element basis.

The potential significance of the Finite Ray Element Method is that it allows to approximate the scattered field with a complexity that depends on the complexity of the field itself rather than on the wave number; in the high frequency approximation, the method resembles geometrical optics. This should allow a fast computation of the scattered field in many cases of practical interest. However, computation of the stiffness matrix requires complicated integrals and is expensive. The stiffness matrix itself has a high condition number already for a modest number of directions, which requires the use of multiple precision arithmetic. Further work is needed to establish when the resulting method is more efficient than standard Finite Element methods.

We consider two variational formulation of the Helmholtz equation: the standard Galerkin form, and First Order System of Least Squares (FOSLS) formulation [2, 4]. The intrinsic advantage of FOSLS is that the matrix of the system is Hermitean and positive definite. However, we have found that in this case, the Galerkin approximation gave better accuracy on the same mesh and with the same number of directions per node.

Related methods that use a uniformly distributed set of directions are known under the names of Microlocal Discretization [3] and Partition of Unity Finite Element Method [1]. Basis functions similar to ours have been used as coarse basis functions in multigrid methods [4, 6].

2. Variational formulation of the problem. Consider the Helmholtz equation

$$\Delta p + K^2 p = 0 \qquad \text{in} \quad \Omega$$

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with boundary conditions

$$\alpha(x)\frac{\partial p}{\partial \nu} + \beta(x)p = \gamma(x) \qquad \text{on} \quad \partial\Omega,$$

ν is outer normal of Ω. This formulation of the boundary conditions includes

- $\alpha(x) = 0$ ... Dirichlet boundary condition
- $\alpha(x) = 1$, $\beta(x) = -iKp$ ... radiation conditon

2.1. Galerkin method. Find $p \in V$, $V = H^1(\Omega)$, such that

$$-\int_\Omega \nabla p \nabla \overline{v} + \int_\Omega K^2 p\overline{v} + \int_{\partial\Omega} g(p-f)\overline{v} = 0, \quad \forall v \in V,$$

where $g(x) = \frac{-\beta(x)}{\alpha(x)}$, $f(x) = \frac{\gamma}{\beta}$. This formulation is suitable for $\alpha \neq 0$, that is, in the absence of Dirichlet boundary condition. The Dirichlet boundary condition can be modeled by a small α; however, this results in very ill-conditioned matrix of the discrete problem. The reason why nonhomogeneous Dirichlet boundary conditions cannot be treated similarly as in usual Finite Element methods is that it is very difficult to expand a given boundary function in the basis functions (2.1).

2.2. First order system of least squares (FOSLS). Following [2], introduce a new independent variable $\mathbf{u} = (u_x, u_y) = \nabla p(x)$. We have to find $p, u_x, u_y \in V$, $V = H^1(\Omega)$ to minimize the least squares functional

$$F(\mathbf{u}, p) = \int_\Omega |\mathbf{u} - \nabla p|^2 + \int_\Omega |\nabla \cdot \mathbf{u} + K^2 p|^2 + \int_\Omega |\nabla \times u|^2 + \int_{\partial\Omega} |\alpha(x)\frac{\partial p}{\partial \nu} + \beta(x)p - \gamma(x)|^2$$

The Galerkin approximation use smaller a number of unknowns, but the bilinear form is not Hermitean positive definite. Problem with Dirichlet can be treated easily. The Least square approximation results in three times more unknowns, but the matrix is Hermitean positive definite.

2.3. Basis function. The finite element space is span of basis functions of the form

$$\phi_j(\mathbf{x})e^{iKe\mathbf{x}\cdot\mathbf{d}_k}, \qquad \mathbf{d}_k = (\cos\theta_k, \sin\theta_k), \tag{2.1}$$

where ϕ_j is a standard $Q1$ basis functions associated with a node x_j, and θ_k runs through a finite set of directions for each node.

The integrals in the computation of the stiffness and the mass matrices are computed by a quadrature formula that uses exact values for integrals of the form $\int_0^1 p(t)e^{\lambda t}$, where $p(t)$ is a polynomial. On a rectangular mesh, such as used here, this results in the quadrature being exact.

3. Computational results with uniform distribution of directions. In all experiments, we use uniform rectangular discretization of a rectangular domain Ω. The exact solution is a plane wave propagating across the domain in the direction

p	h=1	1/2	1/4	1/8	1/16	1/32	1/64	1/128
Q1	0.07996	0.02057	0.00552	0.00089	0.00022	5.6e-05	1.4e-05	3.50e-06
1	0.11422	0.03298	0.00915	0.00186	0.00046	0.00011	2.9e-05	7.30e-06
2	0.07909	0.01993	0.00533	0.00087	0.00021	5.4e-05	1.3e-05	
3	0.00344	0.00024	1.5e-05	9.9e-07	6.4e-08	4.8e-09		
4	0.00032	2.1e-05	1.3e-06					
5	3.0e-06	1.5e-07	8.8e-09					
6	1.6e-06	2.7e-07	1.3e-08					
7	2.1e-05	6.2e-05						

TABLE 3.1

Maximal error (domain: unit square; solution angle = 1.3457; K = 1) Galerkin method

p	h=1	1/2	1/4	1/8	1/16	1/32	1/64
Q1	1.57e+00	6.5e+00	2.2e+01	7.9e+01	2.9e+02	1.12e+03	4.3e+03
1	2.0e+00	7.0e+00	2.2e+01	8.0e+01	2.9e+02	1.1e+03	4.38e+03
2	2.3e+03	2.0e+05	1.1e+07	6.8e+08	4.2e+10	2.6e+12	
3	6.8e+04	6.5e+06	3.4e+08	1.9e+10	1.1e+12	7.0e+13	
4	1.4e+09	2.3e+10	3.1e+12				
5	8.9e+09	2.2e+12	1.8e+15				
6	4.9e+10	7.3e+13	4.2e+16				
7	3.1e+13	1.2e+15					

TABLE 3.2

Condition number (domain: unit square; solution angle = 1.3457; K = 1) Galerkin method

$\alpha = 1.3457$, $v(x) = e^{iK\mathbf{x}\cdot\mathbf{d}}$, $d = (\cos\alpha, \sin\alpha)$. The boundary condition is a radiation condition with right hand side given by this solution, that is,

$$-\frac{\partial p}{\partial \nu} + iKu = iK(1 - \mathbf{d}\cdot\nu).$$

This is the same boundary condition as used in [5].

The error of the numerical and exact solution and the condition number of the discrete problem matrix are summarized in Tables 3.1 – 3.8.

We observe that the maximal error and the condition number are better for the Galerkin discretization than for FOSLS. In addition, for Q1 elements, the Galerkin approximation appears to converge as h^2, while FOSLS approximations only as h.

Because of poor conditioning, all results were obtained in 32 byte extended precision computer arithmetics. Extended or infinite precision was necessary for related results reported in the literature [1] as well (I. Babuška, personal communication).

Because our implementation is only a prototype that does many redundant computations, we do not report CPU times and memory usage.

p	h=1	1/2	1/4	1/8	1/16
1	0.14769	0.05526	0.01651	0.00475	0.00121
2	0.09340	0.04376	0.01632	0.00500	0.00128
3	0.00359	0.00025	1.9e-05	1.3e-06	1.1e-07
4	0.00034	2.4e-05	1.4e-06	8.4e-08	5.1e-09
5	3.4e-06	1.9e-07	1.0e-08	7.3e-10	5.1e-11
6	1.9e-06	2.8e-07	1.2e-08	7.6e-10	
7	1.1e-07	1.3e-08	5.4e-10	4.9e-11	
8	2.7e-09	2.0e-10	1.5e-10	8.2e-11	
9	4.0e-10	1.9e-09	6.4e-10	3.2e-09	
10	1.3e-08	2.1e-09	2.7e-10	1.2e-08	
11	6.3e-09	1.3e-09	8.7e-09	4.4e-09	
12	2.8e-08	3.3e-09	3.3e-09		
13	9.2e-10	1.0e-09	8.6e-09		
14	2.7e-09	5.5e-09			
15	1.2e-09				

TABLE 3.3

Maximal error (domain: unit square; solution angle = 1.3457; K = 1) FOSLS

p h	1	1/2	1/4	1/8	1/16
1	1.9e+01	4.2e+01	1.0e+02	3.4e+02	1.2e+03
2	6.7e+03	2.7e+05	1.3e+07	8.1e+08	5.0e+10
3	2.2e+05	9.8e+06	4.1e+08	2.1e+10	1.2e+12
4	3.9e+09	2.8e+10	3.3e+12	5.8e+14	1.2e+17
5	2.6e+10	2.7e+12	1.9e+15	1.7e+18	1.6e+21
6	1.4e+11	8.8e+13	8.0e+16	2.1e+20	
7	8.9e+13	1.6e+15	6.6e+17	1.7e+20	
8	4.3e+16	8.5e+17	2.7e+19	1.3e+21	
9	6.0e+17	6.2e+19	5.8e+20	5.0e+22	
10	4.1e+19	2.0e+20	1.9e+20	1.6e+21	
11	1.8e+19	1.0e+20	2.2e+21	7.4e+20	
12	1.1e+20	5.9e+20	2.4e+20		
13	2.5e+19	7.8e+19			
14	2.4e+19				
15	4.4e+19				

TABLE 3.4

Condition number (domain: unit square; solution angle = 1.3457; K = 1), FOSLS

p h	1	1/2	1/4	1/8	1/16	1/32	1/64	1/128
Q1	1.36468	1.44736	1.38930	2.45058	1.14497	0.33630	0.08807	0.02200
1	1.18804	1.21853	1.65711	2.06539	1.68717	0.49477	0.12700	0.03194
2	1.32704	1.42395	2.45186	3.30248	1.40016	0.37071	0.09062	
3	1.81748	1.64475	2.76450	1.56787	0.19699	0.01811		
4	1.29928	0.52243	0.22575	0.01906	0.00226	0.00030		
5	0.29225	0.10875	0.01916	0.00154	0.00015	1.6e-05		
6	2.21074	1.94570	0.35341	0.01439	0.00046			
7	2.39380	2.16478	0.13745	0.00787	0.00039			
8	1.26150	0.50052	0.02863	0.00178	7.1e-05			
9	0.14158	0.01532	0.00082	2.9e-05	7.3e-07			
10	0.37624	0.05341	0.00113	3.0e-05				
11	0.77904	0.14062	0.00295	0.00010				
12	0.62793	0.21395	0.00182	5.0e-05				
13	0.46386	0.02711	0.00050	6.9e-06				
14	1.8e-05	1.1e-06	1.9e-08	2.1e-07				
15	0.08744	0.00570	9.6e-05					
16	0.20056	0.00562	9.8e-05					
17	0.10393	0.00285	3.8e-05					
18	0.00249	0.00293	2.8e-06					
19	0.00026	1.7e-05	1.7e-07					
20	0.00131	0.00013	8.5e-07					
21	0.00050	5.1e-05	6.5e-07					
22	6.1e-05	1.1e-05						
23	1.9e-06	3.4e-07						
24	2.1e-06	2.8e-07						
25	1.8e-06	4.8e-07						
26	3.7e-07	8.8e-08						
27	4.5e-08	1.4e-08						
28	1.8e-06	6.9e-06						
29	1.2e-08	5.1e-07						
30	4.0e-08							

TABLE 3.5

Maximal error (domain: unit square; solution angle = 1.3457; K = 20), Galerkin method

p h	1	1/2	1/4	1/8	1/16	1/32	1/64
Q1	8.2e+00	1.2e+01	2.3e+01	1.4e+01	2.0e+01	8.1e+01	3.2e+02
1	3.1e+00	4.7e+00	1.3e+01	1.7e+01	3.4e+01	9.5e+01	3.3e+02
2	5.3e+00	8.6e+00	1.0e+02	4.4e+01	6.7e+02	4.1e+04	
3	5.8e+00	1.4e+01	4.8e+01	5.0e+01	2.0e+03	1.4e+05	
4	5.5e+00	1.1e+01	2.7e+02	1.5e+03	2.6e+05	6.8e+07	
5	1.4e+01	3.6e+01	3.9e+02	2.5e+04	5.6e+07		
6	2.6e+01	1.4e+02	6.5e+02	2.2e+05	2.0e+08		
7	2.9e+01	4.2e+01	1.5e+03	5.7e+05	5.6e+08		
8	2.3e+01	4.2e+01	1.0e+03	1.6e+05	1.0e+08		
9	3.5e+01	3.0e+02	2.4e+04	1.0e+07			
10	5.2e+01	1.5e+03	9.4e+04	3.2e+08			
11	4.8e+01	3.6e+03	2.1e+05	8.8e+08			
12	6.6e+01	2.1e+04	1.0e+06	4.0e+08			
13	4.3e+02	1.9e+04	4.0e+06				
14	2.3e+02	1.3e+05	7.1e+07				
15	7.1e+02	2.3e+05	1.7e+08				
16	1.2e+03	5.4e+05	1.2e+08				
17	1.4e+04	1.3e+06	7.2e+09				
18	9.2e+03	5.4e+06	8.0e+10				
19	3.8e+04	1.3e+07	2.2e+11				
20	2.8e+05	3.7e+07	0.0e+00				
21	2.6e+06	2.4e+08	9.6e+12				
22	2.8e+07	2.3e+09					
23	3.3e+08	1.2e+10					
24	4.3e+09	8.6e+10					
25	6.2e+10	2.0e+12					
26	9.7e+11	3.2e+13					
27	1.6e+13	3.5e+14					
28	3.0e+14	6.4e+15					
29	7.3e+15	2.1e+17					
30	6.8e+16						

TABLE 3.6

Condition number (domain: unit square; solution angle = 1.3457; $K = 20$), Galerkin method

p h	1	1/2	1/4	1/8	1/16
1	1.30575	1.36460	1.50150	1.48131	1.33234
2	1.34333	1.37918	1.57028	1.75570	1.66618
3	3.45319	3.29504	2.83638	2.51312	1.73357
4	1.99176	2.02958	1.83313	1.09600	0.23977
5	0.60486	0.51498	0.37280	0.13507	0.00909
6	3.57309	2.35025	1.76414	0.84981	
7	3.80951	3.99421	2.05738	0.59602	
8	3.41133	1.79664	0.85726	0.11369	
9	0.34382	0.16631	0.04360	0.00077	
10	0.63690	0.40964	0.07065	0.00059	
11	1.69506	1.34893	0.10582		
12	1.14017	0.80025	0.04454		
13	0.29827	0.19596	0.00565		
14	1.45809	6.5e-06	1.5e-07		
15	0.16523	0.03886	0.00089		
16	0.26212	0.03754	0.00096		
17	0.12282	0.01761	0.00027		
18	0.01702	0.00149	2.2e-05		
19	0.00157	0.00011	2.5e-06		
20	0.00362	0.00055	1.1e-05		
21	0.00116	0.00034	4.7e-06		
22	0.00016	5.2e-05	8.5e-07		
23	2.8e-06	1.4e-06	1.9e-08		
24	2.9e-06	1.6e-06	1.8e-08		
25	3.1e-06	1.2e-06	8.1e-07		
26	8.3e-07	4.0e-07	1.7e-08		
27	7.7e-08	4.3e-08	3.2e-07		
28	1.4e-09	1.9e-09			
29	1.1e-08	8.4e-09			
30	1.1e-08	5.8e-09			
31	2.0e-08				
32	2.2e-08				
33	5.3e-08				
34	2.2e-08				
35	2.3e-08				

TABLE 3.7

Maximal error (domain: unit square; solution angle = 1.3457; K = 20), FOSLS

p h	1	1/2	1/4	1/8	1/16
1	1.5e+05	2.0e+05	4.4e+05	1.2e+06	3.0e+06
2	1.6e+05	2.1e+05	4.6e+05	1.3e+06	4.8e+06
3	5.3e+05	5.4e+05	8.1e+05	2.5e+06	1.4e+07
4	1.7e+05	2.4e+05	7.5e+05	5.5e+06	7.0e+07
5	3.3e+06	4.1e+06	4.9e+06	2.0e+07	1.2e+10
6	8.6e+06	8.8e+06	8.5e+06	4.7e+08	
7	7.8e+06	8.0e+06	1.1e+07	1.6e+09	
8	8.7e+06	8.4e+06	1.4e+07	3.5e+08	
9	1.1e+07	1.1e+07	9.2e+07	1.5e+10	
10	1.6e+07	2.4e+07	3.3e+08	2.2e+11	
11	1.8e+07	3.9e+07	6.2e+08		
12	2.0e+07	5.2e+07	1.5e+09		
13	2.5e+07	1.7e+08	7.5e+09		
14	3.8e+07	4.2e+08	9.8e+10		
15	5.7e+07	1.3e+09	2.6e+11		
16	1.3e+08	8.8e+08	2.5e+11		
17	3.6e+08	8.5e+09	1.1e+13		
18	8.1e+08	2.5e+10	1.0e+14		
19	1.7e+09	6.1e+10	3.5e+14		
20	5.5e+09	9.5e+10	5.8e+14		
21	3.1e+10	1.5e+12	1.6e+16		
22	3.1e+11	9.7e+12	1.8e+17		
23	3.2e+12	8.4e+13	1.0e+18		
24	3.8e+13	5.9e+14	1.2e+19		
25	5.6e+14	1.1e+16	2.3e+22		
26	8.9e+15	9.8e+16	6.3e+21		
27	1.4e+17	1.7e+18	2.0e+22		
28	2.5e+18	3.1e+19			
29	5.0e+19	1.3e+21			
30	8.9e+20	9.7e+21			
31	2.2e+22				
32	1.6e+22				
33	1.3e+23				
34	3.0e+22				
35	4.2e+23				

TABLE 3.8

Condition number (domain: unit square; solution angle = 1.3457; K = 20), FOSLS

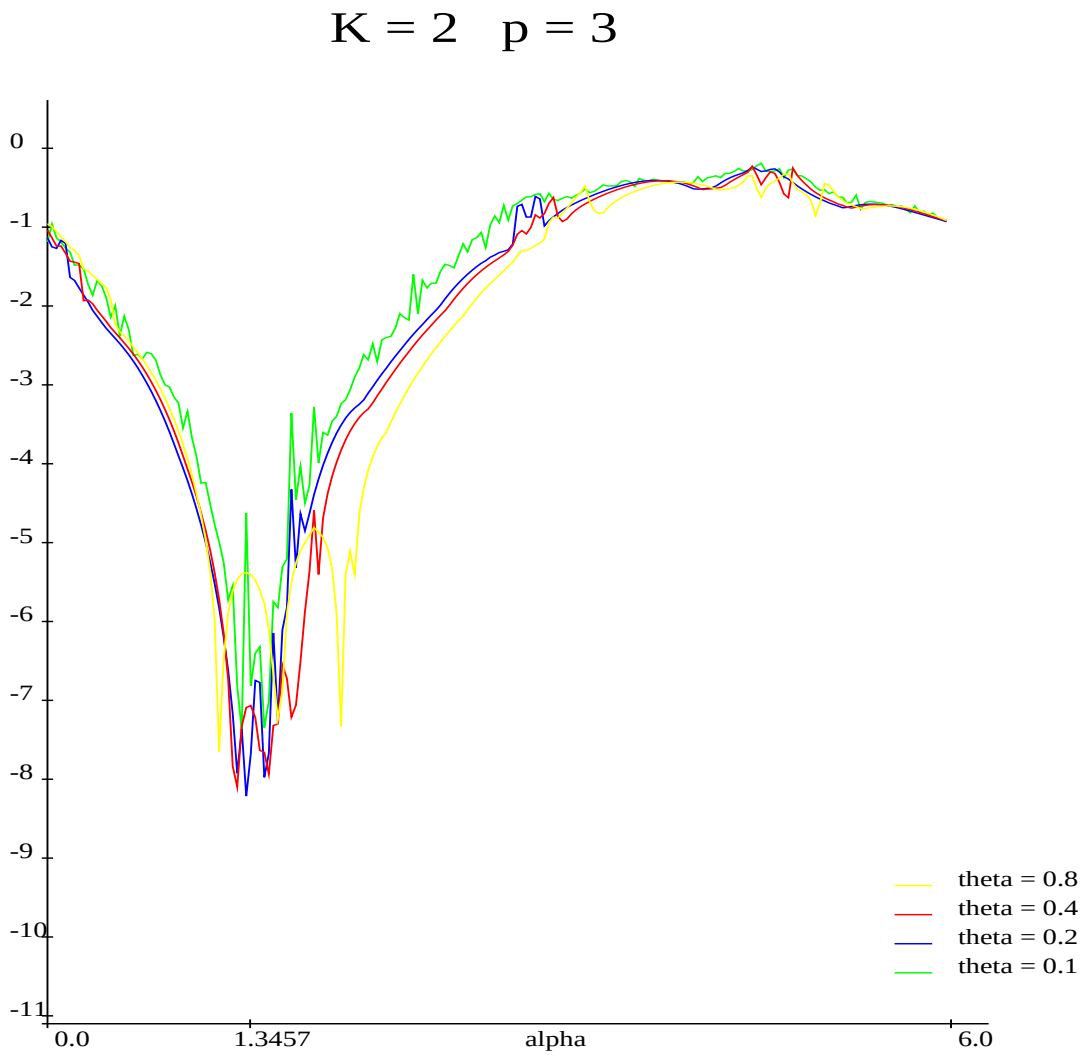

FIG. 4.1. *Maximal error of the approximation of a plane wave in the direction 1.3457 by nodal bases with three directions* α, $\alpha \pm \theta/2$. *Domain: unit square,* $h = 1$.

4. Selection of directions. In this section, we address the choice of approximating the directions $\mathbf{d}$ in the basis functions (2.1). Our test example is as in the previous section, and with $h = 1$, i.e., the domain is just one element.

The first test is to assess how close the choice of the directions has to be for good solution accuracy. We have considered a model problem with the exact solution a plane wave, and the basis with three close directions at each node. One would expect that to obtain a good approximation, the direction of the solution should be close to the directions of the basis functions, and that the accuracy of the solution would increase as the direction of the solution grows closer to one of the basis directions.

angle	node (0,0)	node (0,1)	node (1,0)	node (1,1)
0.00	-0.0149 + I* 0.0092	0.0567 + I* 0.0238	-0.0283 + I*-0.0137	-0.0310 + I* 0.0076
0.42	0.0024 + I*-0.1086	-0.1440 + I* 0.0095	0.0759 + I* 0.1096	0.0925 + I*-0.0557
0.84	0.0029 + I* 0.1002	-0.1267 + I*-0.0028	0.0720 + I*-0.1662	0.0350 + I* 0.0926
1.26	1.0687 + I*-0.1811	0.0991 + I*-1.1415	1.1218 + I* 0.1645	0.5340 + I*-0.8541
1.68	-0.0654 + I* 0.2077	-0.0676 + I*-0.2173	-0.0913 + I* 0.2997	-0.0662 + I*-0.2714
2.09	0.0826 + I*-0.0858	-0.0880 + I* 0.1180	-0.0966 + I* 0.1423	0.0275 + I*-0.0960
2.51	-0.0490 + I* 0.0417	0.0367 + I*-0.0738	0.0213 + I*-0.0794	0.0055 + I* 0.0447
2.93	0.0119 + I*-0.0191	0.0466 + I* 0.0115	-0.0158 + I* 0.0269	-0.0400 + I*-0.0179
3.35	-0.0048 + I*-0.0072	-0.0229 + I*-0.0389	0.0253 + I* 0.0195	0.0029 + I* 0.0261
3.77	0.0115 + I* 0.0223	-0.0021 + I* 0.0050	0.0022 + I*-0.0479	0.0035 + I* 0.0300
4.19	-0.0334 + I*-0.0186	0.0491 + I*-0.0191	0.0081 + I*-0.0073	-0.0271 + I* 0.0261
4.61	0.0647 + I*-0.0110	-0.0448 + I* 0.0049	-0.0368 + I* 0.0241	0.0197 + I*-0.0053
5.03	0.0042 + I* 0.0294	0.0097 + I*-0.0395	-0.0195 + I*-0.0044	0.0158 + I* 0.0155
5.45	0.0076 + I* 0.0578	-0.0512 + I* 0.0065	-0.0497 + I*-0.0137	0.0511 + I*-0.0482
5.86	-0.0843 + I*-0.0200	0.0601 + I* 0.0788	0.0891 + I* 0.0360	-0.0356 + I*-0.0690

TABLE 4.1

Values of the solution on nodes and 15 uniformly distributed basis directions. The exact solution is a plane wave at angle 1.3457. K = 20, nx = ny = 1, hx = hy = 1.

The results of such experiment are summarized in Fig. 4.1. The good news is that accurate solutions are obtained even for quite a wide cone of directions, and that the directions of the basis do not have to match the direction of the solution well. This is an indication of feasibility of a Finite Ray Element method with only a few directions per node that approximate a dominant direction present in the solution.

The next question is how to find dominant directions in a crude approximation to the solution; once those dominant directions are found, we can prune directions that are not useful to decrease cost, and add directions close to the dominant directions for more accuracy.

One obvious approach is to look at the coefficients of an approximate solution. One would expect that largest coefficients should be found at the basis functions that match the direction of the solution best. This is indeed the case for the results in Table 4.1. However, we need to find the dominant direction of the solution with greater accuracy than the spacing of the basis directions, and it is not clear how to do that based on coefficients size.

Therefore, we have developed an alternative approach: if p is the approximate solution, we look for the best L^2 approximation of p by a plane wave within the element. That is, we find angle θ so that

$$e(\theta) = \int_E |p(\mathbf{x}) - e^{iK\mathbf{x}\cdot(\cos\theta,\sin\theta)}|^2 d\mathbf{x} \to \min.$$

This is equivalent to

$$\int_E p(\mathbf{x}) e^{iK\mathbf{x}\cdot(\cos\theta,\sin\theta)} d\mathbf{x} \to \max.$$

The same routines can be used for the computation of this integral as for the computation of the mass matrix.

In Fig. 4.2, we see that this approach has found two dominant directions in a solution quite well.

5. Conclusion. The new method has good approximation properties. Good quality solution is achieved with a very small small number of degrees of freedom.

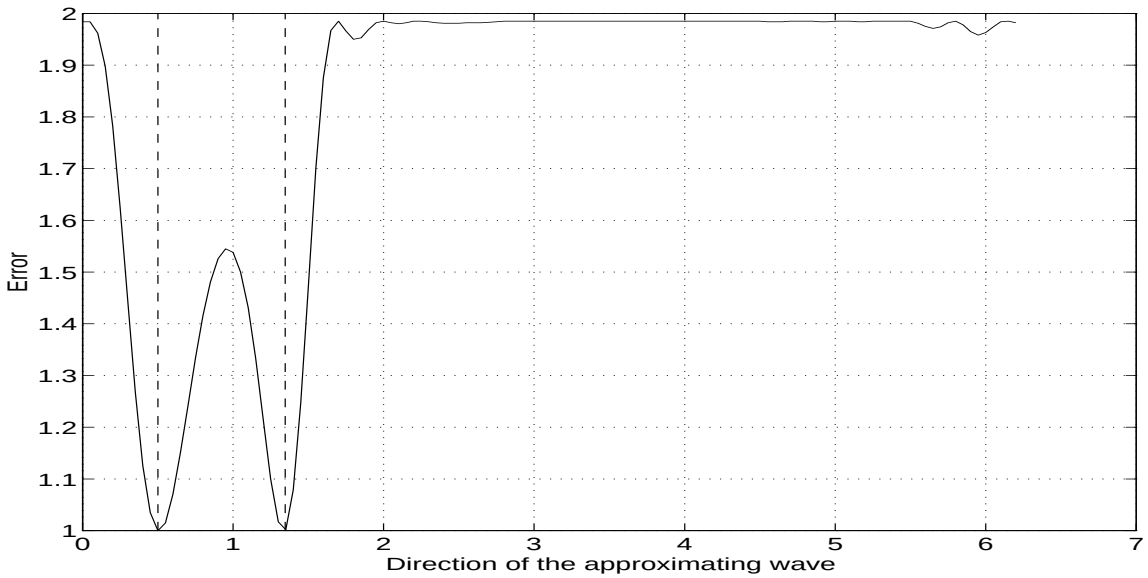

FIG. 4.2. *L^2 error of matching a solution with two dominant directions by a plane wave*

A good solution is obtained also when only a few directions close to the dominant direction of the solution are used. Dominant directions can be found by computing a crude approximation to the solution.

On the other hand, the Galerkin formulation yields a stiffnes matrix that is complex, non-symmetric, indefinite, and extremely ill-conditioned. Problems with such matrix are very difficult to solve, particularly by iterative methods. The FOSLS formulation gives a Hermitean positive definite matrix that is in principle more treatable, but at the cost of lower accuracy and many more variables.

Future developments will include further investigation of the sources of poor conditioning, ways to improve the condition, and design of an adaptive Finite Ray Element method.

REFERENCES

[1] I. BABUŠKA AND J. M. MELENK, *The partition of unity finite element method*, Int. J. Numer. Meths. Engrg., 40 (1997), pp. 727–758.

[2] Z. CAI, T. A. MANTEUFFEL, AND S. F. MCCORMICK, *First-order system least squares for second-order partial differential equations: Part ii.*, SIAM J. Numer. Anal., 34 (1997), pp. 425–454.

[3] A. DE LA BOURDONNAYE, *Une Méthode de discrétisation microlocale et son application à un problème de diffraction*, C. R. Acad. Sci. Paris, Serie I, 318 (1994), pp. 385–388.

[4] B. LEE, T. MANTEUFFEL, S. MCCORMICK, AND J. RUGE, *Multilevel first-order system least squares (FOSLS) for Helmholtz equation*, in Procs. 2nd International Conf. on Approx. and Num. Meths. for the Solution of the Maxwell Equations, Washington, D.C, John Wiley and Sons, 1993.

[5] J. M. MELENK, *On Generalized Finite Element Methods*, PhD thesis, University of Maryland, College Park, 1995.

[6] P. VANĚK, J. MANDEL, AND M. BREZINA, *Solving a two-dimensional Helmholtz problem by algebraic multigrid*, UCD/CCM Report 110, Center for Computational Mathematics, University of Colorado at Denver, October 1997. http://www-math.cudenver.edu/ccmreports/rep110.ps.gz.